Noether Gauge Symmetry Approach to Alternative Theories of Gravity

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OUTLINE

- Motivation behind Modified Gravity (cosmic accelerated expansion)
- Formulation of Noether and Noether Gauge Symmetry (NGS)
- Applying NGS to $f(R)$ Gravity
- NGS Approach to $f(R)-$ Brans-Dicke cosmology
- NGS Approach to Modified Teleparallel Gravity

Cosmic History

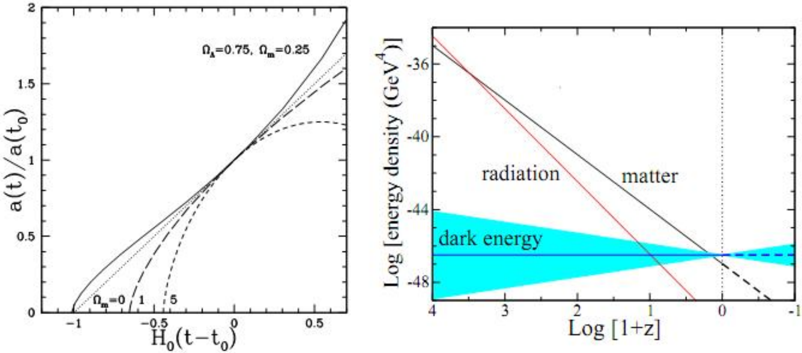

FIGURE 1. Left panel (a): Evolution of the scale factor vs. time for four cosmological models: three matter-dominated models with $\Omega_0 = \Omega_m = 0, 1, 5$, and one with $\Omega_\Lambda = 0.75, \Omega_m = 0.25$. Right panel (b): Evolution of radiation, matter, and dark energy densities with redshift. For dark energy, the band represents $w = -1 \pm 0.2$. From Frieman et al. [13].

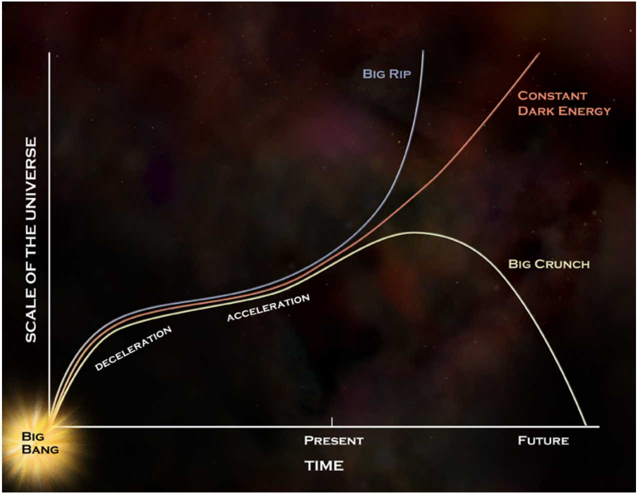

Figure: Future cosmic evolution crucially depends on what kind of dark energy is dominating the Universe at present time.

Possible Explanations of Cosmic Acceleration

There are three possible explanations for cosmic accelerations

- Introduce a "gravitationally repulsive" stress energy tensor , for example $T_{\mu\nu} = \Lambda g_{\mu\nu}$ etc. This approach is termed **Dark Energy**.
- Find the suitable geometric function $f(R)$ instead of the Einstein-Hilbert form. i.e. $R \longrightarrow f(R)$; $f(T)$ where T is torsion of space; $f(R, G)$, where G is Gauss-Bonnet term etc. This approach is termed **Modified Gravity**.
- Ignoring the cosmological principle, try to explain via large scale structure induced apparent acceleration. Possible approach is **Inhomogeneous Cosmology**.
- **Anthropic reasonings**.

Candidates of Dark Energy: Cosmological constant; Quintessence; K-essence; Chaplygin gas; Tachyon field etc

Candidates of Modified Gravity: $f(R)$ theory; $f(R, T)$ theory; $f(T)$ theory; Gauss-Bonnet theory, to name a few.

What is a Symmetry ?

- In simple language: symmetry is a property of a physical system which remains preserved under some change or shift.
- In geometry: a transformation is said to be a symmetry of an object if it leaves the object apparently unchanged.
- "A symmetry of some mathematical structure is a transformation of that structure, of a specified kind, that leaves specified properties of the structure unchanged." (*Ian Stewart*)
- There are various kinds of symmetries such as (1) Local and global symmetries (2) Continuous and discrete symmetries (3) Isometries or spacetime symmetries (4) Supersymmetry.
- A global symmetry is one that holds at all points of spacetime, where as a local symmetry is one that has a different symmetry transformation at different points of spacetime.

- A family of particular transformations may be continuous (such as rotation of a circle). A discrete symmetry is a symmetry that describes non-continuous changes in a system. For instance, a square possesses a discrete rotational symmetry.
- Continuous symmetries can be described by Lie groups while discrete symmetries are described by finite groups (symmetry groups).
- A supersymmetry asserts that each type of boson has, as a supersymmetric partner, a fermion, called a superpartner and vice versa. This symmetry has not been observed so far in experiments.

- An isometry or spacetime symmetry is a direction along with the metric tensor is Lie transported i.e. if ξ is an isometry than $L_\xi(g_{\mu\nu}) = 0$. Killing vector fields can be further generalized to conformal Killing vector fields defined by $L_\xi(g_{\mu\nu}) = \lambda(x) g_{\mu\nu}$.
- Spacetime symmetry is the most interesting class of continuous symmetries that involve transformations of space and time. These may be further classified as: (1)Spatial symmetries: involving only the spatial geometry associated with a physical system, (2) Temporal symmetries: involving only changes in time, (3) Spatio-temporal symmetries: involving changes both in space and time.
- Minkowski spacetime is maximally symmetric spacetime with 10 Killing vectors.

Symmetries have a wide range of applications in various fields, for instance

- to find solution of differential equations or to reduce it to a more simpler form for integration;
- to find solution of a boundary value problem;
- to classify special solutions;
- to deduce new solutions from known ones;
- to find conserved quantities using the most celebrated Noether's theorem.

