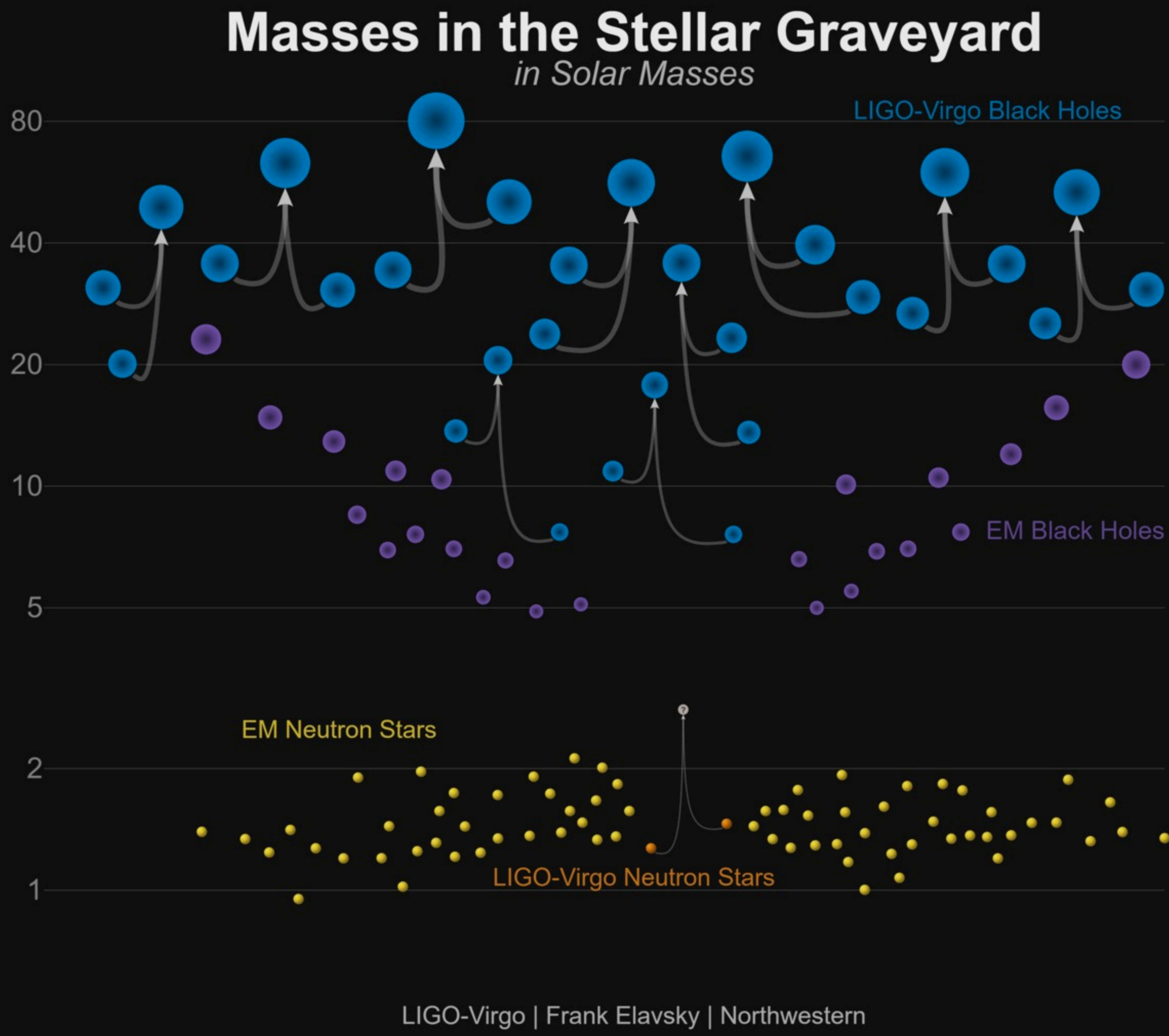
Semi-Analytic Calculation of Gravitational Wave Spectrum induced from Primordial Curvature Perturbations

寺田 隆広

(KEK, JSPS fellow)

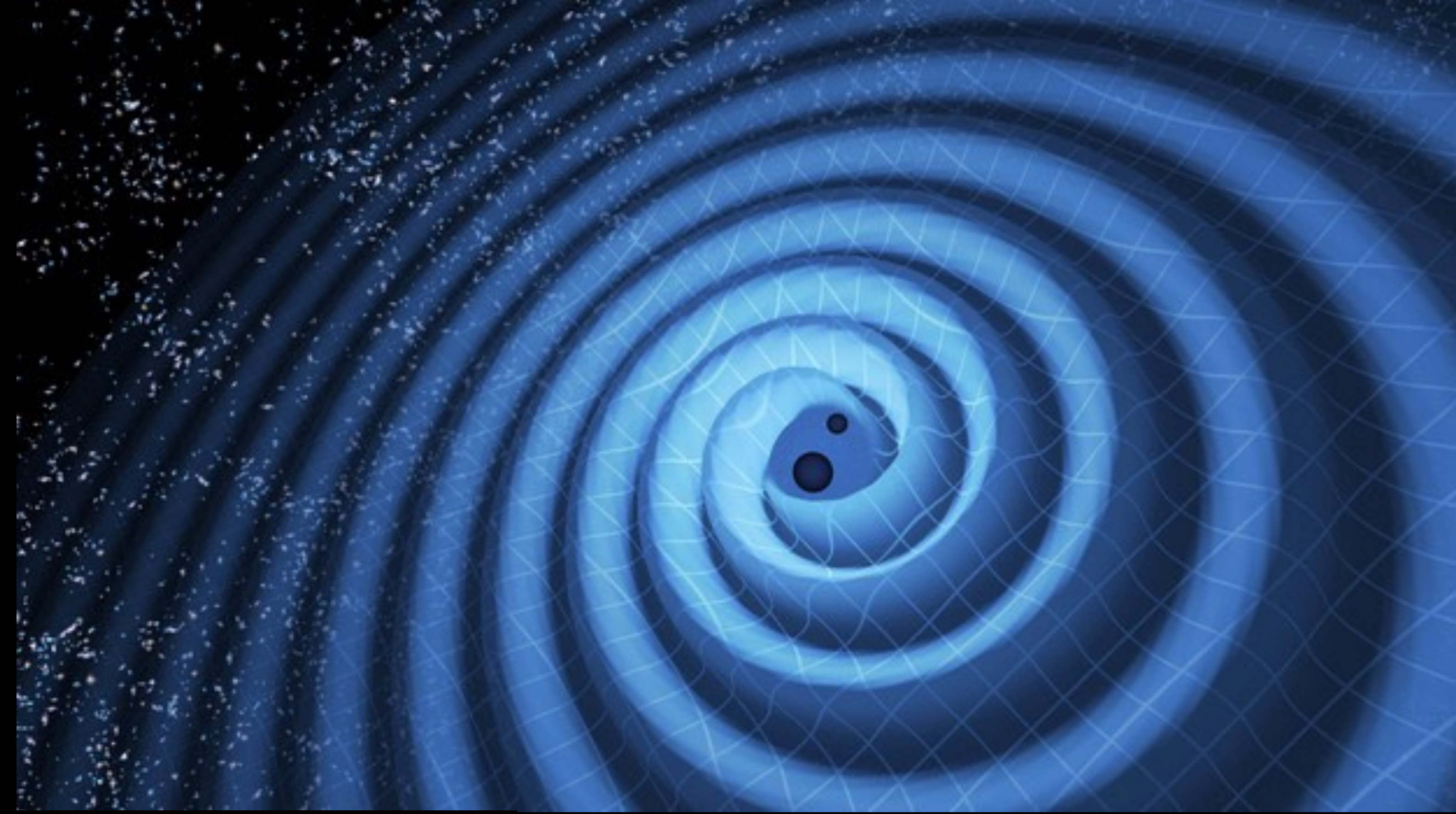
[Kohri, Terada, 1804.08577 [gr-qc], PRD 97 (2018) 123532]


Masses in the Stellar Graveyard
in Solar Masses
80
40
20
10
5
2
1
LIGO-Virgo Black Holes
EM Black Holes
EM Neutron Stars
LIGO-Virgo Neutron Stars
?
LIGO-Virgo | Frank Elavsky | Northwestern

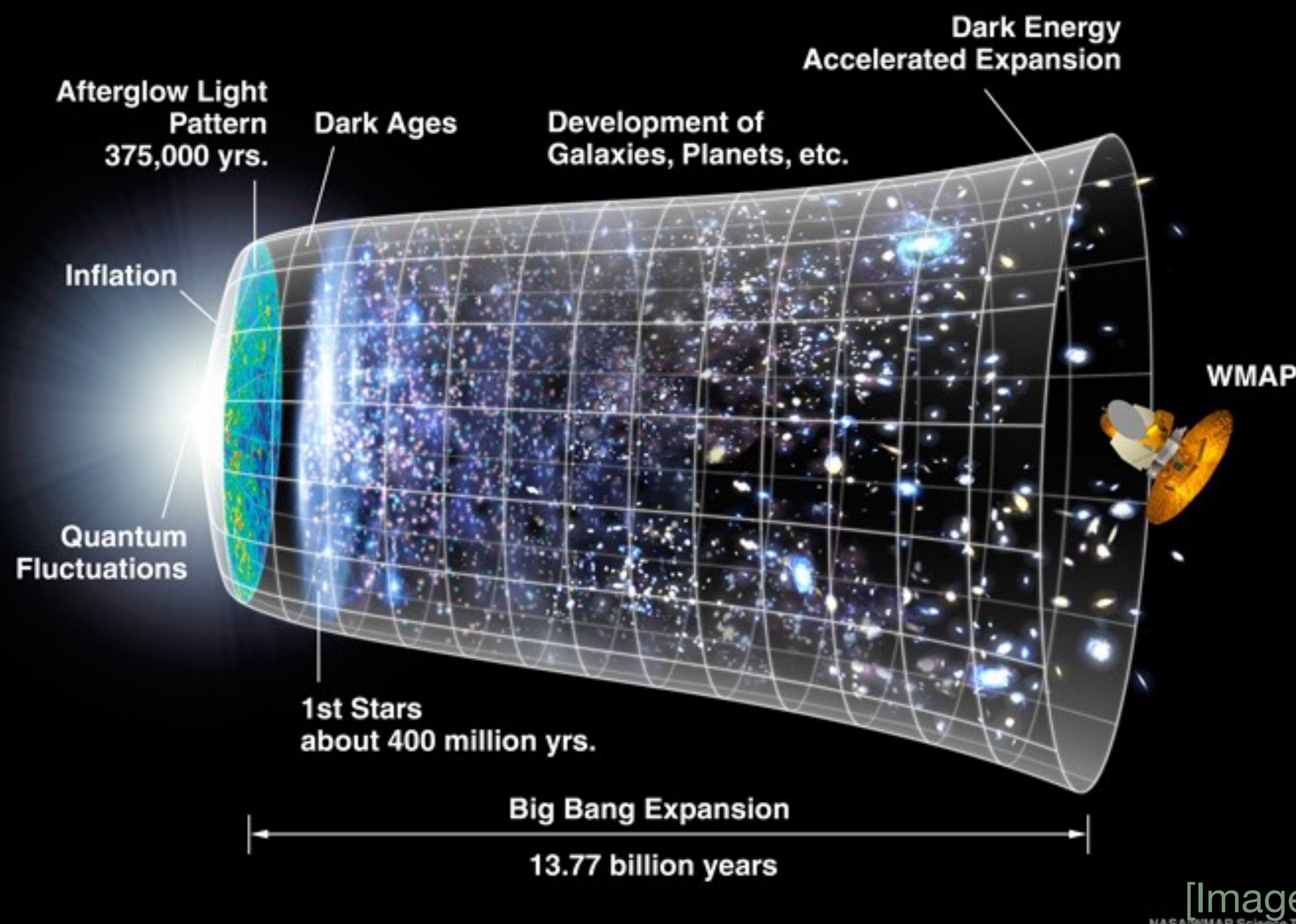
GW Astrophysics

↓

GW Cosmology



[Image credit: LIGO/T. Pyle]



We can probe
the very early Universe
or High Energy Physics.

[Image credit: NASA/WMAP Science Team]

Beginning of a Trend?

- Ananda, Clarkson, Wands, gr-qc/0612013
- Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290
- Assadullahi, Wands, 0901.0989
- Inomata, Kawasaki, Mukaida, Tada, Yanagida, 1611.06130

- Espinosa, Racco, Riotto, 1804.07732
- Kohri, Terada, 1804.08577
- Cai, Pi, Sasaki, 1810.11000
- Bartolo, De Luca, Franciolini, Peloso, Riotto, 1810.12218
- Bartolo, De Luca, Franciolini, Peloso, Racco, Riotto, 1810.12224
- Byrnes, Cole, Patil, 1811.11158
- Inomata, Nakama, 1812.00674

- For other source of stochastic gravitational waves, see e.g. Kuroyanagi, Chiba, Takahashi, 1807.00786.

What is the secondary GW?

[Ananda, Clarkson, Wands, gr-qc/0612013]
[Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]
[Inomata, Kawasaki, Mukaida, Tada, Yanagida, 1611.06130]

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- Inflation produces
  scalar and tensor (1st-order) perturbations

$$\mathrm{d}s^2 = -a^2(1 + 2\Phi)\mathrm{d}\eta^2 + a^2\left((1 - 2\Phi)\delta_{ij} + \frac{1}{2}h_{ij}\right)\mathrm{d}x^i\mathrm{d}x^j$$

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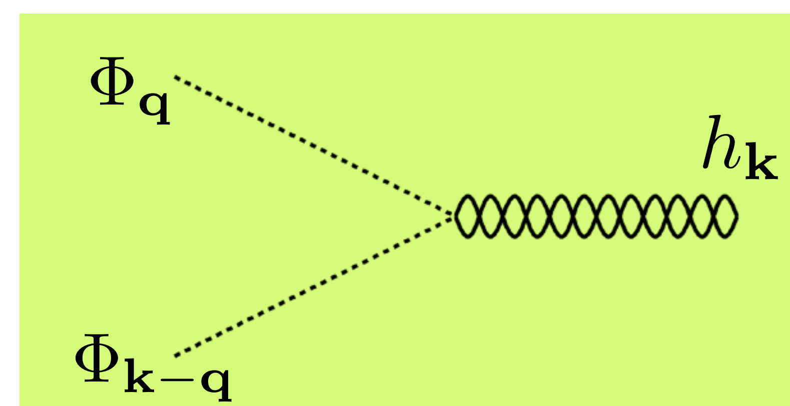
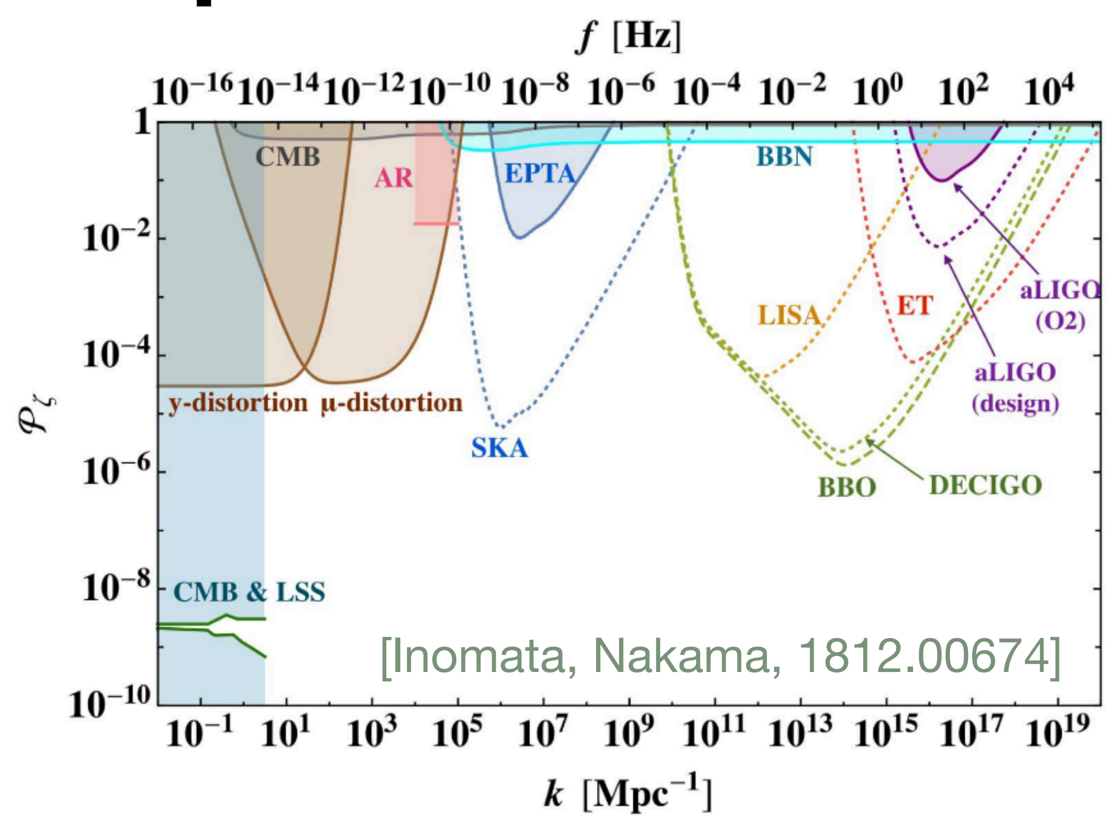
- (2nd-order tensor) = (1st-order scalar)^2

$$h''_{\mathbf{k}}(\eta) + 2\mathcal{H}h'_{\mathbf{k}}(\eta) + k^2 h_{\mathbf{k}}(\eta) = 4S_{\mathbf{k}}(\eta)$$