Noether's Theorem

This theorem describes connections between symmetries of a physical system and their corresponding conserved quantities. It relates a class of conservation laws to symmetries of space, time and internal variables.

More precisely, Noether theorem states that if the Lie derivative of a Lagrangian $\mathcal{L}$ along a vector field Y vanishes i.e. $L_Y\mathcal{L} = 0$, than Y is a symmetry for the Lagrangian and generates the conserved quantity.

Alternatively, Noether symmetries are the symmetries of the Lagrangian.

Basic Formulation of Noether and Noether Gauge Symmetries

Consider the transformations depending upon the infinitesimal parameter ϵ i.e. $Q^i = Q^i(q^j, \epsilon)$ which can generate one parameter Lie group. The corresponding vector field Y with unknowns α^i defined by

$$Y = \alpha^i(q^j)\frac{\partial}{\partial q^i} + \left[\frac{d}{d\zeta}(\alpha^i(q^j))\right]\frac{\partial}{\partial \dot{q}^i},$$

is said to be a Noether symmetry generator for the Lagrangian $\mathcal{L}(t, q^i, \dot{q}^i)$ if it leaves the Lagrangian invariant i.e. $L_Y\mathcal{L} = 0$.
Noether gauge symmetries are the generalizations of Noether symmetries as the presence of some extra symmetries is expected in this case.
Consider the vector field Y such that

$$Y = \tau(t, q^i)\frac{\partial}{\partial t} + \xi^j(t, q^i)\frac{\partial}{\partial q^j},$$

and its first order prolongation is defined as

$$Y^{[1]} = Y + (\xi^j_{,t} + \xi^j_{,i}\dot{q}^i - \tau_{,t}\dot{q}^j - \tau_{,i}\dot{q}^i\dot{q}^j)\frac{\partial}{\partial \dot{q}^j}.$$

Here τ and ξ are unknown functions to be determined and t is the affine parameter.
The vector field Y is said to be a Noether gauge symmetry for the Lagrangian $\mathcal{L}$ if there exists a function (gauge term) $G(t, q^i)$ such that it satisfies

$$Y^{[1]}\mathcal{L} + (D_t\tau)\mathcal{L} = D_t G,$$

where $D_t = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i}$ is the total derivative operator. The corresponding conserved quantity is

$$I = \tau\mathcal{L} + (\xi^j - \dot{q}^j\tau)\frac{\partial \mathcal{L}}{\partial \dot{q}^j} - G.$$

Noether Gauge Symmetry Approach in $f(R)$ Gravity

We start with a (3+1)-dimensional action

$$S = \int d^4x\sqrt{-g}f(R), \tag{1}$$

where R is the scalar curvature and $f(R)$ is an arbitrary non-linear function of R.

Variation of the action (1) with respect to the metric yields the field equations

$$\frac{1}{2}g_{\mu\nu}f(R) - R_{\mu\nu}f'(R) + \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R) = 0. \tag{2}$$

The geometry of the spacetime is given by the flat FLRW line element

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \tag{3}$$

With the consideration of this model the field equations become

$$\begin{aligned} 2\dot{H} + 3H^2 &= -\frac{1}{f'}\left[f'''\dot{R}^2 + f''(2H\dot{R} + \ddot{R}) + \frac{1}{2}(f - Rf')\right], & (4) \\ H^2 &= \frac{1}{6f'}[f'R - f - 6\dot{R}Hf''], & (5) \end{aligned}$$

The action (1) can be written as

$$S = \int dt\mathcal{L}(a, \dot{a}, R, \dot{R}) = \int dt\Big[a^3 f(R) - \lambda\Big\{R - 6\Big(H^2 + \frac{\ddot{a}}{a}\Big)\Big\}\Big], \quad (6)$$

Varying the Lagrangian with respect to R yields

$$\lambda = a^3 f' \quad (7)$$

In order to apply the NGS approach, one may easily verify that, in the FRW model, the Lagrangian related to the above action takes the form

$$\mathcal{L}(a, \dot{a}, R, \dot{R}) = 6(\dot{a}^2 a f' + \dot{a}\dot{R}a^2 f'') + a^3(f'R - f). \quad (8)$$

A vector field

$$\mathbf{X} = \xi(t,a,R)\frac{\partial}{\partial t} + \eta(t,a,R)\frac{\partial}{\partial a} + \beta(t,a,R)\frac{\partial}{\partial R}, \tag{9}$$

whose first prolongation is

$$\mathbf{X}^{[1]} = \mathbf{X} + \dot{\eta}(t,a,R)\frac{\partial}{\partial \dot{a}} + \dot{\beta}(t,a,R)\frac{\partial}{\partial \dot{R}}, \tag{10}$$

where

$$\begin{aligned}
\dot{\eta} &\equiv \frac{\partial \eta}{\partial t} + \dot{a}\left(\frac{\partial \eta}{\partial a} - \frac{\partial \xi}{\partial t}\right) + \dot{R}\frac{\partial \eta}{\partial R} - \dot{a}^2\frac{\partial \xi}{\partial a} - \dot{a}\dot{R}\frac{\partial \xi}{\partial R}, \\
\dot{\beta} &\equiv \frac{\partial \beta}{\partial t} + \dot{a}\frac{\partial \beta}{\partial a} + \dot{R}\left(\frac{\partial \beta}{\partial R} - \frac{\partial \xi}{\partial t}\right) - \dot{R}^2\frac{\partial \xi}{\partial R} - \dot{a}\dot{R}\frac{\partial \xi}{\partial a},
\end{aligned} \tag{11}$$

is called a NGS if the following condition holds

$$\mathbf{X}^{[1]}\mathcal{L} + (\mathbf{D}\xi)\mathcal{L} = \mathbf{D}A(t,a,R). \tag{12}$$

Here A is the gauge function and

$$\mathbf{D} \equiv \frac{\partial}{\partial t} + \dot{a}\frac{\partial}{\partial a} + \dot{R}\frac{\partial}{\partial R}. \tag{13}$$

Using the Lagrangian (7) in (12) and after the separation of monomials we obtain the following system of determining equations

$$\xi_{,a} = 0, \quad \xi_{,R} = 0, \tag{14}$$

$$\eta f' + \beta a f'' + 2af'\eta_{,a} - \xi_{,t} a f' + a^2 f'' \beta_{,a} = 0, \tag{15}$$

$$12af'\eta_{,t} + 6a^2 f''\beta_{,t} = A_{,a}, \tag{16}$$

$$f''\eta_{,R} = 0, \tag{17}$$

$$6a^2 f''\eta_{,t} = A_{,R}, \tag{18}$$

$$2af''\eta + a^2 f'''\beta + a^2 f''(\eta_{,a} + \beta_{,R} - \xi_{,t}) + 2af'\eta_{,R} = 0, \tag{19}$$

$$a^2(3\eta + a\xi_{,t})(f'R - f) + a^3 f'' R\beta = A_{,t} \tag{20}$$

where "," denotes partial derivative.

There are three cases in the solution of the above determining equations (14)-(20). These are given below.

Case-I. If f is arbitrary, the solution of the above system (14)-(20) gives rise to the following NGS generator

$$\mathbf{X} = \frac{\partial}{\partial t}, \tag{21}$$

and the gauge term is a constant which can be taken as zero. The energy type first integral is

$$I = 6\dot{a}^2 a f' + 6\dot{a}\dot{R}a^2 f'' - a^3(f'R - f). \tag{22}$$

Case-II. If f is the fractional power law, viz.

$$f = f_0 R^{3/2}, \tag{23}$$

then the solution of the determining equations (14)-(20) yields

$$\xi = b_1 t + b_2,$$
$$\eta = \frac{2}{3} b_1 a + b_3 a^{-1},$$
$$\beta = -2R(b_1 + b_3 a^{-2}). \tag{24}$$

The gauge term is zero. The corresponding generators are

$$\mathbf{X}_0 = \frac{\partial}{\partial t}, \mathbf{X}_1 = t\frac{\partial}{\partial t} + \frac{2}{3}a\frac{\partial}{\partial a} - 2R\frac{\partial}{\partial R}, \mathbf{X}_2 = a^{-1}\frac{\partial}{\partial a} - 2Ra^{-2}\frac{\partial}{\partial R}, \tag{25}$$

which constitute a well-known three-dimensional algebra with commutation relations

$$[\mathbf{X}_0, \mathbf{X}_1] = 0,\ [\mathbf{X}_0, \mathbf{X}_2] = 0,\ [\mathbf{X}_1, \mathbf{X}_2] = -\frac{4}{3}\mathbf{X}_2.$$

Here too the gauge term is zero. The corresponding first integrals are

$$\begin{aligned} I_0 &= 9\dot{a}^2 a f_0 R^{1/2} + \frac{9}{2}\dot{a}\dot{R}a^2 f_0 R^{-1/2} - \frac{1}{2}a^3 f_0 R^{3/2}, \\ I_1 &= -9\dot{a}^2 a t f_0 R^{1/2} + \frac{1}{2} f_0 t a^3 R^{3/2} + 3\dot{a}a^2 f_0 R^{1/2} \\ &\quad +3\dot{R}a^3 f_0 R^{-1/2} - \frac{9}{2}\dot{a}\dot{R}ta^2 f_0 R^{-1/2}, \\ I_2 &= 9\dot{a} f_0 R^{1/2} + \frac{9}{2} a f_0 R^{-1/2}\dot{R}. \end{aligned} \tag{26}$$

Case-III. If f is a general power law of the form

$$f = f_0 R^{\nu}, \quad \nu \neq 0, 1, \frac{3}{2}, \tag{27}$$

then the above linear determining equation system has solution

$$\xi = b_1 t + b_2, \quad \eta = \frac{2\nu - 1}{3} a b_1, \quad \beta = -2Rb_1. \tag{28}$$

Here the gauge term is zero and f_0 is a constant.
Note that if $\nu = 0, 1, \frac{3}{2}$, then f is a constant, linear or fractional power law of Case-II. These are thus excluded in Case-III. It should also be stated that for Case-III there are two Noether symmetries. The Noether symmetry generators are given by

$$\mathbf{X}_0 = \frac{\partial}{\partial t}, \quad \mathbf{X}_1 = t\frac{\partial}{\partial t} + \frac{2\nu - 1}{3} a \frac{\partial}{\partial a} - 2R\frac{\partial}{\partial R}. \tag{29}$$

The Lie algebra is the two-dimensional Abelian Lie algebra, $[\mathbf{X}_0, \mathbf{X}_1] = 0$. We note that for the fractional power law the algebra is three-dimensional and for the arbitrary power law it is two-dimensional and thus the symmetry breaks by one.

The corresponding first integrals for this case are

$$\begin{aligned} I_0 &= 6\dot{a}^2 a\nu f_0 R^{\nu-1} + 6\dot{a}\dot{R}a^2\nu(\nu-1)f_0 R^{\nu-2} \\ &\quad -a^3 f_0 R^\nu(\nu-1), \\ I_1 &= -6\dot{a}^2 atf_0\nu R^{\nu-1} + a^3 tf_0(\nu-1)R^\nu \\ &\quad +4\nu(2-\nu)\dot{a}a^2 f_0 R^{\nu-1} + 2\nu(\nu-1)(2\nu-1) \\ &\quad \times\dot{R}a^3 f_0 R^{\nu-2} - 6\nu(\nu-1)\dot{a}\dot{R}ta^2 f_0 R^{\nu-2}. \end{aligned} \tag{30}$$

Noether Gauge Symmetry approach in $f(R)$-Brans-Dicke Cosmology

We begin with the action

$$S = \int d^4x\sqrt{-g}\left(\phi f(R) - \frac{\omega}{\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right), \qquad (31)$$

In the background of flat FRW spacetime, the action becomes

$$S = \int d^4x a^3(t)\left(\phi f(R) + \frac{\omega}{\phi}\dot{\phi}^2 - V(\phi)\right). \qquad (32)$$

The Lagrangian is

$$\begin{aligned} L(a, R, \phi) &= a^3(t)\left(\phi f(R) + \frac{\omega}{\phi}\dot{\phi}^2 - V(\phi)\right) \\ &\quad -\lambda\left(R - 6\left(\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2\right)\right), \end{aligned} \qquad (33)$$

where λ is a Lagrange multiplier.