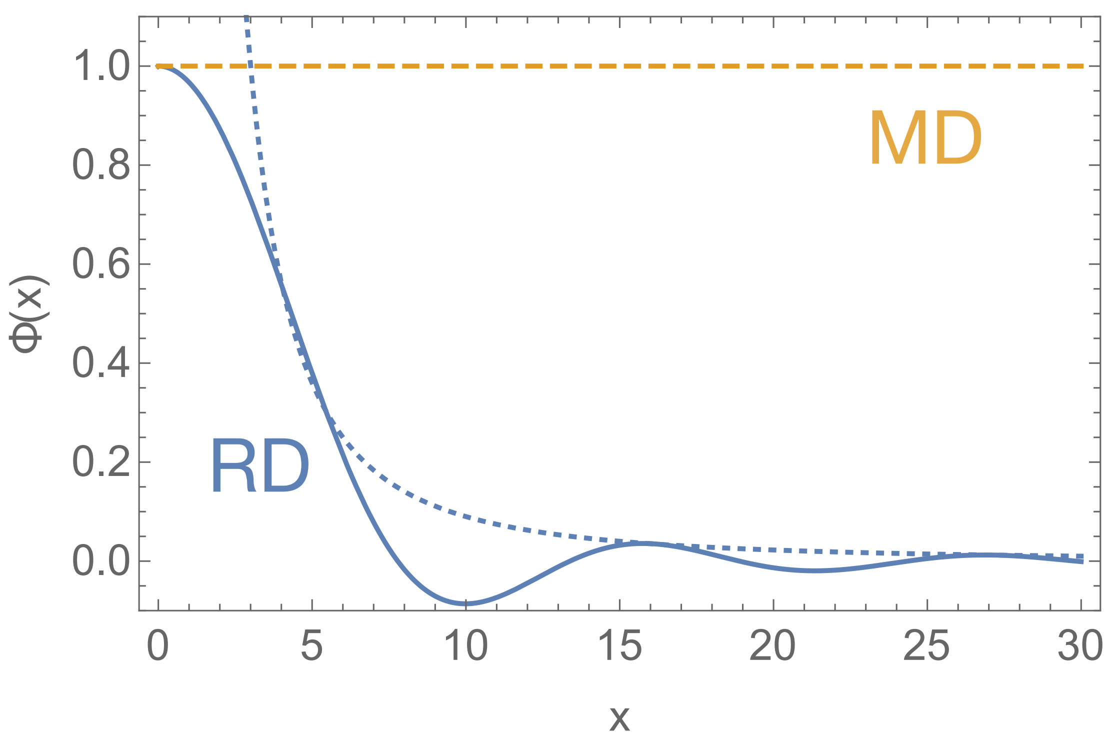
$$S_{\mathbf{k}} = \int \frac{\mathrm{d}^3 q}{(2\pi)^{3/2}} e_{ij}(\mathbf{k}) q_i q_j \left(2\Phi_{\mathbf{q}}\Phi_{\mathbf{k}-\mathbf{q}} + \frac{4}{3(1+w)}\left(\mathcal{H}^{-1}\Phi'_{\mathbf{q}} + \Phi_{\mathbf{q}}\right)\left(\mathcal{H}^{-1}\Phi'_{\mathbf{k}-\mathbf{q}} + \Phi_{\mathbf{k}-\mathbf{q}}\right)\right)$$

Enhanced scalar perturbations



Initial condition

Enhanced primordial curvature perturbations are allowed at small scales.

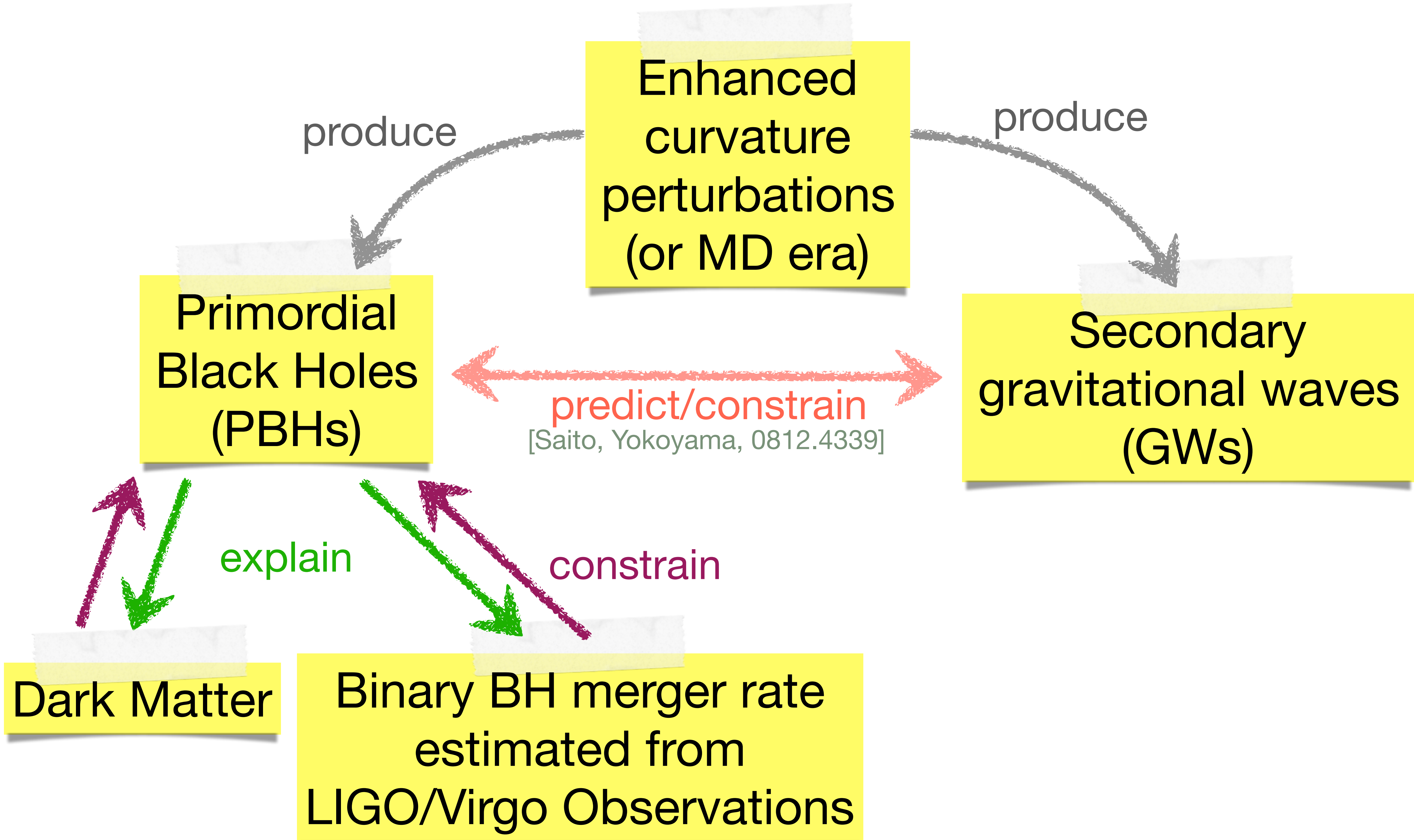


$x \equiv k\eta$

Dynamics

In a radiation-dominated (RD) era, the source decays.
In a matter-dominated (MD) era, the source does not decay.

Relation to PBH scenario



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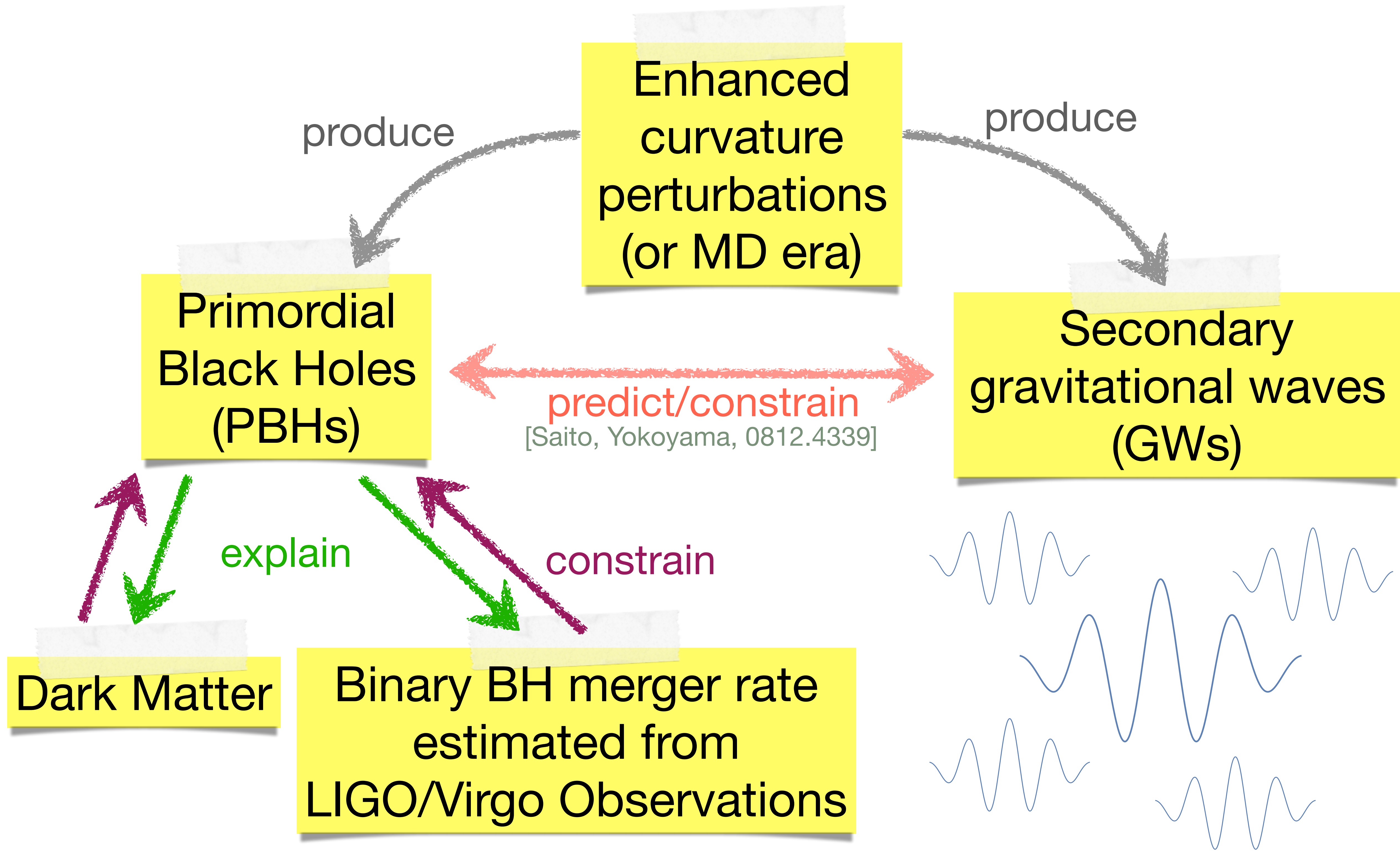