By varying the action w.r.t R we obtain:

$$\lambda = \phi f'(R). \tag{34}$$

The action in the final form becomes

$$\begin{aligned} L(a, R, \phi) \;\; &= \;\; a^3(t)\left(\phi f(R) + \frac{\omega}{\phi}\dot{\phi}^2 - V(\phi)\right) \\ &\quad -\phi f'(R)\left(R - 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)\right). \end{aligned} \tag{35}$$

At first this Lagrangian must be converted to a new Lagrangian containing only $a, \dot{a}$ and not $\ddot{a}$. Its done by taking a part-by-part integration from the last term

$$\int dt a^3(t)\left(\phi f'(R)\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)\right) = \int dt \dot{a}\{a\dot{a}\phi f'(R) - \frac{d}{dt}(a^2\phi f'(R))\}.$$

Thus the total Lagrangian of our model becomes

$$L(a,R,\phi,\dot{a},\dot{\phi},\dot{R}) = \quad a^3(t)\Big(\phi f(R) + \tfrac{\omega}{\phi}\dot{\phi}^2 - V(\phi) - \phi f'(R)R\Big) + 6\dot{a}\{a\dot{a}\phi f'(R) - \tfrac{d}{dt}(a^2\phi f'(R))\}. \quad (37)$$

To calculate the Noether symmetries, we define it first. A vector field

$$\mathbf{X} = \mathcal{T}(t,a,R,\phi)\frac{\partial}{\partial t} + \alpha(t,a,R,\phi)\frac{\partial}{\partial a} + \beta(t,a,R,\phi)\frac{\partial}{\partial R} + \gamma(t,a,R,\phi)\frac{\partial}{\partial \cdots}$$

is a Noether symmetry corresponding to a Lagrangian $L(t,a,R,\phi,\dot{a},\dot{R},\dot{\phi})$ if

$$\mathbf{X}^{[1]}L + LD_t(\mathcal{T}) = D_t B, \quad (39)$$

holds, where $\mathbf{X}^{[1]}$ is the first prolongation of the generator $\mathbf{X}$, $B(t,a,R,\phi)$ is a gauge function and D_t is the total derivative operator

$$D_t = \frac{\partial}{\partial t} + \dot{a}\frac{\partial}{\partial a} + \dot{R}\frac{\partial}{\partial R} + \dot{\phi}\frac{\partial}{\partial \phi}. \quad (40)$$

The prolonged vector field is given by

$$\mathbf{X}^{[1]} = \mathbf{X} + \alpha_t \frac{\partial}{\partial \dot{t}} + \beta_t \frac{\partial}{\partial \dot{R}} + \gamma_t \frac{\partial}{\partial \dot{\phi}}, \tag{41}$$

where

$$\begin{aligned} \alpha_t &= D_t \alpha - \dot{a} D_t \mathcal{T}, \quad \beta_t = D_t \beta - \dot{R} D_t \mathcal{T}, \\ \gamma_t &= D_t \gamma - \dot{\phi} D_t \mathcal{T}. \end{aligned} \tag{42}$$

If $\mathbf{X}$ is the Noether symmetry corresponding to the Lagrangian $L(t, a, R, \phi, \dot{a}, \dot{\phi}, \dot{R})$, then

$$\mathbf{I} = \mathcal{T} L + (\alpha - \mathcal{T} \dot{a}) \frac{\partial L}{\partial \dot{a}} + (\beta - \mathcal{T} \dot{R}) \frac{\partial L}{\partial \dot{R}} + (\gamma - \mathcal{T} \dot{\phi}) \frac{\partial L}{\partial \dot{\phi}} - B, \tag{43}$$

is a first integral or an invariant or a conserved quantity associated with $\mathbf{X}$.

The Noether condition results in the over-determined system of equations

$$\mathcal{T}_a = 0, \quad \mathcal{T}_\phi = 0, \quad \mathcal{T}_R = 0, \quad \alpha_R = 0, \tag{44}$$

$$12\alpha_t a\phi f' + 6\beta_t a^2\phi f'' + 6\gamma_t a^2 f' = B_a, \tag{45}$$

$$6\alpha_t a^2\phi f'' + B_R = 0, \tag{46}$$

$$6\alpha_t a^2\phi f' - 2\gamma_t a^3\omega + \phi B_\phi = 0, \tag{47}$$

$$3\alpha_\phi \phi^2 f'' - a\omega\gamma_R = 0, \tag{48}$$

$$\begin{aligned} &3\alpha a^2\left(\phi f - V - \phi R f'\right) - \beta a^3\phi R f'' + \gamma a^3 (f \\ &- V' - Rf') + a^3\left(\phi f - V - \phi R f'\right)\mathcal{T}_t = B_t, \end{aligned} \tag{49}$$

$$\begin{aligned} &\alpha\phi f' + \beta a\phi f'' + \gamma a f' + 2a\phi f'\alpha_a - a\phi f'\mathcal{T}_t \\ &+ \beta_a a^2\phi f'' + \gamma_a a^2 f' = 0, \end{aligned} \tag{50}$$

$$3\alpha\omega\phi - \gamma a\omega - 6\alpha_\phi f'\phi^2 + 2a\omega\phi\gamma_\phi - a\omega\phi\mathcal{T}_t = 0, \tag{51}$$

$$2\alpha\phi f'' + \beta a\phi f''' + \gamma a f'' + a\phi f''\alpha_a - a\phi f''\mathcal{T}_t$$
$$+ a\phi f''\beta_R + af'\gamma_R = 0, \tag{52}$$
$$6\alpha\phi f' + 3\beta\phi a f'' + 6\alpha_\phi\phi^2 f' + 3a\alpha_a\phi f'$$
$$- 3a\phi f'\mathcal{T}_t + 3\beta_\phi a\phi^2 f'' + 3a\gamma_\phi\phi f' - \gamma_a a^2\omega = 0. \tag{53}$$

With the help of the Computer Algebra System (Maple), the solutions of the above set of linear Partial differential equations for $f(R), V(\phi), \mathcal{T}, \alpha, \beta$ and γ are obtained as follows:

$$f(R) = rR + c_1, \tag{54}$$
$$V(\phi) = c_1\phi + V_\circ\phi^2, \tag{55}$$
$$\mathcal{T} = t + c_2, \tag{56}$$
$$\alpha = a, \tag{57}$$
$$\beta = 0, \tag{58}$$
$$\gamma = -2\phi, \tag{59}$$

where r, c_1, c_2 and $V_\circ$ are constants and the gauge function B is zero here. The Lagrangian admits two Noether symmetry generators

$$\mathbf{X}_1 = \frac{\partial}{\partial t}, \tag{60}$$

$$\mathbf{X}_2 = t\frac{\partial}{\partial t} + a\frac{\partial}{\partial a} - 2\phi\frac{\partial}{\partial \phi}. \tag{61}$$

The first symmetry $\mathbf{X}_1$ (invariance under time translation) gives the energy conservation $\mathbf{I}_1$ of the dynamical system, while the second symmetry $\mathbf{X}_2$ (scaling symmetry) and a corresponding conserved quantity $\mathbf{I}_2$. The two first integrals (conserved quantities) which are

$$\mathbf{I}_1 = 6a\dot{a}^2\phi r - \frac{a^3 w}{\phi}\dot{\phi}^2 - a^3 V_\circ \phi^2, \tag{62}$$

$$\mathbf{I}_2 = 6ta\dot{a}^2\phi r - t\frac{a^3 w}{\phi}\dot{\phi}^2 - ta^3 V_\circ \phi^2 + 12a^2\phi r(1-\dot{a}) - 4a^3 w\dot{\phi}. \tag{63}$$

When we take $r = 1$ and $c_1 = 0$ then

$$f(R) = R, \quad V(\phi) = V_\circ \phi^2, \tag{64}$$

and Noether symmetries remains unchanged while first integrals changes accordingly.

Torsion Tensor and Torsion Scalar

General Relativity (GR) is a geometrical theory of gravitation in which gravitational field appears as a manifestation of spacetime curvature. According to GR, spacetime is represented by a (3+1)-dimensional differentiable manifold endowed with a Lorentzian metric and a Levi-Civita connection.

A more general geometrical setting: the metric and connection may be taken as independent structures. A general metric-compatible connection has two important properties: curvature and torsion. Torsion is priori absent absent in the geometrical framework of GR, although it was taken into consideration by Einstein for a unified theory of gravitation and electromagnetism. Since then, the meaning and role of torsion in gravitation have been a recurring theme.

Teleparallel Gravity

- As an alternative gravitational theory to general relativity, "teleparallelism" has been considered. It is constructed by using the Weitzenböck connection.
- The action is described by the torsion scalar T and not the Ricci scalar curvature R as in general relativity formulated with the Levi-Civita connection.
- It is known that the modified teleparallel gravity, so-called $f(T)$ gravity, can realize both inflation and the late-time cosmic acceleration. (See for review arXiv:1202.4057 [gr-qc])

Brief Review of Teleparallel Gravity

- In TG, dynamical variables are vierbein or tetrad fields, $\mathbf{e}_a(x^\mu)$, which form an orthonormal basis for the tangent space at each point of the manifold with spacetime coordinates x^μ.
- $\mathbf{e}_a(x^\mu)$ is a vector in tangent space, and can be described in a coordinate basis by its components e_a^μ, so that e_a^μ is a vector in a spacetime.
- The spacetime metric is given by

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b,$$

  where $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$, is the Minkowski metric for the tangent space.
- Tetrad fields satisfy orthonormal conditions

$$e_a^\mu e_\nu^a = \delta_\nu^\mu, \quad e_a^\mu e_\mu^b = \delta_a^b$$

- In general relativity (GR), we use the Levi-Civita connection

$$\Gamma^{\lambda}_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}),$$

  in which commas denotes partial derivatives. This leads to nonzero spacetime curvature but zero torsion.
- In TG, the the Weitzenbock connection $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ (tilded to distinguish it from $\Gamma^{\lambda}_{\mu\nu}$),

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} \equiv e_b^{\lambda}\partial_\nu e^b_\mu = -e^b_\mu \partial_\nu e_b^{\lambda}$$

  which leads to zero curvature but nonzero torsion.
- The torsion tensor reads

$$T^{\lambda}_{\mu\nu} \equiv \tilde{\Gamma}^{\lambda}_{\nu\mu} - \tilde{\Gamma}^{\lambda}_{\mu\nu}.$$

- The contorsion tensor reads

$$K^{\rho}_{\mu\nu} = \tilde{\Gamma}^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\nu}.$$

- To construct a Lagrangian density out of torsion tensor for TG, we need to define another tensor

$$S^{\rho\mu\nu} \equiv K^{\mu\nu\rho} - g^{\rho\nu} T^{\sigma\mu}_{\sigma} + g^{\rho\mu} T^{\sigma\nu}_{\sigma}.$$

- The teleparallel Lagrangian density is given by

$$\mathcal{L} \equiv \frac{e}{16\pi G} T = S^{\rho\mu\nu} T_{\rho\mu\nu},$$

  where $e = \sqrt{-g}$ is the determinant of e^{λ}_{a} and $g = det(g_{\mu\nu})$.
- The $f(T)$ gravity theory generalizes T in the lagrangian density to an arbitrary function of T. The equations of motion for $f(T)$ gravity are

$$e^{-1}\partial_{\mu}(eS_{i}{}^{\mu\nu})(1+f_{T}) - e_{i}^{\lambda} T^{\rho}{}_{\mu\lambda} S_{\rho}{}^{\nu\mu} f_{T}$$
$$+S_{i}{}^{\mu\nu}\partial_{\mu}(T) f_{TT} - \frac{1}{4} e^{\nu}_{i}(1+f(T)) = 4\pi G e_{i}^{\rho} T_{\rho}{}^{\nu}.$$