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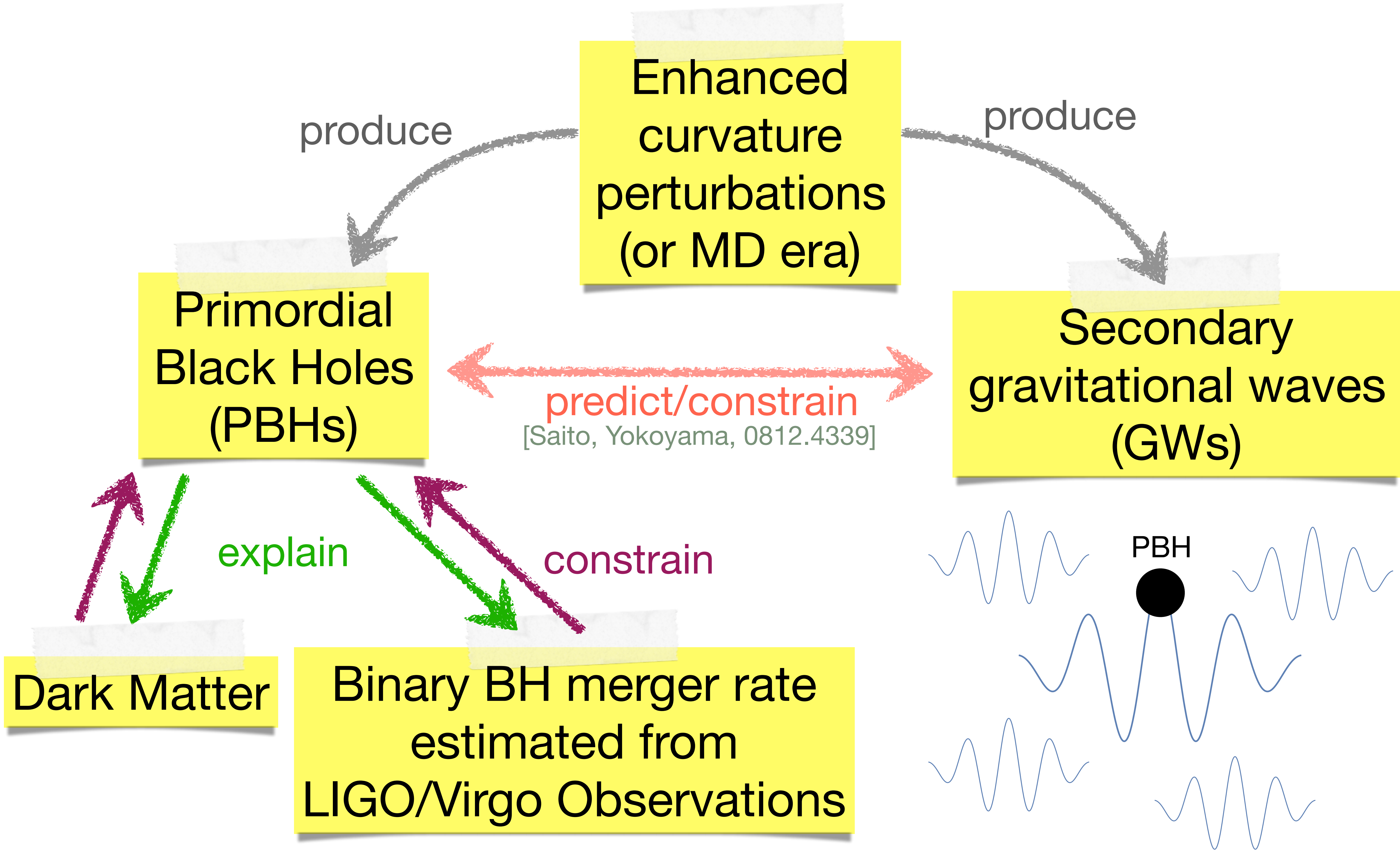
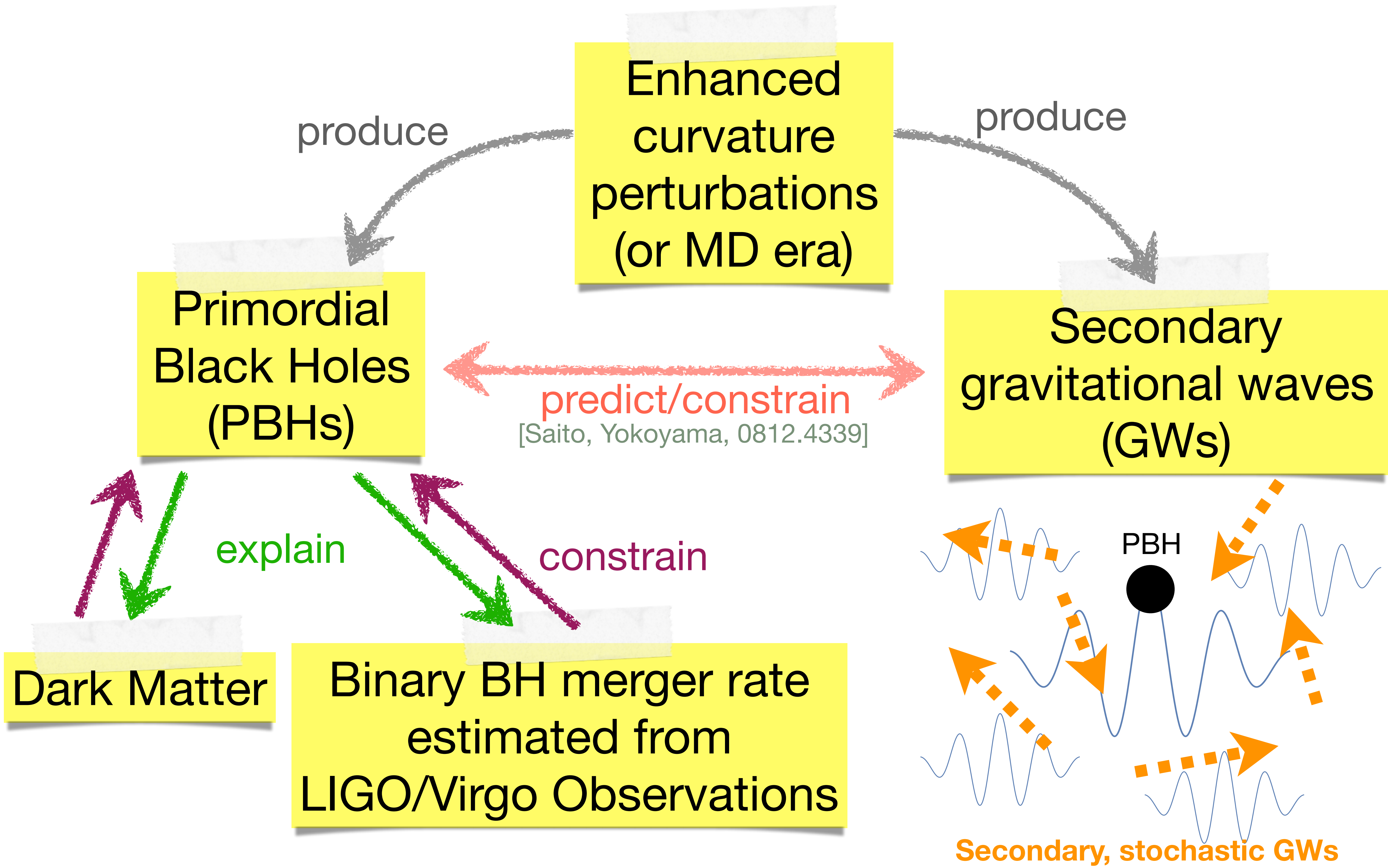
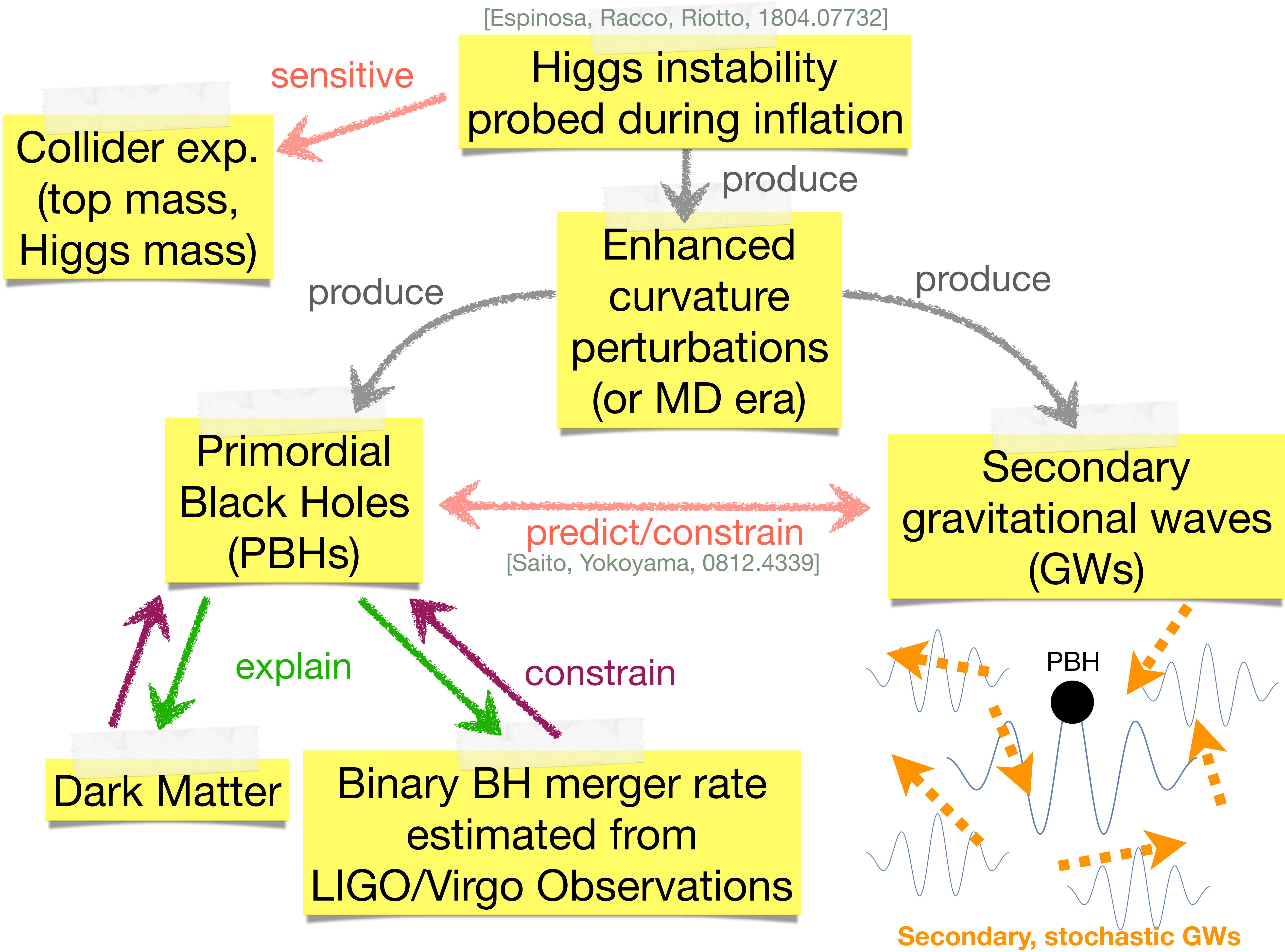


Relation to PBH scenario




[Espinosa, Racco, Riotto, 1804.07732]
Higgs instability probed during inflation
sensitive
Collider exp. (top mass, Higgs mass)
produce
Enhanced curvature perturbations (or MD era)
produce
produce
Primordial Black Holes (PBHs)
predict/constrain
[Saito, Yokoyama, 0812.4339]
Secondary gravitational waves (GWs)
explain
constrain
Dark Matter
Binary BH merger rate estimated from LIGO/Virgo Observations
PBH
Secondary, stochastic GWs

Motivation (What we do)

The secondary GW power spectrum is obtained by **multiple integral** of an **oscillatory function**.

$$\mathcal{P}_h \sim \int \mathrm{d}k \int \mathrm{d}k' \left(\int \mathrm{d}t f(k,k',t) \right)^2 \mathcal{P}_\zeta(k)\mathcal{P}_\zeta(k')$$

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"universal part"

$$\mathcal{P}_h \sim \int \mathrm{d}k \int \mathrm{d}k' \underbrace{\boxed{\left(\int \mathrm{d}t\, f(k,k',t)\right)^2}}_{\text{describing time evolution}} \underbrace{\mathcal{P}_\zeta(k)\mathcal{P}_\zeta(k')}_{\text{primordial quantity}}$$

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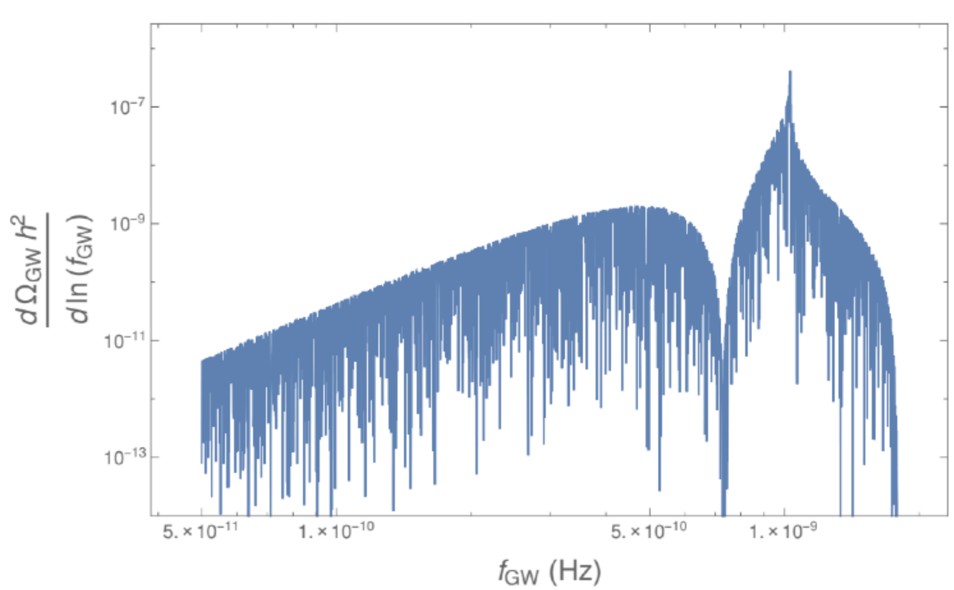
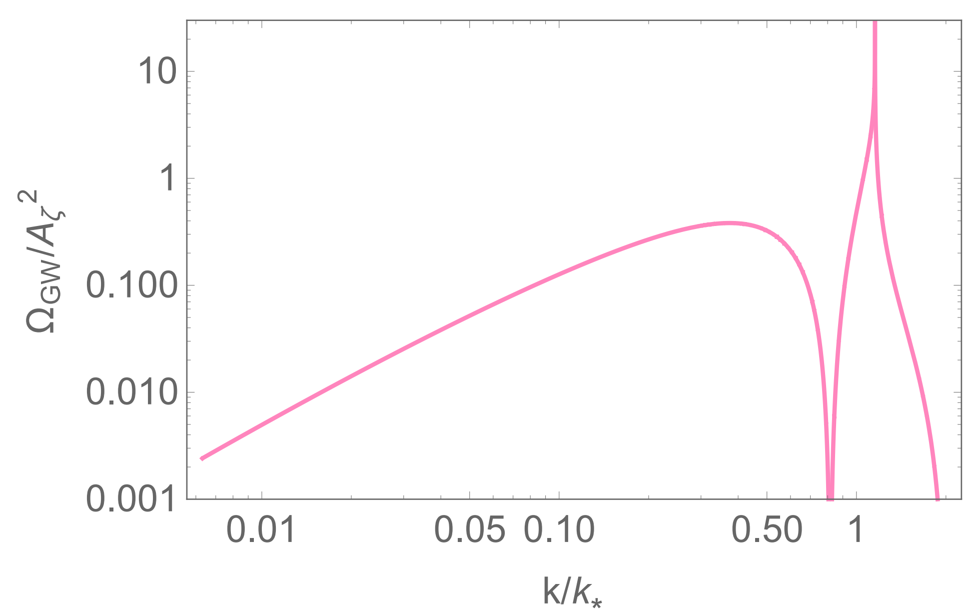
describing time evolution   primordial quantity

[Orlofsky, Pierce, Wells, 1612.05279]





Details (1/3)

[Ananda, Clarkson, Wands, gr-qc/0612013] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]
[Inomata, Kawasaki, Mukaida, Tada, Yanagida, 1611.06130]

$$\mathcal{P}_h(\eta, k) = 4 \int_0^\infty \mathrm{d}v \int_{|1-v|}^{1+v} \mathrm{d}u \left(\frac{4v^2 - (1+v^2-u^2)^2}{4vu} \right)^2 I^2(v,u,x) \mathcal{P}_\zeta(kv) \mathcal{P}_\zeta(ku)$$

where $x \equiv k\eta \qquad u = |\mathbf{k} - \tilde{\mathbf{k}}|/k \qquad v = \tilde{k}/k$

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We want to analytically calculate this function.

Definitions: $I(v, u, x) = \int_0^x \mathrm{d}\bar{x} \frac{a(\bar{\eta})}{a(\eta)} k G_k(\eta, \bar{\eta}) f(v, u, \bar{x})$

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$$\begin{aligned} f(v,u,\bar{x}) =& \frac{6(w+1)}{3w+5}\Phi(v\bar{x})\Phi(u\bar{x}) + \frac{6(1+3w)(w+1)}{(3w+5)^2}\left(\bar{x}\partial_{\bar{\eta}}\Phi(v\bar{x})\Phi(u\bar{x}) + \bar{x}\partial_{\bar{\eta}}\Phi(u\bar{x})\Phi(v\bar{x})\right) \\ &+ \frac{3(1+3w)^2(1+w)}{(3w+5)^2}\bar{x}^2\partial_{\bar{\eta}}\Phi(v\bar{x})\partial_{\bar{\eta}}\Phi(u\bar{x}) \end{aligned}$$

We have used $\mathcal{H} = aH = 2/((1+3w)\eta) \qquad \bar{x} \equiv k\bar{\eta} \qquad w \equiv P/\rho$

Details (2/3)

For definiteness, consider w = 1/3 (RD era).

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$$\Phi(x) = \frac{9}{x^2}\left(\frac{\sin(x/\sqrt{3})}{x/\sqrt{3}} - \cos(x/\sqrt{3})\right)$$

$$f_{\mathrm{RD}}(v,u,x) = \frac{12}{u^3v^3x^6}\left(18uvx^2\cos\frac{ux}{\sqrt{3}}\cos\frac{vx}{\sqrt{3}} + (54-6(u^2+v^2)x^2+u^2v^2x^4)\sin\frac{ux}{\sqrt{3}}\sin\frac{vx}{\sqrt{3}}\right.$$
$$\left.+2\sqrt{3}ux(v^2x^2-9)\cos\frac{ux}{\sqrt{3}}\sin\frac{vx}{\sqrt{3}} + 2\sqrt{3}vx(u^2x^2-9)\sin\frac{ux}{\sqrt{3}}\cos\frac{vx}{\sqrt{3}}\right)$$

Details (3/3)

For definiteness, consider w = 1/3 (RD era).

$$
\begin{aligned}
I_{\rm RD}(v,u,x) =& \frac{3}{4u^3v^3x}\left(-\frac{4}{x^3}\left(uv(u^2+v^2-3)x^3\sin x - 6uvx^2\cos\frac{ux}{\sqrt{3}}\cos\frac{vx}{\sqrt{3}}\right.\right. \\
&+6\sqrt{3}ux\cos\frac{ux}{\sqrt{3}}\sin\frac{vx}{\sqrt{3}} + 6\sqrt{3}vx\sin\frac{ux}{\sqrt{3}}\cos\frac{vx}{\sqrt{3}} - 3(6+(u^2+v^2-3)x^2)\sin\frac{ux}{\sqrt{3}}\sin\frac{vx}{\sqrt{3}}\Bigg) \\
&+(u^2+v^2-3)^2\left(\sin x\left(\mathrm{Ci}\left(\left(1-\frac{v-u}{\sqrt{3}}\right)x\right)+\mathrm{Ci}\left(\left(1+\frac{v-u}{\sqrt{3}}\right)x\right)\right.\right. \\
&-\mathrm{Ci}\left(\left|1-\frac{v+u}{\sqrt{3}}\right|x\right)-\mathrm{Ci}\left(\left(1+\frac{v+u}{\sqrt{3}}\right)x\right)+\log\left|\frac{3-(u+v)^2}{3-(u-v)^2}\right|\Bigg) \\
&+\cos x\left(-\mathrm{Si}\left(\left(1-\frac{v-u}{\sqrt{3}}\right)x\right)-\mathrm{Si}\left(\left(1+\frac{v-u}{\sqrt{3}}\right)x\right)\right. \\
&\left.\left.\left.+\mathrm{Si}\left(\left(1-\frac{v+u}{\sqrt{3}}\right)x\right)+\mathrm{Si}\left(\left(1+\frac{v+u}{\sqrt{3}}\right)x\right)\right)\right)\right)
\end{aligned}
$$

$$\mathrm{Si}(x) = \int_0^x \mathrm{d}\bar{x}\frac{\sin\bar{x}}{\bar{x}}$$

$$\mathrm{Ci}(x) = -\int_x^\infty \mathrm{d}\bar{x}\frac{\cos\bar{x}}{\bar{x}}$$

See also [Espinosa, Racco, Riotto, 1804.07732]