Modified Teleparallel Gravity Minimally Coupled with a Canonical Scalar Field

One suitable form of the action for $f(T)$ gravity in Weitzenböck spacetime is

$$\mathcal{S} = \int d^4xe\Big((T + f(T)) + \mathcal{L}_m\Big)\,, \tag{65}$$

where $f(T)$ is an arbitrary function of torsion T and $e = \det(e^i_\mu)$. The dynamical quantity of the model is the scalar torsion T and the matter Lagrangian $\mathcal{L}_m$. The total action reads:

$$\mathcal{S} = \int d^4x\; e\Big(T + f(T) + \lambda(T + 6H^2) + V(\phi)\phi_{;\mu}\phi^{;\mu} - W(\phi)\Big). \tag{66}$$

Here trace of the torsion tensor is $T = -6H^2$, $e = det(e^\mu_i)$, λ is the Lagrange multiplier and $H = \frac{\dot{a}}{a}$ is the Hubble parameter. For our convenience, we keep the original Saez-Ballester scalar field in our effective action. Varying action with respect to T, we obtain

$$\lambda = -(1 + f'(T)). \tag{67}$$

Integrating over the spatial volume we get the following reduced Lagrangian:

$$\begin{aligned}\mathcal{L}(a,\phi,T,\dot{a},\dot{\phi}) &= a^3\Big[T+f(T)-[1+f'(T)]\Big(T+6(\frac{\dot{a}}{a})^2\Big) \\ &\quad +V(\phi)\dot{\phi}^2-W(\phi)\Big]. \end{aligned} \tag{68}$$

For the Lagrangian, the equations of motion read

$$f_{TT}(T+6H^2)=0, \tag{69}$$

$$\frac{\ddot{a}}{a}=-\frac{1}{4(1+f_T)}\Big[f-Tf_T-\frac{T}{3}(1+f_T)+V(\phi)\dot{\phi}^2-W(\phi)+4H\dot{T}f_{TT}\Big], \tag{70}$$

$$\ddot{\phi}+3H\dot{\phi}+\frac{1}{2V}(V'\dot{\phi}^2+W')=0. \tag{71}$$

Here we have two possibilities: (1) $f_{TT}=0$, which gives the teleparallel gravity and we are not interested in this case. (2) Another possibility is $T=-6H^2$ which is the standard definition of the torsion scalar in $f(T)$ gravity.

To calculate the Noether symmetries, we define it first. A vector field

$$\mathbf{X} = \mathcal{T}(t,a,T,\phi)\frac{\partial}{\partial t} + \alpha(t,a,T,\phi)\frac{\partial}{\partial a} + \beta(t,a,T,\phi)\frac{\partial}{\partial T} + \gamma(t,a,T,\phi)\frac{\partial}{\partial \phi}$$

is a Noether gauge symmetry corresponding to a Lagrangian $\mathcal{L}(t,a,T,\phi,\dot{a},\dot{T},\dot{\phi})$ if

$$\mathbf{X}^{[1]}\mathcal{L} + \mathcal{L}D_t(\mathcal{T}) = D_t B, \tag{73}$$

holds, where $\mathbf{X}^{[1]}$ is the first prolongation of the generator $\mathbf{X}$, $B(t,a,T,\phi)$ is a gauge function and D_t is the total derivative operator

$$D_t = \frac{\partial}{\partial t} + \dot{a}\frac{\partial}{\partial a} + \dot{T}\frac{\partial}{\partial T} + \dot{\phi}\frac{\partial}{\partial \phi}. \tag{74}$$

The prolonged vector field is given by

$$\mathbf{X}^{[1]} = \mathbf{X} + \alpha_t\frac{\partial}{\partial \dot{t}} + \beta_t\frac{\partial}{\partial \dot{T}} + \gamma_t\frac{\partial}{\partial \dot{\phi}}, \tag{75}$$

where

$$\alpha_t = D_t\alpha - \dot{a}D_t\mathcal{T}, \quad \beta_t = D_t\beta - \dot{T}D_t\mathcal{T}, \quad \gamma_t = D_t\gamma - \dot{\phi}D_t\mathcal{T}. \tag{76}$$

If $\mathbf{X}$ is the Noether symmetry corresponding to the Lagrangian $\mathcal{L}(t, a, T, \phi, \dot{a}, \dot{T}, \dot{\phi})$, then

$$\mathbf{I} = \mathcal{T}\mathcal{L} + (\alpha - \mathcal{T}\dot{a})\frac{\partial\mathcal{L}}{\partial\dot{a}} + (\beta - \mathcal{T}\dot{T})\frac{\partial\mathcal{L}}{\partial\dot{T}} + (\gamma - \mathcal{T}\dot{\phi})\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - B, \tag{77}$$

is a first integral or an invariant or a conserved quantity associated with $\mathbf{X}$.

The Noether condition results in the over-determined system of equations

$$\mathcal{T}_a = 0, \quad \mathcal{T}_\phi = 0, \quad \mathcal{T}_T = 0, \quad \alpha_T = 0, \tag{78}$$

$$\gamma_T = 0, \quad B_T = 0, \quad 2a^3 V \gamma_t = B_\phi, \tag{79}$$

$$6(1+f')\alpha_\phi - a^2 V \gamma_a = 0, \tag{80}$$

$$12a(1+f')\alpha_t + B_a = 0, \tag{81}$$

$$3V\alpha + aV'\gamma + 2aV\gamma_\phi - aV\mathcal{T}_t = 0, \tag{82}$$

$$(1+f')(\alpha + 2a\alpha_a - a\mathcal{T}_t) + af''\beta = 0, \tag{83}$$

$$3a^2(f - Tf' - W)\alpha - a^3 Tf''\beta - a^3 W'\gamma + a^3(f - Tf' - W)\mathcal{T}_t = B_t. \tag{84}$$

We obtain the solution of the above system of linear partial differential equations for $f(T)$, $V(\phi)$, $W(\phi)$, $\mathcal{T}$, α, β and γ. We have:

$$f(T) = \frac{1}{2}t_0 T^2 - T + c_2, \tag{85}$$

$$V(\phi) = V_0\phi^{-4}, \tag{86}$$

$$W(\phi) = W_0\phi^{-4} + c_2, \tag{87}$$

$$\mathcal{T} = t + c_1, \tag{88}$$

$$\alpha = a, \tag{89}$$

$$\beta = -2T, \tag{90}$$

$$\gamma = \phi, \tag{91}$$

where t_0, V_0, W_0, c_2 and c_1 are constants.

It is clear that the Lagrangian admits two Noether symmetry generators

$$\mathbf{X}_1 = \frac{\partial}{\partial t}, \tag{92}$$

$$\mathbf{X}_2 = t\frac{\partial}{\partial t} + a\frac{\partial}{\partial a} - 2T\frac{\partial}{\partial T} + \phi\frac{\partial}{\partial \phi}. \tag{93}$$

The two first integrals (conserved quantities) which are

$$\mathbf{I}_1 = -\frac{1}{2}t_0 T^2 a^3 + 6t_0 Ta\dot{a}^2 - V_0 a^3\phi^{-4}\dot{\phi}^2 - W_0 a^3\phi^{-4}, \tag{94}$$

$$\mathbf{I}_2 = -\frac{1}{2}t_0 t T^2 a^3 + 6t_0 t Ta\dot{a}^2 - V_0 ta^3\phi^{-4}\dot{\phi}^2 - W_0 ta^3\phi^{-4} - 12t_0 Ta^2\dot{a} + 2V_0 a^3\phi^{-3}\dot{\phi}. \tag{95}$$

Also the commutator of generators satisfies $[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1$ which shows that the algebra of generators is closed.

$F(T)$ Cosmology with Canonical Scalar Fields

Consider Noether symmetry in $F(T)$ cosmology in the present of matter and scalar field. The Lagrangian of our model is

$$\mathcal{S} = \int d^4x \; e(\mathcal{L}_F + \mathcal{L}_m + \mathcal{L}_\phi), \tag{96}$$

Specifically the total action reads (in chosen units $16\pi G = \hbar = c = 1$)

$$\mathcal{S} = \int dt \; a^3 \Big[F(T) - \rho_m + \frac{1}{2}\epsilon \phi_{,\mu}\phi^{,\mu} - V(\phi) \Big]. \tag{97}$$

In FRW cosmological background, the Lagrangian is

$$\begin{aligned} \mathcal{S} &= \int dt \; a^3 \Big[F(T) - \lambda(T + 6H^2) - \frac{\rho_{m0}}{a^3} \\ &\quad + \frac{1}{2}\epsilon \phi_{,\mu}\phi^{,\mu} - V(\phi) \Big]. \end{aligned} \tag{98}$$

The point-like Lagrangian reads (ignoring a constant factor $2\pi^2$)

$$\mathcal{L}(a,\dot{a},\phi,\dot{\phi},T)=a^3(F-TF_T)-6F_T a\dot{a}^2-\rho_{m0}-a^3\Big(\frac{1}{2}\epsilon\dot{\phi}^2+V(\phi)\Big).(99$$

Moreover for a given dynamical system, the Euler-Lagrange equation is

$$\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\Big)-\frac{\partial\mathcal{L}}{\partial q_i}=0, \tag{100}$$

where $q_i=a,\phi,T$ are the generalized coordinates of the configuration space $\mathcal{Q}=\{a,\phi,T\}$. The three equations of motion (corresponding to variations of $\mathcal{L}$ with respect to T,ϕ,a respectively) are

$$a^3F_{TT}(T+6H^2)=0, \tag{101}$$

$$\epsilon(\ddot{\phi}+3H\dot{\phi})-\frac{\partial V}{\partial\phi}=0, \tag{102}$$

$$4\frac{\ddot{a}}{a}(F_T+2TF_T)+4H^2(F_T-2TF_T)+F-TF_{TT}=\rho_\phi, \tag{103}$$

where

$$\rho_\phi\equiv\frac{1}{2}\epsilon\dot{\phi}^2+V(\phi).$$

The Noether symmetry generator is a vector field defined by

$$\mathbf{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial \phi} + \eta\frac{\partial}{\partial T} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{\phi}} + \dot{\eta}\frac{\partial}{\partial \dot{T}}, \tag{104}$$

where dot represents the total derivative given by

$$\frac{d}{dt} \equiv \dot{\phi}\frac{\partial}{\partial \phi} + \dot{a}\frac{\partial}{\partial a} + \dot{T}\frac{\partial}{\partial T}. \tag{105}$$

The vector field $\mathbf{X}$ can be thought of as a vector field on $\mathcal{TQ} = (a, \dot{a}, \phi, \dot{\phi}, T, \dot{T})$ is the related tangent bundle on which $\mathcal{L}$ is defined.

A Noether symmetry $\mathbf{X}$ of a Lagrangian $\mathcal{L}$ exists if the Lie derivative of $\mathcal{L}$ along the vector field $\mathbf{X}$ vanishes i.e.

$$L_X\mathcal{L} = \mathbf{X}\mathcal{L} = \alpha\frac{\partial \mathcal{L}}{\partial a} + \beta\frac{\partial \mathcal{L}}{\partial \phi} + \eta\frac{\partial \mathcal{L}}{\partial T} + \dot{\alpha}\frac{\partial \mathcal{L}}{\partial \dot{a}} + \dot{\beta}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} + \dot{\eta}\frac{\partial \mathcal{L}}{\partial \dot{T}} = 0. \tag{106}$$