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&+6\sqrt{3}ux\cos\frac{ux}{\sqrt{3}}\sin\frac{vx}{\sqrt{3}} + 6\sqrt{3}vx\sin\frac{ux}{\sqrt{3}}\cos\frac{vx}{\sqrt{3}} - 3(6+(u^2+v^2-3)x^2)\sin\frac{ux}{\sqrt{3}}\sin\frac{vx}{\sqrt{3}}\Bigg)\\
&+(u^2+v^2-3)^2\Bigg(\sin x\Bigg(\mathrm{Ci}\left(\left(1-\frac{v-u}{\sqrt{3}}\right)x\right) + \mathrm{Ci}\left(\left(1+\frac{v-u}{\sqrt{3}}\right)x\right)\\
&-\mathrm{Ci}\left(\left|1-\frac{v+u}{\sqrt{3}}\right|x\right) - \mathrm{Ci}\left(\left(1+\frac{v+u}{\sqrt{3}}\right)x\right) + \log\left|\frac{3-(u+v)^2}{3-(u-v)^2}\right|\Bigg)\\
&+\cos x\Bigg(-\mathrm{Si}\left(\left(1-\frac{v-u}{\sqrt{3}}\right)x\right) - \mathrm{Si}\left(\left(1+\frac{v-u}{\sqrt{3}}\right)x\right)\\
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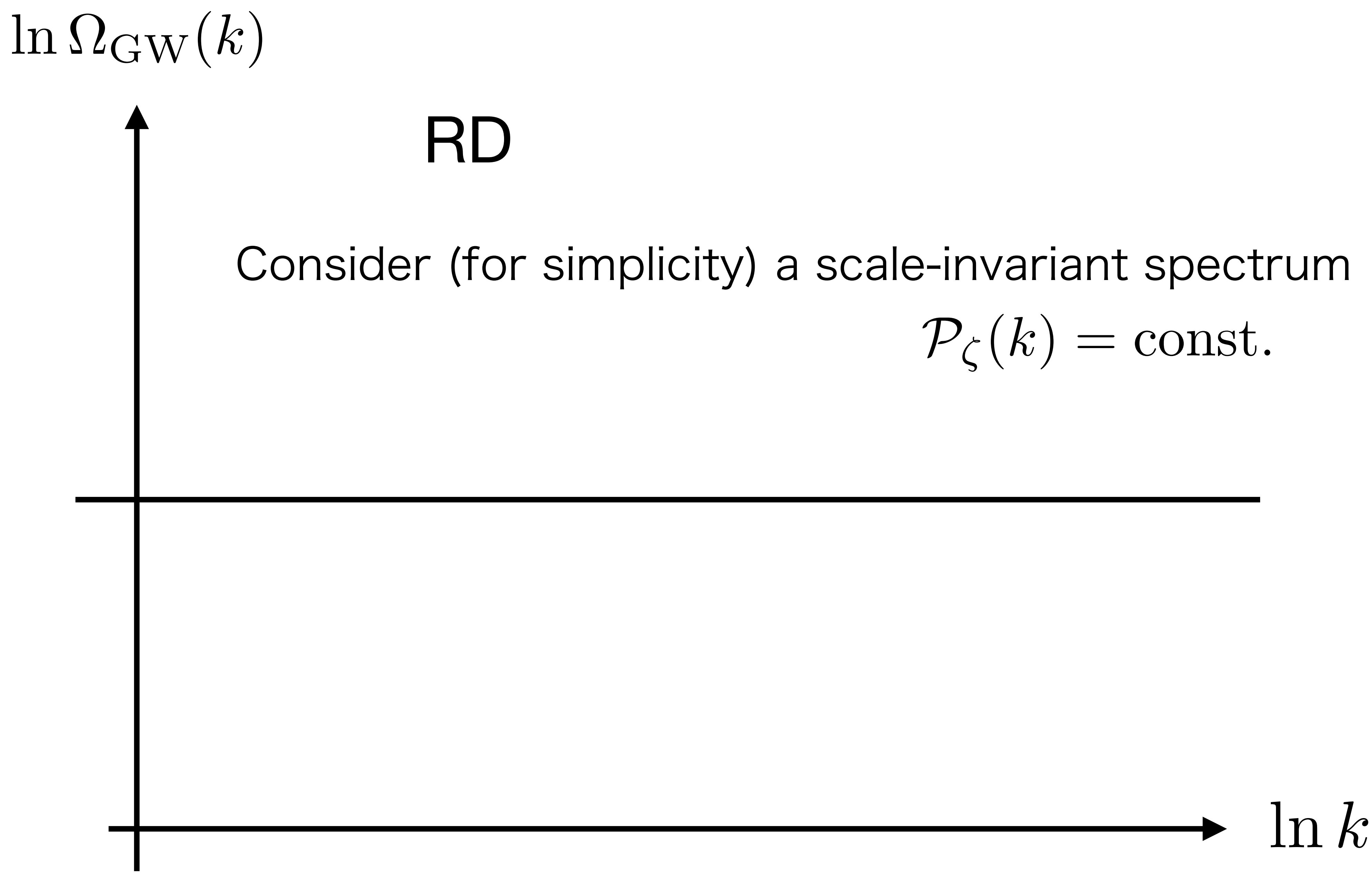
Oscillation average in the late-time limit

$$
\begin{aligned}
\overline{I^2_{\rm RD}(v,u,x\to\infty)} = &\frac{1}{2}\left(\frac{3(u^2+v^2-3)}{4u^3v^3x}\right)^2\Bigg(\left(-4uv+(u^2+v^2-3)\log\left|\frac{3-(u+v)^2}{3-(u-v)^2}\right|\right)^2\\
&+\pi^2(u^2+v^2-3)^2\Theta(v+u-\sqrt{3})\Bigg)
\end{aligned}
$$

See also [Espinosa, Racco, Riotto, 1804.07732]

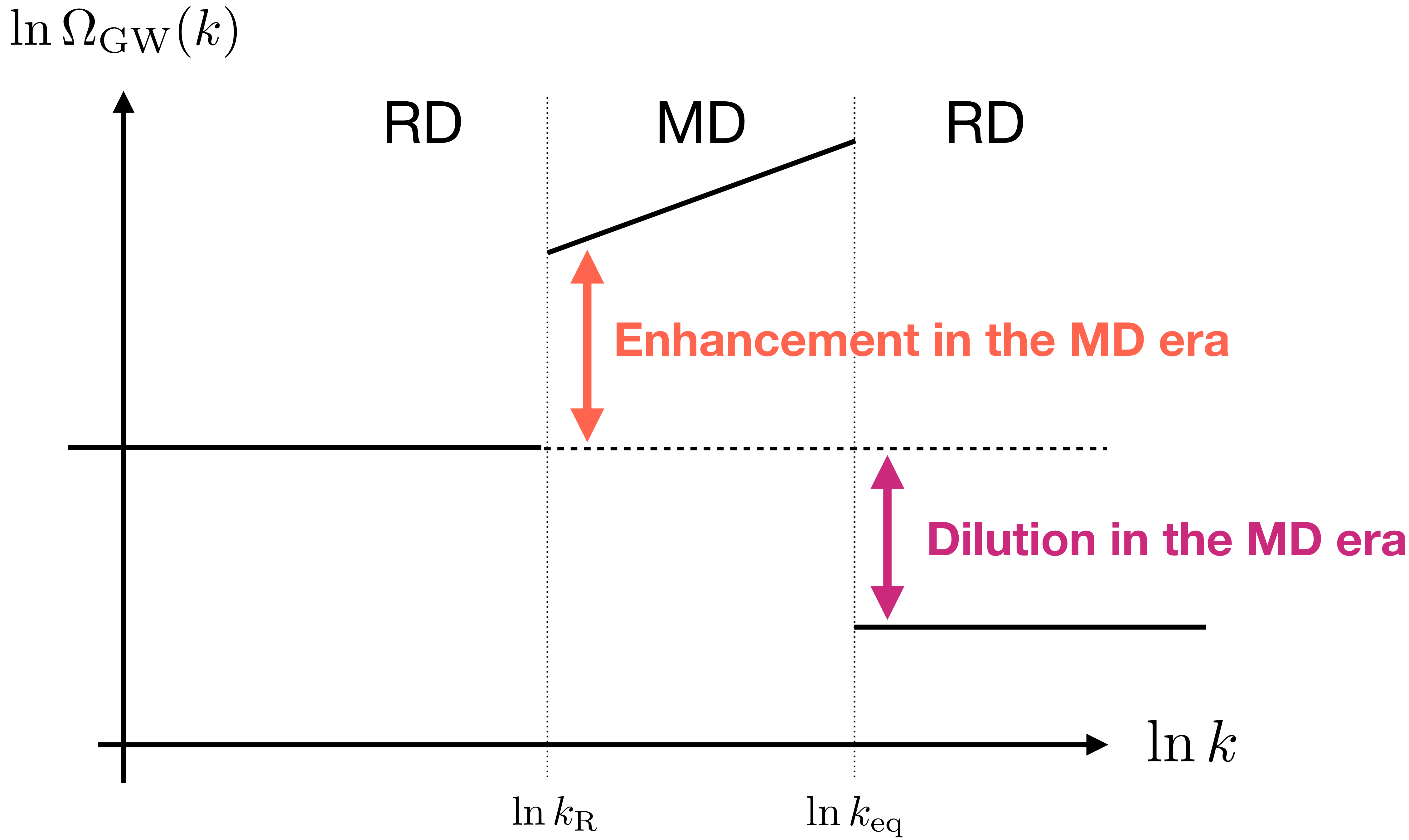
General cases



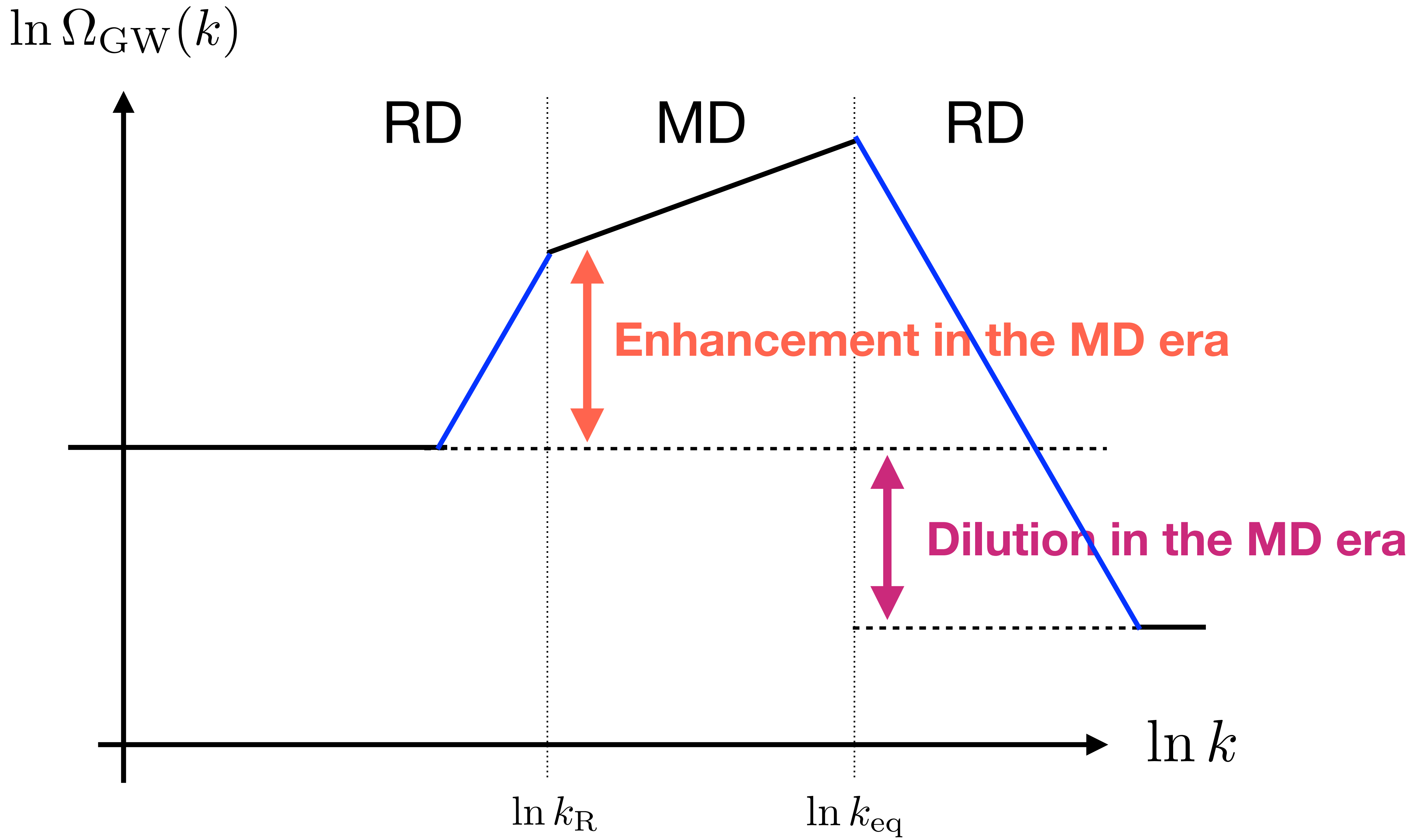
Consider (for simplicity) a scale-invariant spectrum