By requiring the coefficients of $\dot{a}^2$, $\dot{\phi}^2$, $\dot{T}^2$, $\dot{a}\dot{\phi}$, $\dot{a}\dot{T}$ and $\dot{\phi}\dot{T}$ to be zero, we find

$$\begin{aligned}
3\alpha F - 3\alpha T F_T - 3\alpha V(\phi) - \eta a T F_{TT} - \beta a V'(\phi) &= 0, \quad (107)\\
\alpha F_T + \eta a F_{TT} + 2 a F_T \frac{\partial \alpha}{\partial a} &= 0, \quad (108)\\
\frac{3}{2}\alpha + a\frac{\partial \beta}{\partial \phi} &= 0, \quad (109)\\
12 F_T \frac{\partial \alpha}{\partial \phi} + \epsilon a^2 \frac{\partial \beta}{\partial a} &= 0, \quad (110)\\
12 a F_T \frac{\partial \alpha}{\partial T} &= 0, \quad (111)\\
\epsilon a^3 \frac{\partial \beta}{\partial T} &= 0. \quad (112)
\end{aligned}$$

By assuming $F_T \neq 0$, from Eqs. (111) and (112), we conclude

$$\alpha = \alpha(a, \phi), \quad \beta = \beta(a, \phi). \tag{113}$$

Now we must solve the system of equations (107)-(110). The non-trivial solution for this system reads as the following form (Model-I)

$$\begin{aligned}
F(T) &= \frac{4}{3} c_1 T^{\frac{3}{4}} + c_3, && (114)\\
V(\phi) &= c_4 + c_5 (c_1 \phi + c_2)^2, && (115)\\
\alpha(a, \phi) &= -\frac{2}{3} c_1 a, && (116)\\
\beta(a, \phi, T) &= c_1 \phi + c_2, && (117)\\
\eta(a, \phi, T) &= \frac{8}{3} c_1 T. && (118)
\end{aligned}$$

Using (116)-(118), the Noether symmetries are

$$\mathbf{X}_1=-\frac{2}{3}a\frac{\partial}{\partial a}+\frac{8}{3}T\frac{\partial}{\partial T}+\phi\frac{\partial}{\partial\phi},\quad \mathbf{X}_2=\frac{\partial}{\partial T}. \tag{119}$$

The symmetry $\mathbf{X}_1$ represents the scaling i.e. the Lagrangian remains invariant under scaling transformation while the second symmetry $\mathbf{X}_2$ shows that Lagrangian is invariant under T translation.

These NS generators form a two dimensional closed algebra

$$[\mathbf{X}_1,\mathbf{X}_2]=-\frac{8}{3}\mathbf{X}_2. \tag{120}$$

The conjugate momenta for the variables of configuration space $\mathcal{Q}$ can be defined as

$$p_a = \frac{\partial\mathcal{L}}{\partial\dot{a}}=-12a\dot{a}F_T, \tag{121}$$

$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}=\epsilon\dot{a}^3\dot{\phi}, \tag{122}$$

$$p_T = \frac{\partial\mathcal{L}}{\partial\dot{T}}=0. \tag{123}$$

Notice that $p_T = 0$ on account of symmetry $\mathbf{X}_2$. The Noether charge of the system reads

$$Q = \alpha p_a + \beta p_\phi + \eta p_T = 8c_1 a^2 \dot{a} F_T + (c_1\phi + c_2)\epsilon \dot{a}^3 \dot{\phi}. \qquad (124)$$

Using (114) in (124), we get

$$Q = -\frac{32}{36^{1/4}} c_1^2 (\dot{a}a^5)^{1/2} - a^3 \epsilon \dot{\phi}(c_1\phi + c_2). \qquad (125)$$

- *Remark:* One obvious solution of the system (107)-(112) is

$$\alpha = \eta = 0, \quad \beta = constant, \quad V(\phi) = constant \qquad (126)$$

In this case a, T are cyclic coordinates and we have the following constant charge

$$Q = \beta p_\phi = -\beta \epsilon a^3 \dot{\phi}. \qquad (127)$$

This is the same as the Noether symmetry analysis of purely scalar fields in general relativity. For $Q = 0$, $\phi =$constant, which corresponds to 'cosmological constant'. But if $Q \neq 0$ then we have $\dot{\phi} \propto a^{-3}$ and the scalar field dilutes with the expansion of the Universe.

We find another interesting solution of a teleparallel gravity with a scalar field (Model-II)

$$
\begin{aligned}
F(T) &= -\frac{3}{16}\epsilon c_1^2 T + c_4, && (128)\\
V(\phi) &= c_5 e^{-2\frac{\phi}{c_1}}(1 + e^{\frac{\phi+c_2}{c_1}})^2, && (129)\\
\alpha(a,\phi) &= \frac{2}{3}\sqrt{\frac{-c_3}{a}}\sinh\left(\frac{\phi+c_2}{c_1}\right), && (130)\\
\beta(a,\phi) &= \sqrt{\frac{-c_3}{a}}\cosh\left(\frac{\phi+c_2}{c_1}\right), && (131)\\
\eta(a,\phi,t) &= arbitrary. && (132)
\end{aligned}
$$

The corresponding Noether charge is given by

$$
Q = \sqrt{\frac{-c_3}{a}}\left[-\frac{3}{2}\epsilon c_1^2 a\dot{a}\sinh\left(\frac{\phi+c_2}{c_1}\right) + \epsilon a^3\dot{\phi}\cosh\left(\frac{\phi+c_2}{c_1}\right)\right]. \quad (133)
$$

Final Remarks

- The Noether Gauge Symmetry approach has been extensively used in Modified gravity theories, in particular, to determine the unknown functions appearing in the gravitational Lagrangian; Noether symmetries and the conserved quantities.
- Since the solutions are obtained using symmetries, hence the solutions are more physically relevant.
- The NGS approach will also be enormously useful for analysis in Horndeski's generalized scalar tensor theory.

References

- M. Jamil, F.M. Mahomed and D. Momeni : Phys. Lett. B 702 (2011) 315319.
- M. Jamil, D. Momeni and R. Myrzakulov : Eur. Phys. J. C 72 (2012) 2137.
- A. Aslam, M. Jamil, D. Momeni and R. Myrzakulov : Can. J. Phys. 91 (2013) 9397.
- A. Aslam, M. Jamil and R. Myrzakulov : Phys. Scr. 88 (2013) 025003.

THANK YOU FOR YOUR ATTENTION