$$\mathcal{P}_\zeta(k) = \text{const.}$$

General cases

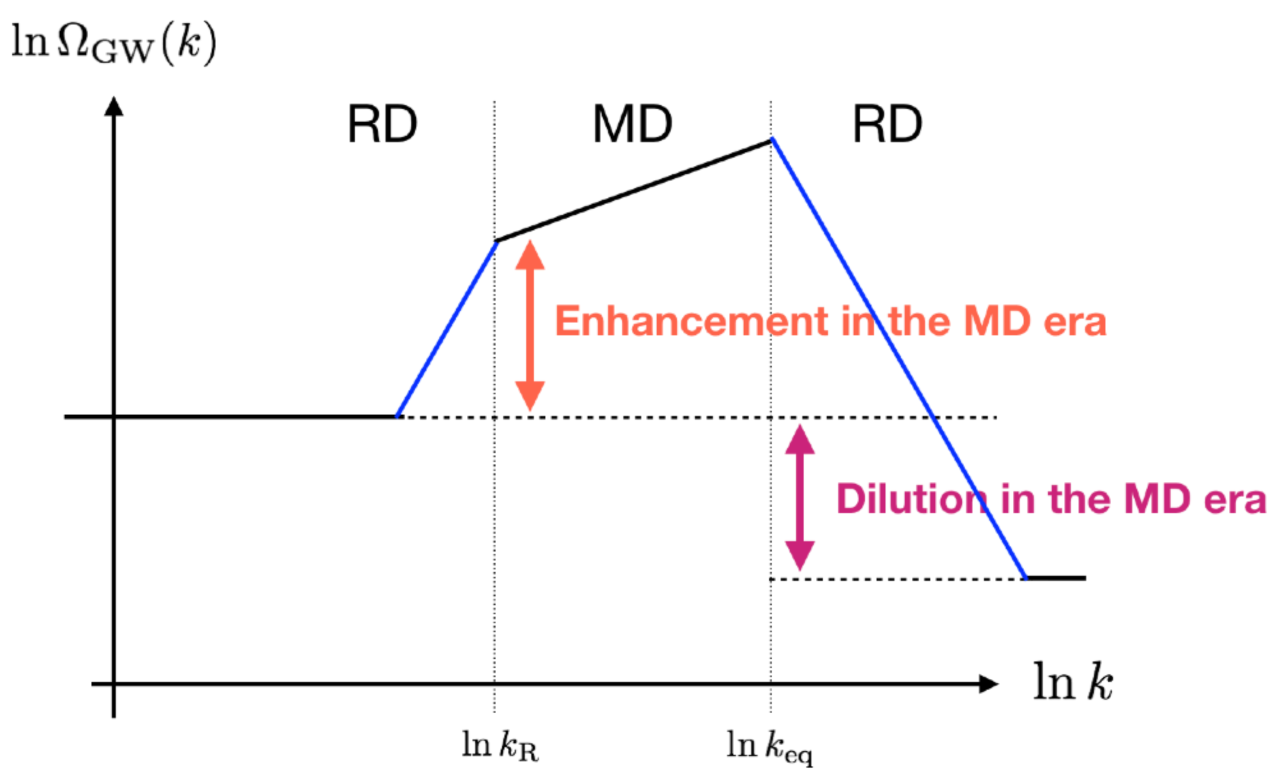


General cases

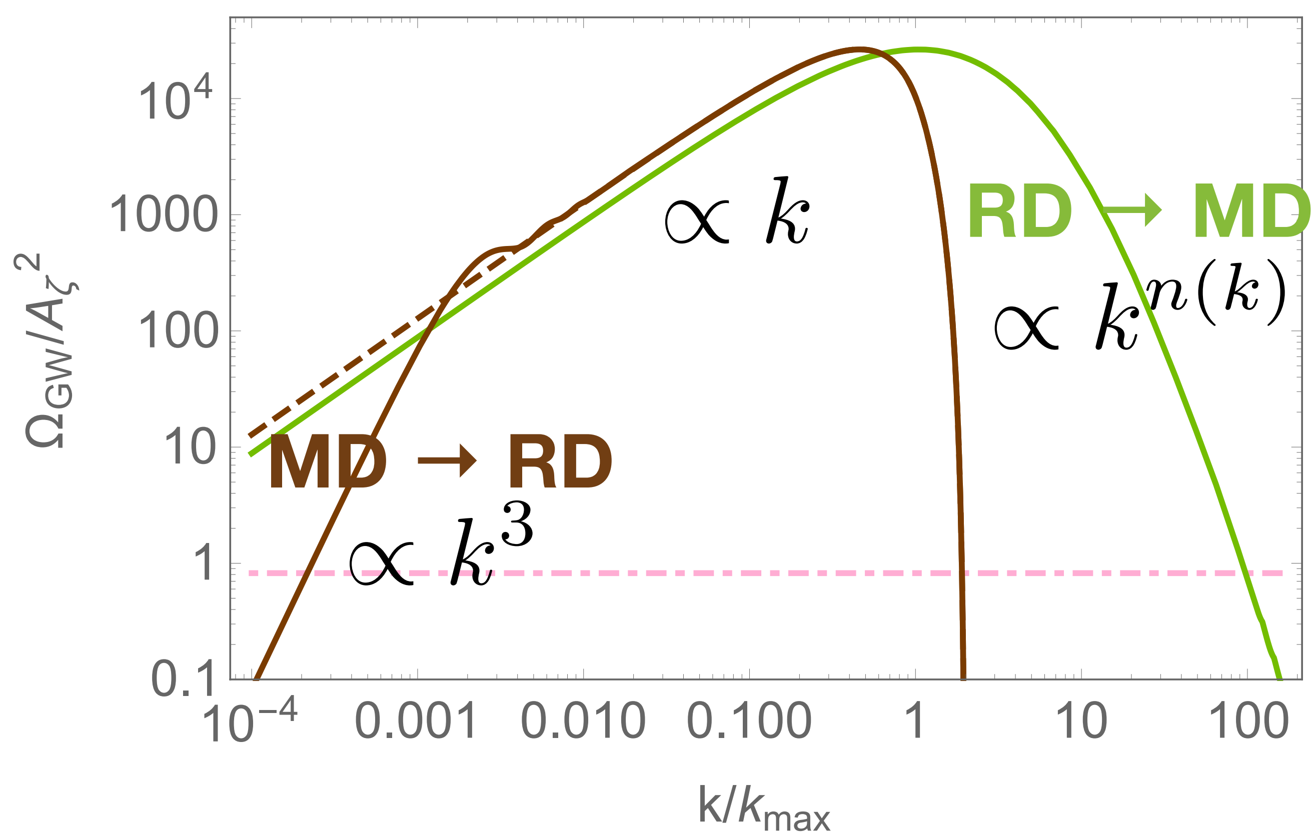


General cases

Schematic



Precise

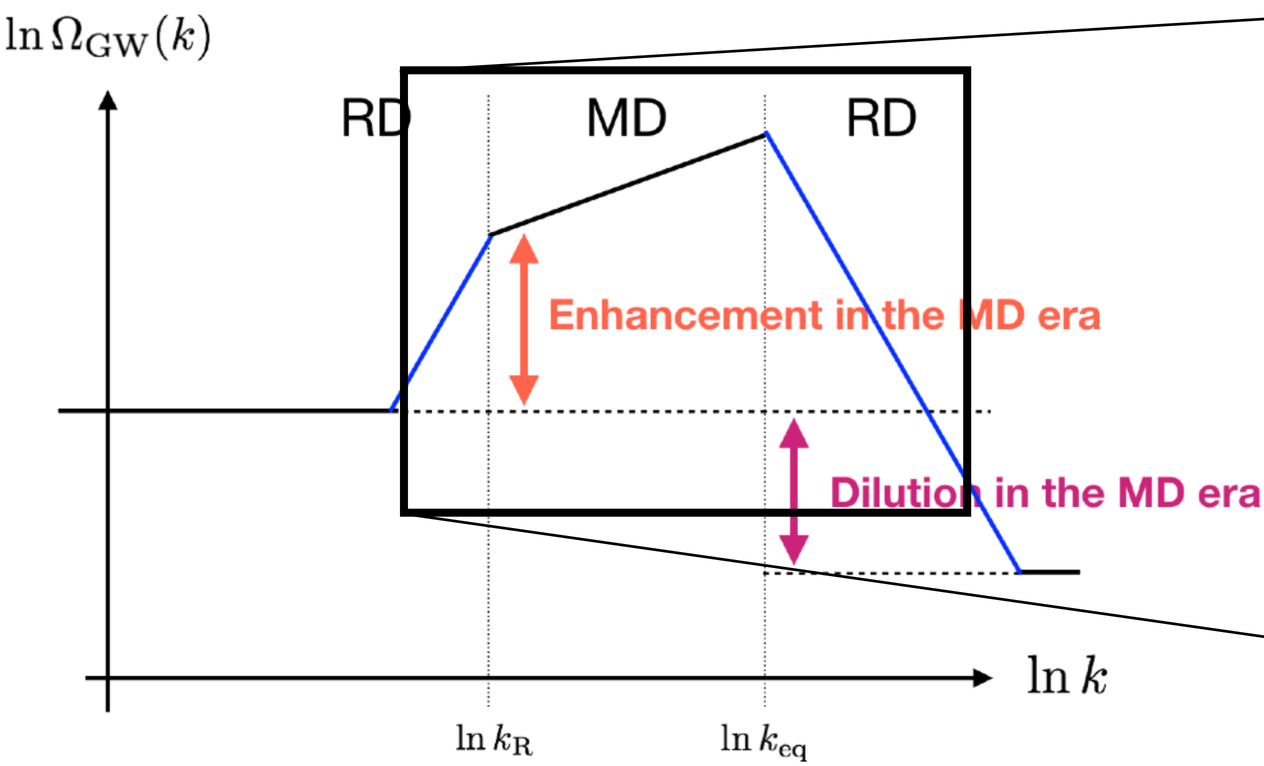


$-6 \lesssim n(k) \lesssim -4$

Scale-invariant primordial spectrum is assumed as a simple example.

General cases

Schematic



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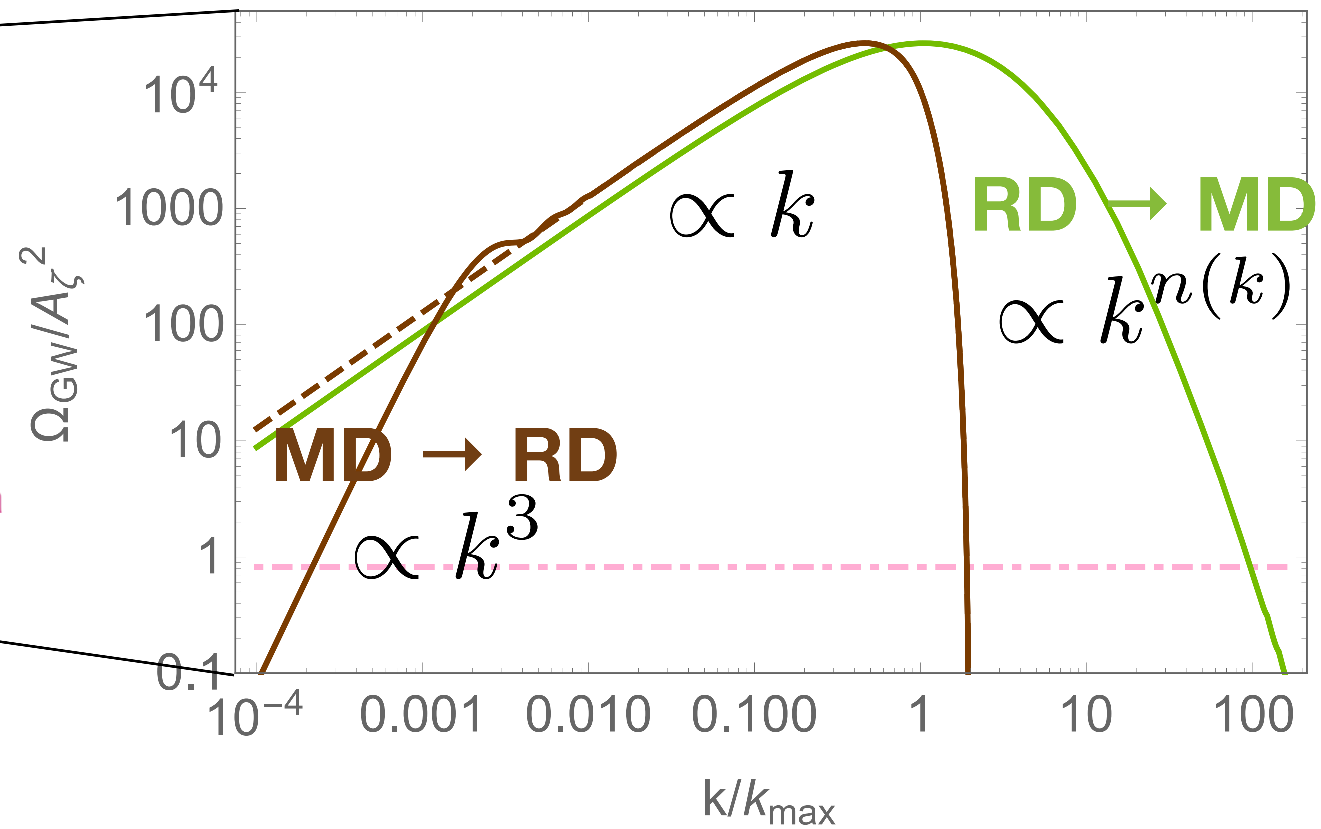

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Summary

- **Secondary GW** is induced by primordial curvature perturbations. (important e.g. in PBH scenarios)
- We **analytically calculated** the **universal part** of the power spectrum of the secondary GW.
- Applicable to GWs induced in **RD**, **MD**, and **more general cases**.
- **Please use our results to write your papers!**

Appendix:
More quantitative explanations

- Basic things
- Radiation-dominated (RD) era
- Matter-dominated (MD) era
- More general cases

Observable & Basic definitions

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Energy fraction

$$\Omega_{\rm GW}(\eta,k) = \frac{1}{\rho_{\rm tot}(\eta)}\frac{\mathrm{d}\rho_{\rm GW}(\eta)}{\mathrm{d}\ln k} = \frac{1}{24}\left(\frac{k}{a(\eta)H(\eta)}\right)^2 \overline{\mathcal{P}_h(\eta,k)}$$

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Power spectrum

$$\langle h^{\lambda}_{\mathbf{k}}(\eta)h^{\lambda'}_{\mathbf{k}'}(\eta)\rangle = \delta_{\lambda\lambda'}\delta^3(\mathbf{k}+\mathbf{k}')\frac{2\pi^2}{k^3}\mathcal{P}_h(\eta,k)$$

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Gravitational field: Green's function method

$$a(\eta)h_{\mathbf{k}}(\eta) = 4\int^{\eta} \mathrm{d}\bar{\eta}\, G_{\mathbf{k}}(\eta,\bar{\eta})a(\bar{\eta})S_{\mathbf{k}}(\bar{\eta})$$

$$G''_{\mathbf{k}}(\eta,\bar{\eta}) + \left(k^2 - \frac{a''(\eta)}{a(\eta)}\right)G_{\mathbf{k}}(\eta,\bar{\eta}) = \delta(\eta-\bar{\eta})$$

Power spectrum & “universal part”

[Ananda, Clarkson, Wands, gr-qc/0612013] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]
[Inomata, Kawasaki, Mukaida, Tada, Yanagida, 1611.06130]

$$\mathcal{P}_h(\eta, k) = 4\int_0^\infty \mathrm{d}v \int_{|1-v|}^{1+v} \mathrm{d}u \left(\frac{4v^2 - (1+v^2-u^2)^2}{4vu}\right)^2 I^2(v,u,x)\mathcal{P}_\zeta(kv)\mathcal{P}_\zeta(ku)$$

where $x \equiv k\eta \qquad u = |\mathbf{k} - \tilde{\mathbf{k}}|/k \qquad v = \tilde{k}/k$

Power spectrum & “universal part”

[Ananda, Clarkson, Wands, gr-qc/0612013] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]
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We want to analytically calculate this function.

Definitions: $I(v,u,x) = \int_0^x \mathrm{d}\bar{x}\,\frac{a(\bar{\eta})}{a(\eta)} kG_k(\eta,\bar{\eta})f(v,u,\bar{x})$

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$$\begin{aligned} f(v,u,\bar{x}) =& \frac{6(w+1)}{3w+5}\Phi(v\bar{x})\Phi(u\bar{x}) + \frac{6(1+3w)(w+1)}{(3w+5)^2}\left(\bar{x}\partial_{\bar{\eta}}\Phi(v\bar{x})\Phi(u\bar{x}) + \bar{x}\partial_{\bar{\eta}}\Phi(u\bar{x})\Phi(v\bar{x})\right) \\ &+ \frac{3(1+3w)^2(1+w)}{(3w+5)^2}\bar{x}^2\partial_{\bar{\eta}}\Phi(v\bar{x})\partial_{\bar{\eta}}\Phi(u\bar{x}) \end{aligned}$$

We have used $\mathcal{H} = aH = 2/((1+3w)\eta) \qquad \bar{x} \equiv k\bar{\eta} \qquad w \equiv P/\rho$

Analytic Calculation

For definiteness, consider w = 1/3 (RD era).

$$I(v,u,x) = \int_0^x \mathrm{d}\bar{x}\, \frac{a(\bar{\eta})}{a(\eta)} k G_k(\eta,\bar{\eta}) f(v,u,\bar{x})$$

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$$a(\bar{\eta})/a(\eta) = \bar{x}/x$$

$$kG_{\mathbf{k}}(\eta,\bar{\eta}) = \sin(x-\bar{x})$$

$$\Phi(x) = \frac{9}{x^2}\left(\frac{\sin(x/\sqrt{3})}{x/\sqrt{3}} - \cos(x/\sqrt{3})\right)$$

$$f_{\mathrm{RD}}(v,u,x) = \frac{12}{u^3v^3x^6}\left(18uvx^2\cos\frac{ux}{\sqrt{3}}\cos\frac{vx}{\sqrt{3}} + (54-6(u^2+v^2)x^2+u^2v^2x^4)\sin\frac{ux}{\sqrt{3}}\sin\frac{vx}{\sqrt{3}}\right.$$
$$\left. +2\sqrt{3}ux(v^2x^2-9)\cos\frac{ux}{\sqrt{3}}\sin\frac{vx}{\sqrt{3}} + 2\sqrt{3}vx(u^2x^2-9)\sin\frac{ux}{\sqrt{3}}\cos\frac{vx}{\sqrt{3}}\right)$$

Analytic Calculation

For definiteness, consider w = 1/3 (RD era).

$$
\begin{aligned}
I_{\mathrm{RD}}(v,u,x) =& \frac{3}{4u^3v^3x}\Bigg(-\frac{4}{x^3}\Bigg(uv(u^2+v^2-3)x^3\sin x - 6uvx^2\cos\frac{ux}{\sqrt{3}}\cos\frac{vx}{\sqrt{3}} \\
&+6\sqrt{3}ux\cos\frac{ux}{\sqrt{3}}\sin\frac{vx}{\sqrt{3}} + 6\sqrt{3}vx\sin\frac{ux}{\sqrt{3}}\cos\frac{vx}{\sqrt{3}} - 3(6+(u^2+v^2-3)x^2)\sin\frac{ux}{\sqrt{3}}\sin\frac{vx}{\sqrt{3}}\Bigg) \\
&+(u^2+v^2-3)^2\Bigg(\sin x\Bigg(\mathrm{Ci}\left(\left(1-\frac{v-u}{\sqrt{3}}\right)x\right) + \mathrm{Ci}\left(\left(1+\frac{v-u}{\sqrt{3}}\right)x\right) \\
&-\mathrm{Ci}\left(\left|1-\frac{v+u}{\sqrt{3}}\right|x\right) - \mathrm{Ci}\left(\left(1+\frac{v+u}{\sqrt{3}}\right)x\right) + \log\left|\frac{3-(u+v)^2}{3-(u-v)^2}\right|\Bigg) \\
&+\cos x\Bigg(-\mathrm{Si}\left(\left(1-\frac{v-u}{\sqrt{3}}\right)x\right) - \mathrm{Si}\left(\left(1+\frac{v-u}{\sqrt{3}}\right)x\right) \\
&+\mathrm{Si}\left(\left(1-\frac{v+u}{\sqrt{3}}\right)x\right) + \mathrm{Si}\left(\left(1+\frac{v+u}{\sqrt{3}}\right)x\right)\Bigg)\Bigg)\Bigg)
\end{aligned}
$$

$$\mathrm{Si}(x) = \int_0^x \mathrm{d}\bar{x}\frac{\sin\bar{x}}{\bar{x}}$$

$$\mathrm{Ci}(x) = -\int_x^\infty \mathrm{d}\bar{x}\frac{\cos\bar{x}}{\bar{x}}$$

Analytic Calculation

For definiteness, consider w = 1/3 (RD era).

$$I_{\rm RD}(v,u,x) = \frac{3}{4u^3v^3x}\left(-\frac{4}{x^3}\left(uv(u^2+v^2-3)x^3\sin x - 6uvx^2\cos\frac{ux}{\sqrt{3}}\cos\frac{vx}{\sqrt{3}}\right.\right.$$
$$\left. + 6\sqrt{3}ux\cos\frac{ux}{\sqrt{3}}\sin\frac{vx}{\sqrt{3}} + 6\sqrt{3}vx\sin\frac{ux}{\sqrt{3}}\cos\frac{vx}{\sqrt{3}} - 3(6+(u^2+v^2-3)x^2)\sin\frac{ux}{\sqrt{3}}\sin\frac{vx}{\sqrt{3}}\right)$$
$$+ (u^2+v^2-3)^2\left(\sin x\left(\mathrm{Ci}\left(\left(1-\frac{v-u}{\sqrt{3}}\right)x\right) + \mathrm{Ci}\left(\left(1+\frac{v-u}{\sqrt{3}}\right)x\right)\right.\right.$$
$$\left. - \mathrm{Ci}\left(\left|1-\frac{v+u}{\sqrt{3}}\right|x\right) - \mathrm{Ci}\left(\left(1+\frac{v+u}{\sqrt{3}}\right)x\right) + \log\left|\frac{3-(u+v)^2}{3-(u-v)^2}\right|\right)$$
$$+ \cos x\left(-\mathrm{Si}\left(\left(1-\frac{v-u}{\sqrt{3}}\right)x\right) - \mathrm{Si}\left(\left(1+\frac{v-u}{\sqrt{3}}\right)x\right)\right.$$
$$\left.\left.\left. + \mathrm{Si}\left(\left(1-\frac{v+u}{\sqrt{3}}\right)x\right) + \mathrm{Si}\left(\left(1+\frac{v+u}{\sqrt{3}}\right)x\right)\right)\right)\right)$$

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Oscillation average in the late-time limit

$$\overline{I_{\rm RD}^2(v,u,x\to\infty)} = \frac{1}{2}\left(\frac{3(u^2+v^2-3)}{4u^3v^3x}\right)^2\left(\left(-4uv+(u^2+v^2-3)\log\left|\frac{3-(u+v)^2}{3-(u-v)^2}\right|\right)^2\right.$$
$$\left. + \pi^2(u^2+v^2-3)^2\Theta(v+u-\sqrt{3})\right)$$

Analytically

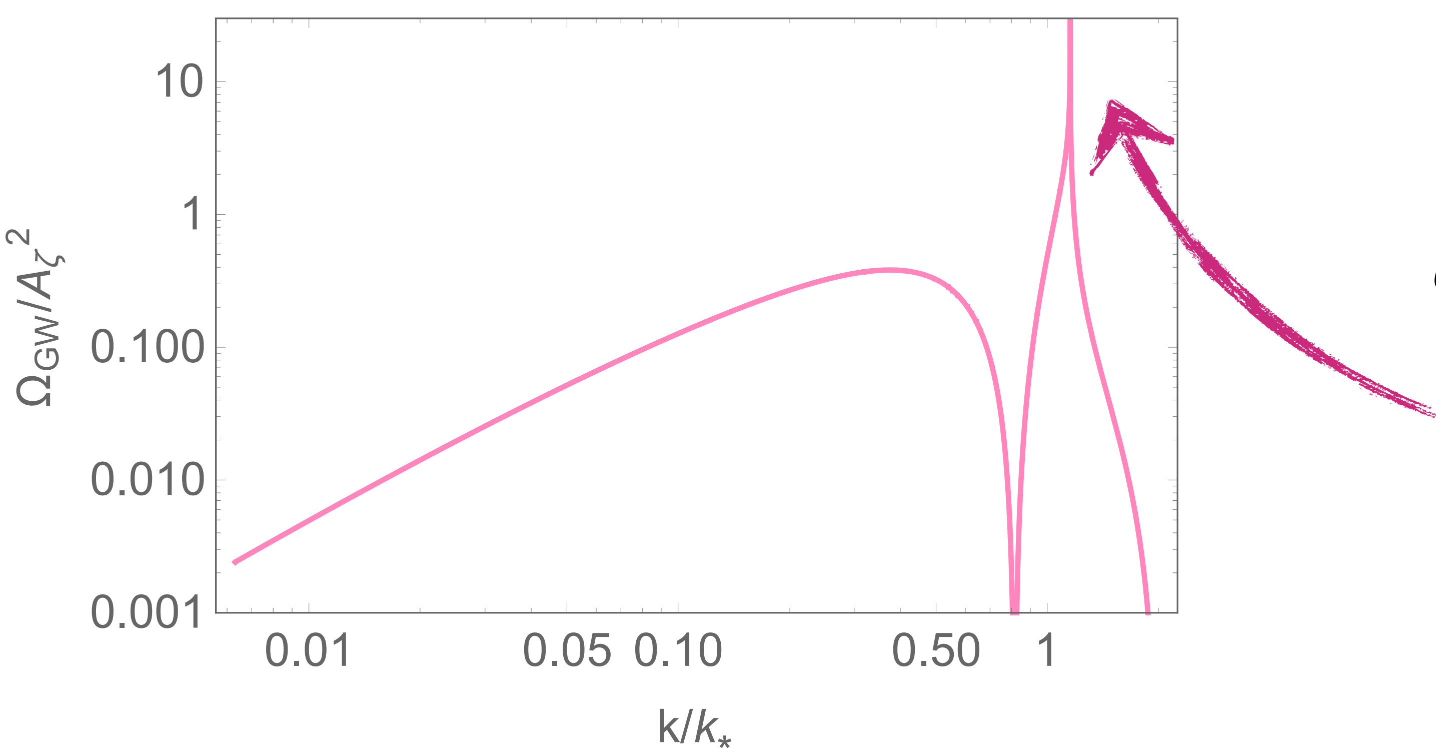
Calculable GW spectrum

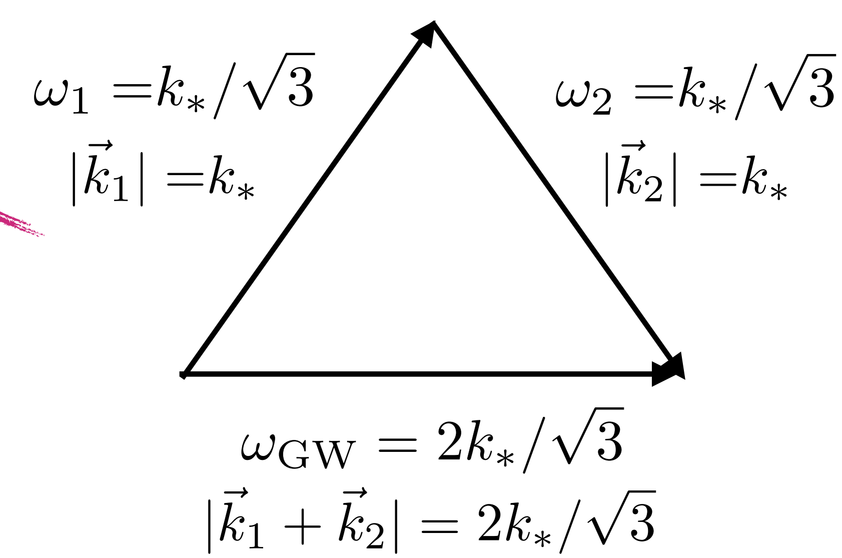
<u>Example 1 in RD: Monochromatic case</u> $\mathcal{P}_\zeta(k) = A_\zeta \delta(\log k/k_*)$

$$\Omega_{\rm GW}(\eta, k) = \frac{3A_\zeta^2}{64}\left(\frac{4-\tilde{k}^2}{4}\right)^2 \tilde{k}^2 \left(3\tilde{k}^2-2\right)^2$$

$$\times\left(\pi^2(3\tilde{k}^2-2)^2\Theta(2\sqrt{3}-3\tilde{k}) + \left(4+(3\tilde{k}^2-2)\log\left|1-\frac{4}{3\tilde{k}^2}\right|\right)^2\right)\Theta(2-\tilde{k})$$

$$\tilde{k} \equiv k/k_*$$





Analytically

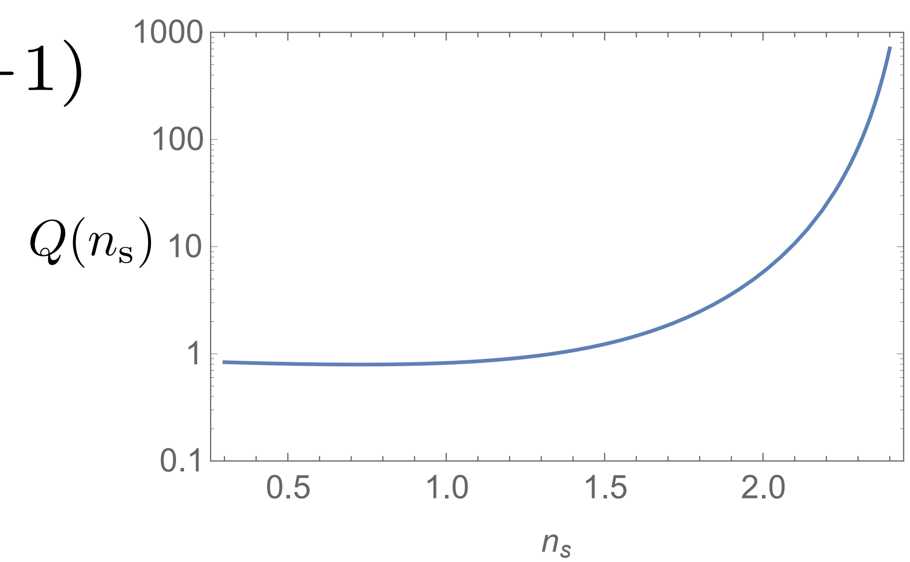
Calculable GW spectrum

Example 2 in RD: Scale-invariant case $\mathcal{P}_\zeta(k) = A_\zeta$

$$\Omega_{\rm GW}(\eta, k) \simeq 0.8222 A_\zeta^2$$

Example 3 in RD: Power-law case $\mathcal{P}_\zeta = A_\zeta \left(\frac{k}{k_*}\right)^{n_{\rm s}-1}$

$$\Omega_{\rm GW}(\eta, k) = Q(n_{\rm s}) A_\zeta^2 \left(\frac{k}{k_*}\right)^{2(n_{\rm s}-1)}$$



Enhancement in MD era

[Assadullahi, Wands, 0901.0989] [Baumann, Steinhardt, Takahashi, Ichiki, hep-th/0703290]

Next, consider w = 0 (MD era).

$$I(v,u,x) = \int_0^x \mathrm{d}\bar{x} \frac{a(\bar{\eta})}{a(\eta)} k G_k(\eta,\bar{\eta}) f(v,u,\bar{x})$$

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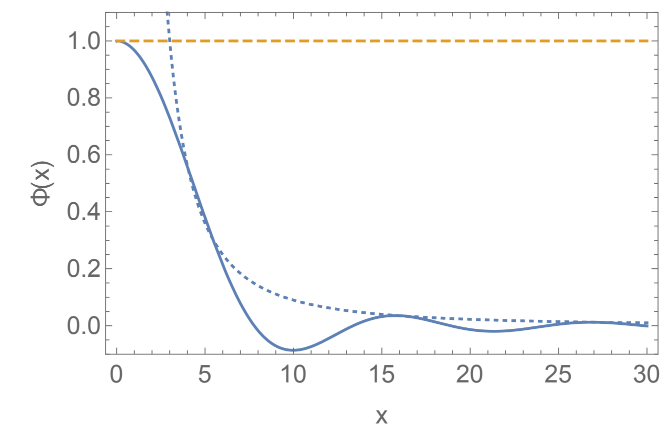

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$$f_{\mathrm{MD}}(v,u,x) = \frac{6}{5}$$

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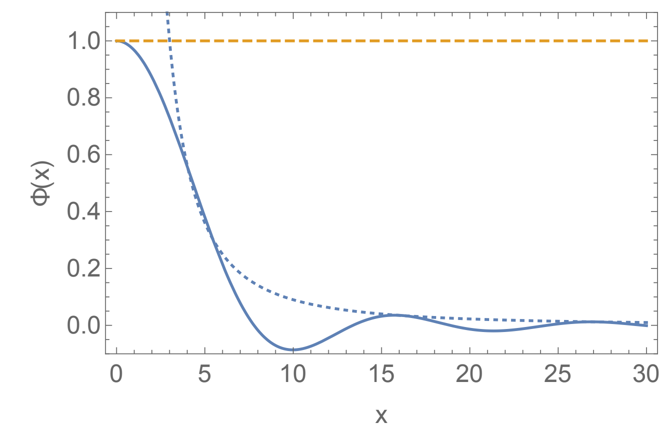

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Non-decaying source!

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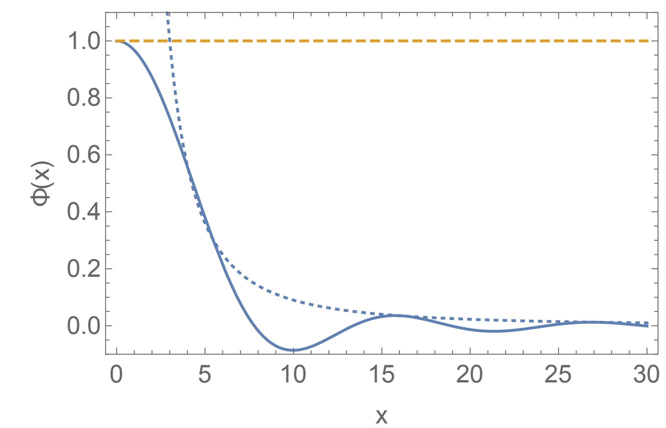

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$$f_{\mathrm{MD}}(v,u,x) = \frac{6}{5}$$

Non-decaying source!

$$\overline{I^2_{\mathrm{MD}}(v,u,x\to\infty)} = \frac{18}{25}$$

← **Not diluted!**

Enhancement in MD era

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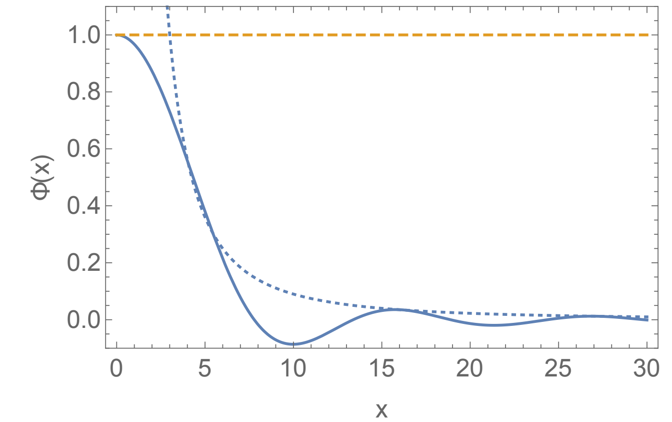

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Constant metric distortion After MD → **Propagating GW**

Calculable GW spectrum

Example 1 in MD: Monochromatic case $\mathcal{P}_\zeta(k) = A_\zeta \delta(\log k/k_*)$

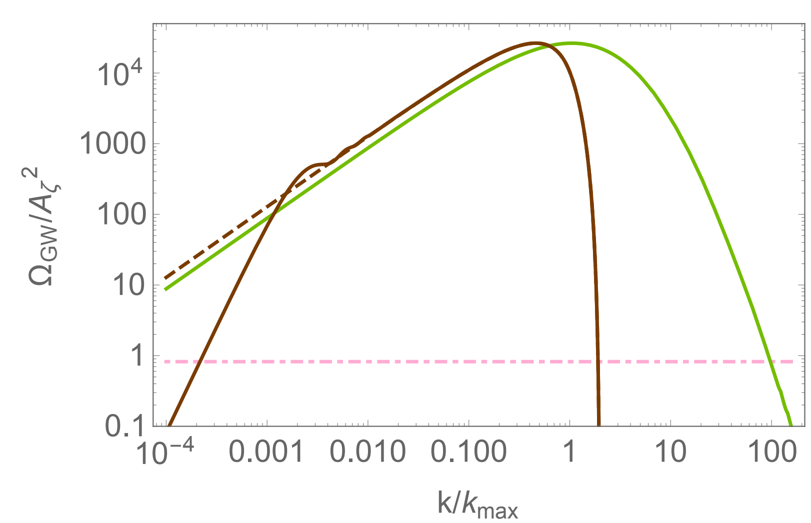
$$\Omega_{\rm GW} = \frac{3}{25}\left(\frac{k_*}{aH}\right)^2 \left(1 - \left(\frac{k}{2k_*}\right)^2\right)^2 A_\zeta^2 \Theta(2k_* - k)$$

Example 2 in MD: Scale-invariant case with a cutoff

$$\mathcal{P}_\zeta(k) = A_\zeta \Theta(k_{\rm max} - k)$$

$$\Omega_{\rm GW} = \frac{A_\zeta^2}{14000}\left(\frac{k}{aH}\right)^2 \times \begin{cases} \left(1792\tilde{k}^{-1} - 2520 + 768\tilde{k} + 105\tilde{k}^2\right) & (0 < k \leq k_{\rm max}) \\ \left(1 - 2\tilde{k}^{-1}\right)^4 \left(105\tilde{k}^2 + 72\tilde{k} + 16 - 32\tilde{k}^{-1} - 16\tilde{k}^{-2}\right) & (k_{\rm max} < k \leq 2k_{\rm max}) \end{cases}$$

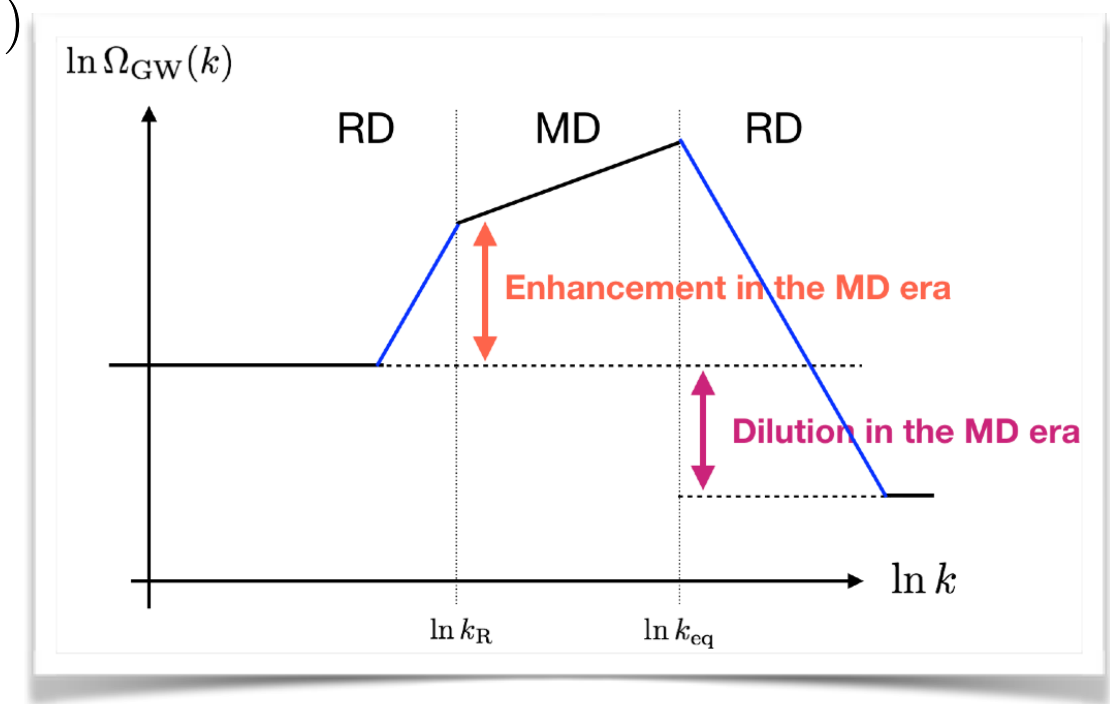
$$\tilde{k} \equiv k/k_*$$



General cases

The MD → RD transition

$$I(v,u,x) = \int_0^{x_{\rm R}} {\rm d}\bar{x} \left(\frac{x_{\rm R}}{x}\right)\left(\frac{\bar{x}}{x_{\rm R}}\right)^2 kG_k^{\rm MD\to RD}(\eta,\bar{\eta}) f_{\rm MD}(v,u,\bar{x}) + \int_{x_{\rm R}}^{x} {\rm d}\bar{x} \left(\frac{\bar{x}}{x}\right) kG_k^{\rm RD}(\eta,\bar{\eta}) f_{\rm MD\to RD}(v,u,\bar{x})$$



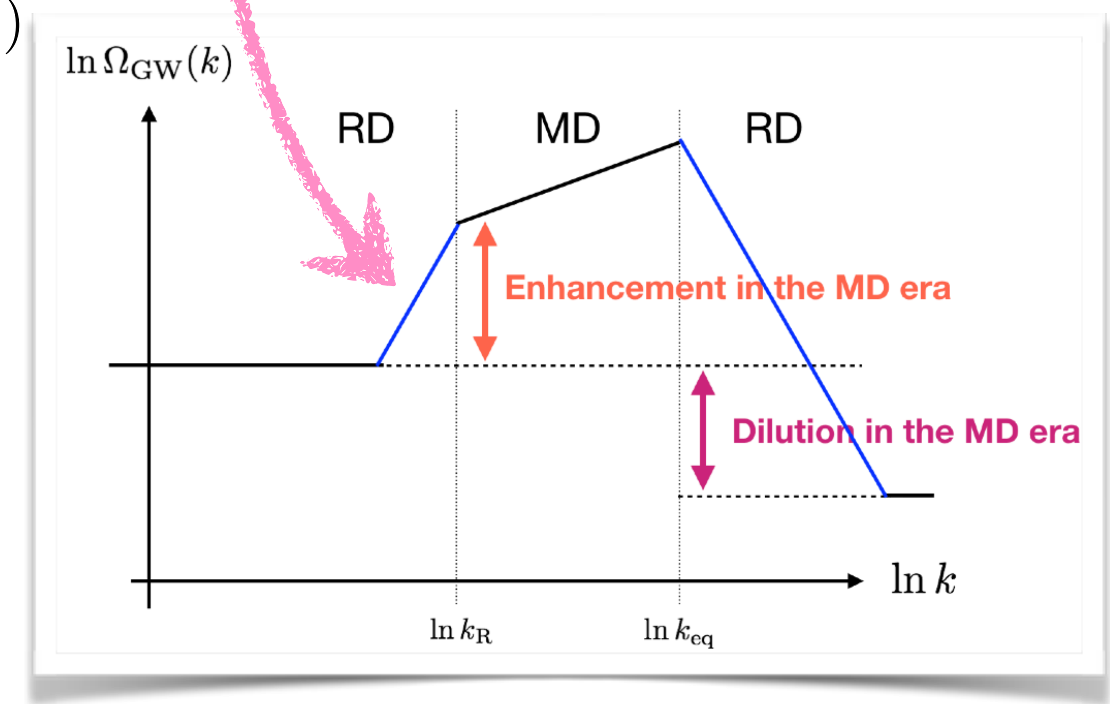
The RD → MD transition

$$I(v,u,x) = \int_0^{x_{\rm eq}} {\rm d}\bar{x} \left(\frac{x_{\rm eq}}{x}\right)^2\left(\frac{\bar{x}}{x_{\rm eq}}\right) kG_k^{\rm RD\to MD}(\eta,\bar{\eta}) f_{\rm RD}(v,u,\bar{x}) + \int_{x_{\rm eq}}^{x} {\rm d}\bar{x} \left(\frac{\bar{x}}{x}\right)^2 kG_k^{\rm MD}(\eta,\bar{\eta}) f_{\rm RD\to MD}(v,u,\bar{x})$$

General cases

<u>The MD → RD transition</u>

$$I(v,u,x) = \boxed{\int_0^{x_\mathrm{R}} \mathrm{d}\bar{x} \left(\frac{x_\mathrm{R}}{x}\right)\left(\frac{\bar{x}}{x_\mathrm{R}}\right)^2 k G_k^{\mathrm{MD}\to\mathrm{RD}}(\eta,\bar{\eta}) f_\mathrm{MD}(v,u,\bar{x})}$$
$$+ \int_{x_\mathrm{R}}^{x} \mathrm{d}\bar{x} \left(\frac{\bar{x}}{x}\right) k G_k^{\mathrm{RD}}(\eta,\bar{\eta}) f_{\mathrm{MD}\to\mathrm{RD}}(v,u,\bar{x})$$



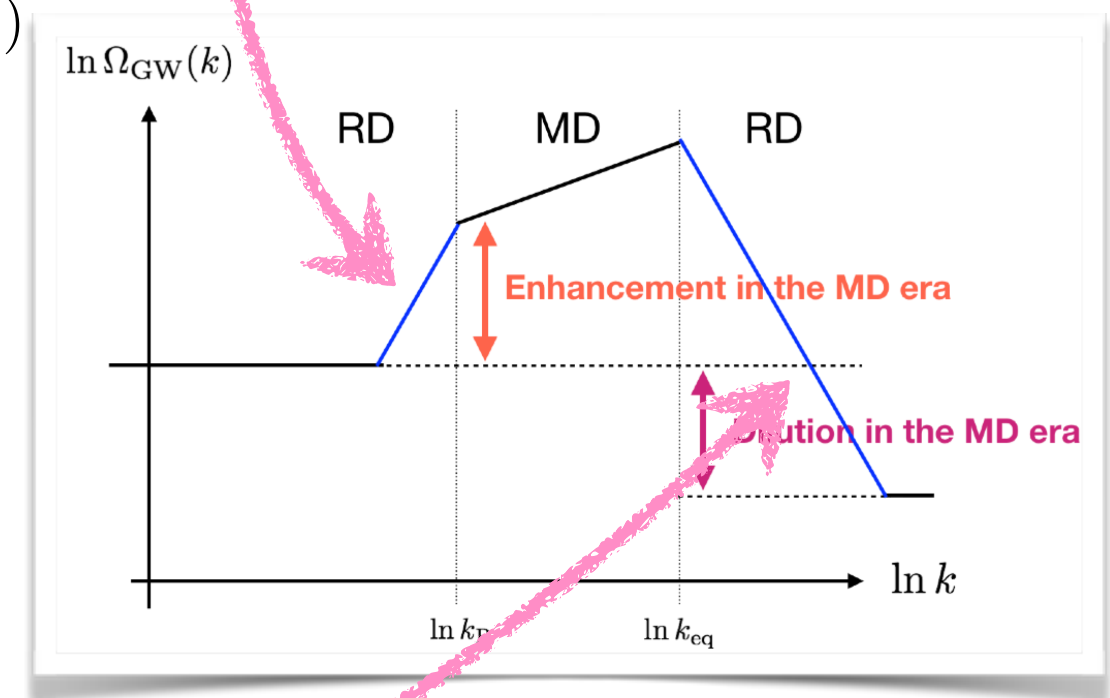
<u>The RD → MD transition</u>

$$I(v,u,x) = \int_0^{x_\mathrm{eq}} \mathrm{d}\bar{x} \left(\frac{x_\mathrm{eq}}{x}\right)^2\left(\frac{\bar{x}}{x_\mathrm{eq}}\right) k G_k^{\mathrm{RD}\to\mathrm{MD}}(\eta,\bar{\eta}) f_\mathrm{RD}(v,u,\bar{x})$$
$$+ \int_{x_\mathrm{eq}}^{x} \mathrm{d}\bar{x} \left(\frac{\bar{x}}{x}\right)^2 k G_k^{\mathrm{MD}}(\eta,\bar{\eta}) f_{\mathrm{RD}\to\mathrm{MD}}(v,u,\bar{x})$$

General cases

<u>The MD $\rightarrow$ RD transition</u>

$$I(v,u,x) = \boxed{\int_0^{x_\mathrm{R}} \mathrm{d}\bar{x} \left(\frac{x_\mathrm{R}}{x}\right)\left(\frac{\bar{x}}{x_\mathrm{R}}\right)^2 kG_k^{\mathrm{MD}\rightarrow\mathrm{RD}}(\eta,\bar{\eta}) f_\mathrm{MD}(v,u,\bar{x})}$$
$$+ \int_{x_\mathrm{R}}^{x} \mathrm{d}\bar{x} \left(\frac{\bar{x}}{x}\right) kG_k^{\mathrm{RD}}(\eta,\bar{\eta}) f_{\mathrm{MD}\rightarrow\mathrm{RD}}(v,u,\bar{x})$$



<u>The RD $\rightarrow$ MD transition</u>

$$I(v,u,x) = \int_0^{x_\mathrm{eq}} \mathrm{d}\bar{x} \left(\frac{x_\mathrm{eq}}{x}\right)^2\left(\frac{\bar{x}}{x_\mathrm{eq}}\right) kG_k^{\mathrm{RD}\rightarrow\mathrm{MD}}(\eta,\bar{\eta}) f_\mathrm{RD}(v,u,\bar{x})$$
$$\boxed{+ \int_{x_\mathrm{eq}}^{x} \mathrm{d}\bar{x} \left(\frac{\bar{x}}{x}\right)^2 kG_k^{\mathrm{MD}}(\eta,\bar{\eta}) f_{\mathrm{RD}\rightarrow\mathrm{MD}}(v,u,\bar{x})}$$

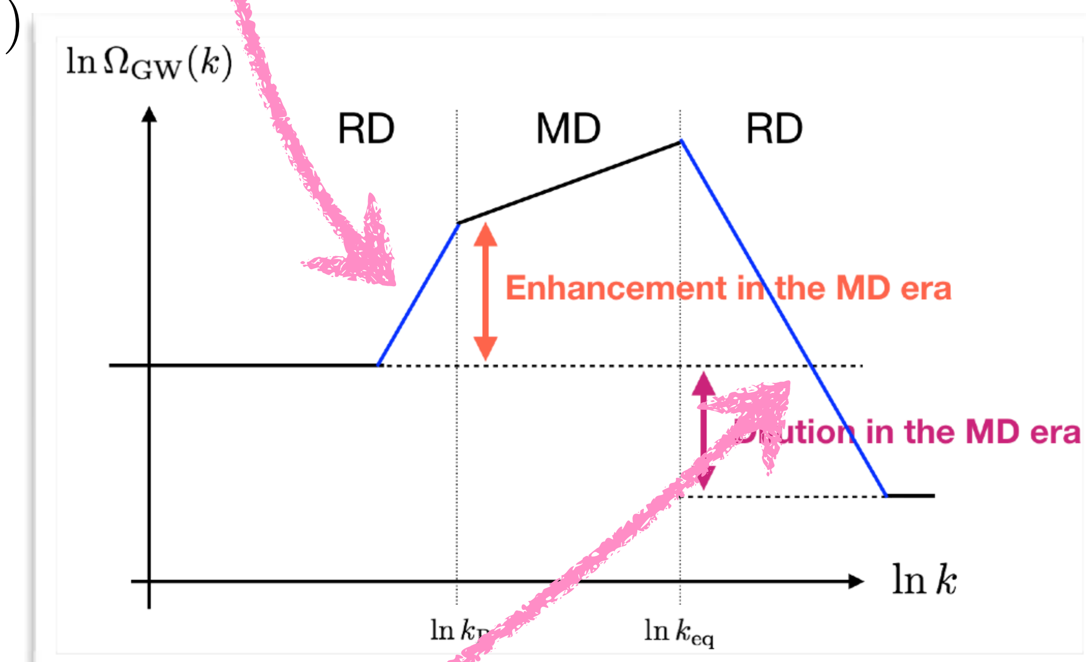
General cases

The MD → RD transition

$$I(v,u,x) = \int^{x_{\mathrm{R}}} \mathrm{d}\bar{x} \left(\frac{x_{\mathrm{R}}}{\cdots}\right) \cdots (v,u,x)$$

$$\cdots \int_{x_{\mathrm{R}}} \cdots \left(\frac{\cdots}{x}\right) kG_k^{\mathrm{RD}}(\eta,\bar{\eta}) f_{\mathrm{MD}\to\mathrm{RD}}(v,u,\bar{x})$$

$$\frac{3}{5xx_{\mathrm{R}}^3}\left(3(2x_{\mathrm{R}}^2-1)\cos x - 6x_{\mathrm{R}}\sin x + 2x_{\mathrm{R}}^4\cos(x-x_{\mathrm{R}}) + 4x_{\mathrm{R}}^3\sin(x-x_{\mathrm{R}}) + 3\cos(x-2x_{\mathrm{R}})\right)$$



The RD → MD transition

$$I(v,u,x) = \int_0^{x_{\mathrm{eq}}} \mathrm{d}\bar{x} \left(\frac{x_{\mathrm{eq}}}{x}\right)^2 \left(\frac{\bar{x}}{x_{\mathrm{eq}}}\right) kG_k^{\mathrm{RD}\to\mathrm{MD}}(\eta,\bar{\eta}) f_{\mathrm{RD}}(v,u,\bar{x})$$

$$+\int_{x_{\mathrm{eq}}}^{x} \mathrm{d}\bar{x} \left(\frac{\bar{x}}{x}\right)^2 kG_k^{\mathrm{MD}}(\eta,\bar{\eta}) f_{\mathrm{RD}\to\mathrm{MD}}(v,u,\bar{x})$$

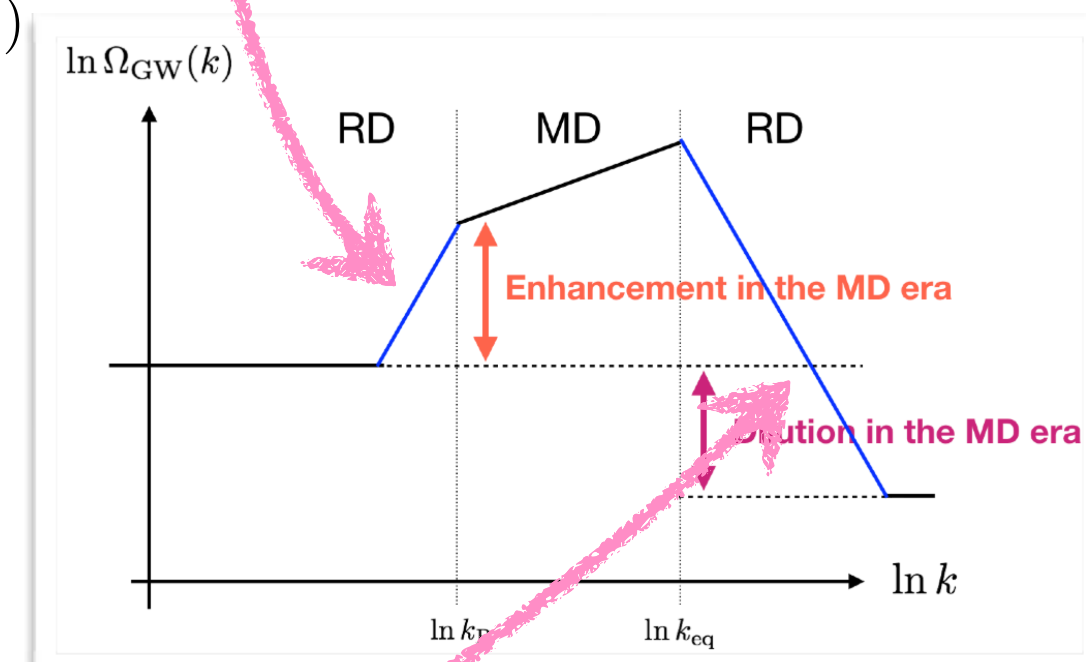
General cases

The MD → RD transition

$$I(v,u,x) = \int^{x_{\rm R}} \mathrm{d}\bar{x} \left(\frac{x_{\rm R}}{\ } \right) \cdots {}_{\rm D}(v,u,x)$$

$$\frac{3}{5 x x_{\rm R}^3}\left(3(2x_{\rm R}^2-1)\cos x - 6x_{\rm R}\sin x + 2x_{\rm R}^4\cos(x-x_{\rm R}) + 4x_{\rm R}^3\sin(x-x_{\rm R}) + 3\cos(x-2x_{\rm R})\right)$$

$$\int_{x_{\rm R}} \cdots \left(\frac{\ }{x}\right) k G_k^{\rm RD}(\eta,\bar{\eta}) f_{\rm MD\to RD}(v,u,\bar{x})$$



The RD → MD transition

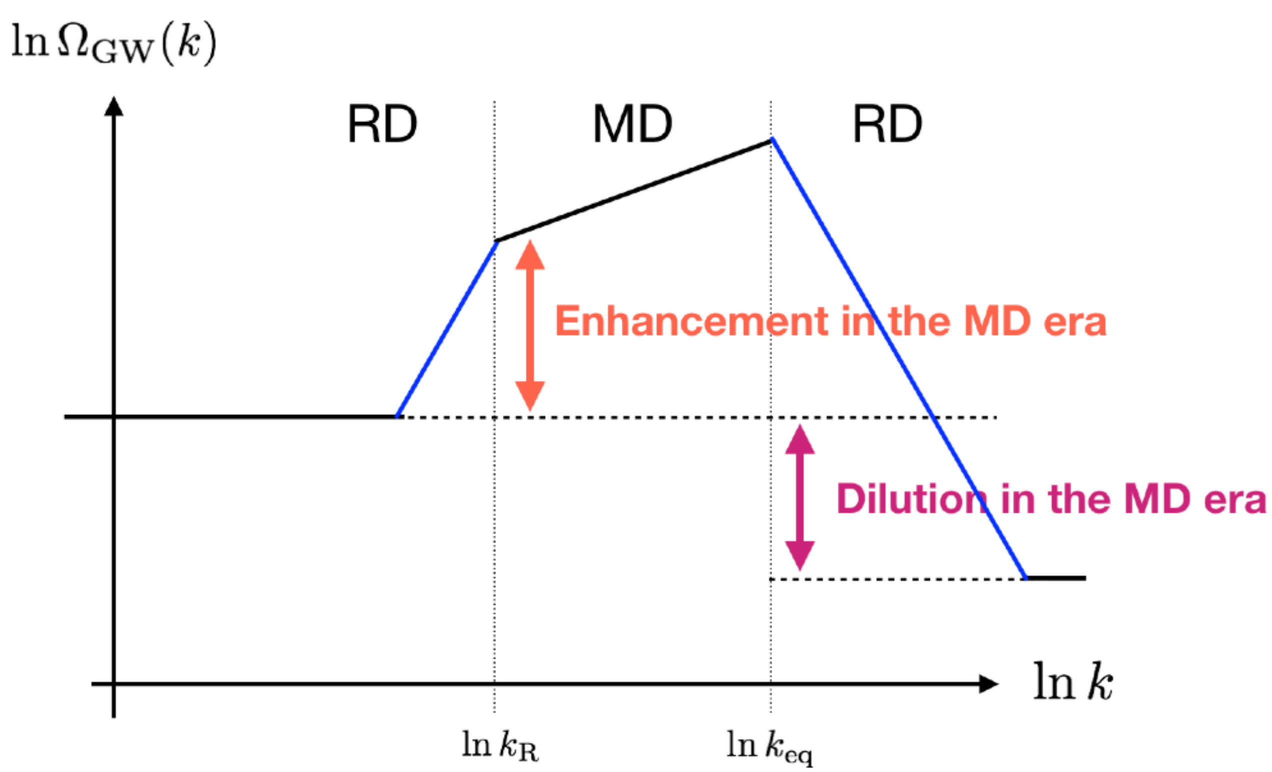
$$\frac{(x^3 - 3(x-x_{\rm eq}) - x x_{\rm eq}^2)\cos(x-x_{\rm eq}) - (3+3x x_{\rm eq} - x_{\rm eq}^2)\sin(x-x_{\rm eq})}{x^3} \times \frac{6}{5}\frac{\ln(c_1 u x_{\rm eq})}{(c_2 u x_{\rm eq})^2}\frac{\ln(c_1 v x_{\rm eq})}{(c_2 v x_{\rm eq})^2}$$

$$\cdots {}^{\rm RD\to MD}(\eta,\bar{\eta}) f_{\rm RD}(v,u,\bar{x})$$

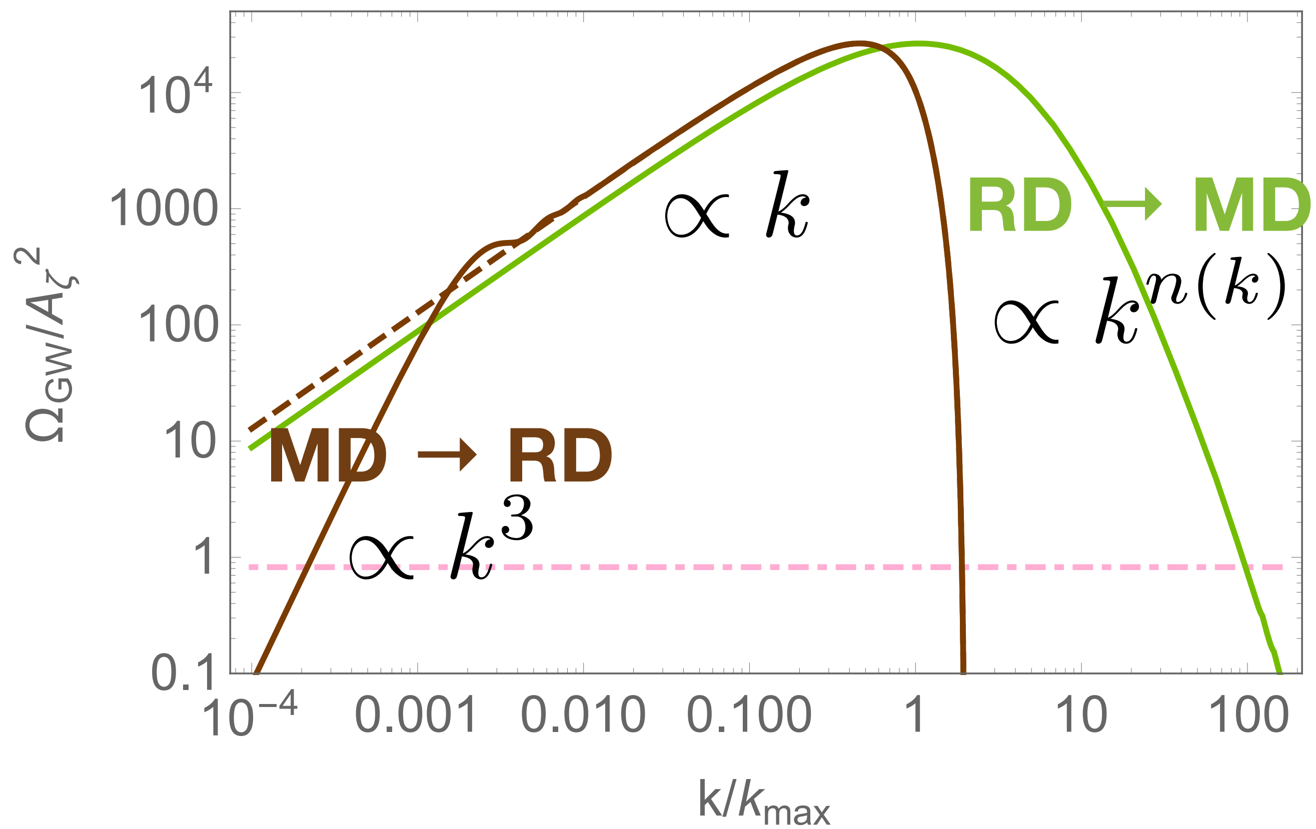
$$+\int_{x_{\rm eq}}^{x} \mathrm{d}\bar{x}\left(\frac{\bar{x}}{x}\right) k G_k \cdots$$

General cases

Schematic



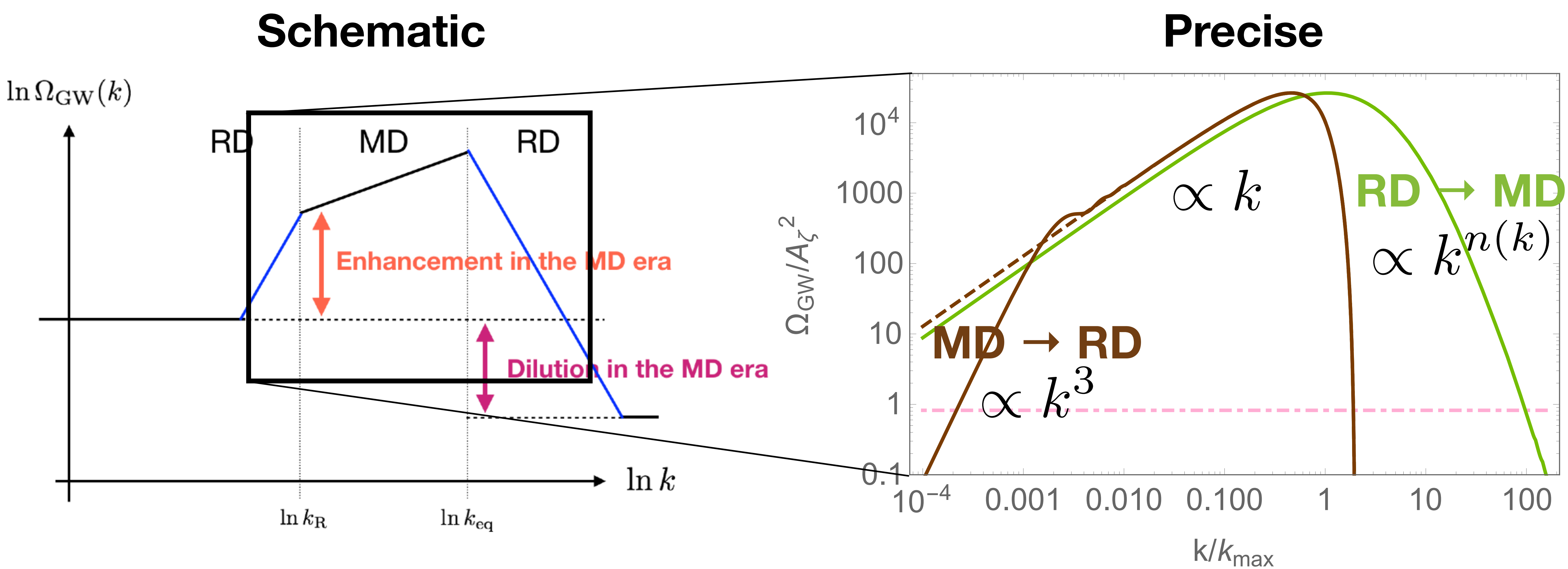
Precise



$-6 \lesssim n(k) \lesssim -4$

Scale-invariant primordial spectrum is assumed as a simple example.

General cases



$$-6 \lesssim n(k) \lesssim -4$$

Scale-invariant primordial spectrum is assumed as a simple example.

APPLICATION:

PRIMORDIAL BLACK HOLE SCENARIOS

[Kohri, Terada, 1802.06785]

Current PBH abundance

PBH fraction in CDM

$$f_{\mathrm{PBH}}(M) = \frac{1}{\rho_{\mathrm{CDM}}} \frac{\mathrm{d}\rho_{\mathrm{PBH}}}{\mathrm{d}\ln M}$$

$$= \left(\frac{g_*(T)}{g_*(T_{\mathrm{eq}})} \frac{g_{*,s}(T_{\mathrm{eq}})}{g_{*,s}(T)} \frac{T}{T_{\mathrm{eq}}} \gamma \beta(\sigma(k(M))) \right) \Bigg|_{T=\mathrm{Min}[T_M, T_{\mathrm{R}}]} \frac{\Omega_{\mathrm{m}}}{\Omega_{\mathrm{CDM}}}$$

Current PBH abundance

PBH fraction in CDM

PBH formation probability

$$f_{\mathrm{PBH}}(M) = \frac{1}{\rho_{\mathrm{CDM}}} \frac{\mathrm{d}\rho_{\mathrm{PBH}}}{\mathrm{d}\ln M}$$

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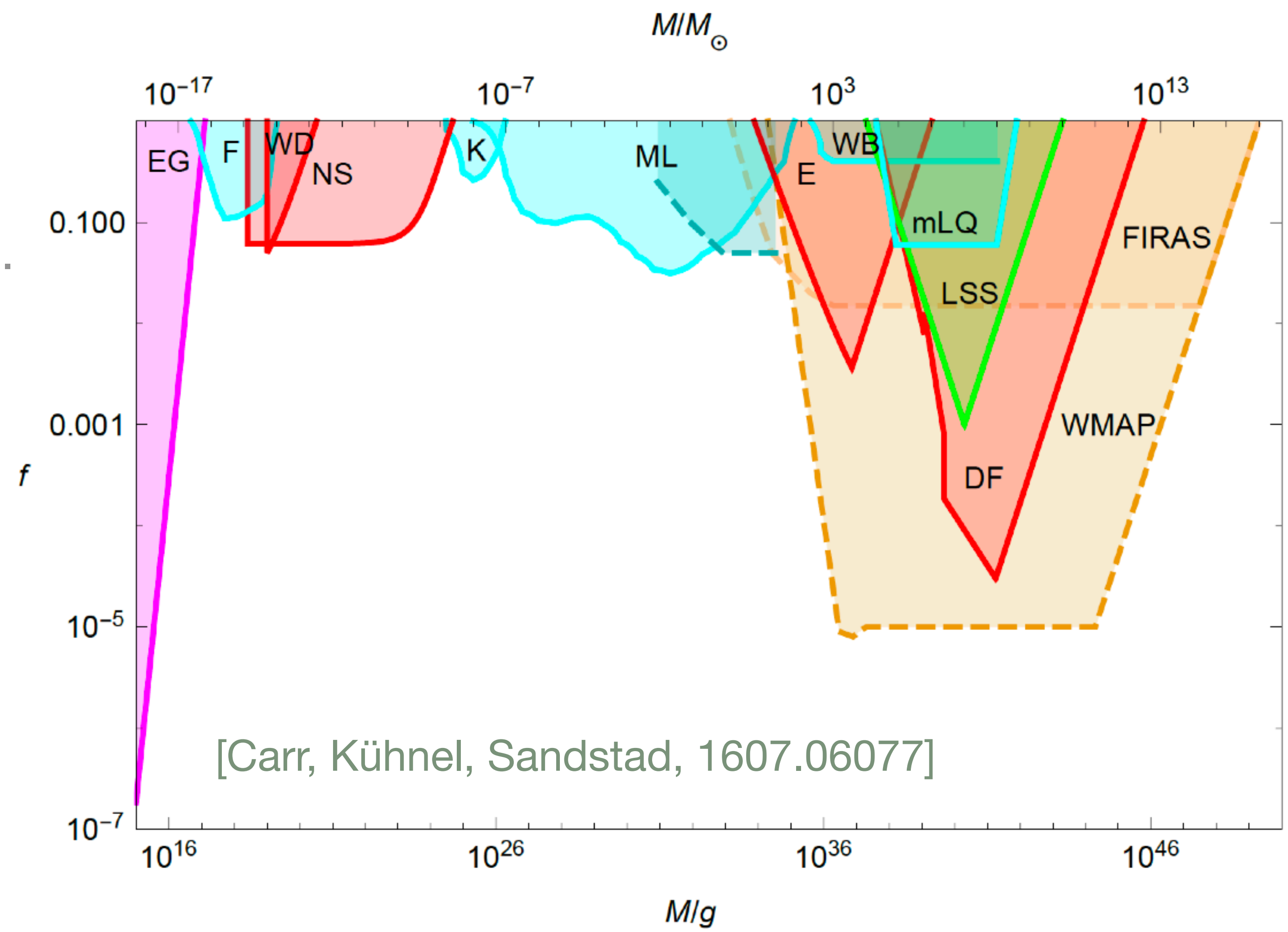
Current PBH abundance

Observational constraints

Note: These are not the latest constraints.



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PBH formation probability

Coarse-grained perturbations

[Young, Byrnes, Sasaki, 1405.7023]
see also [Ando, Inomata, Kawasaki, 1802.06393]

$$\sigma^2(k) = \int_{-\infty}^{\infty} \mathrm{d}\ln q\, w^2\left(\frac{q}{k}\right) \frac{4(1+w_{\mathrm{eos}})^2}{(5+3w_{\mathrm{eos}})^2} \left(\frac{q}{k}\right)^4 T^2(q/k) P_\zeta(q)$$

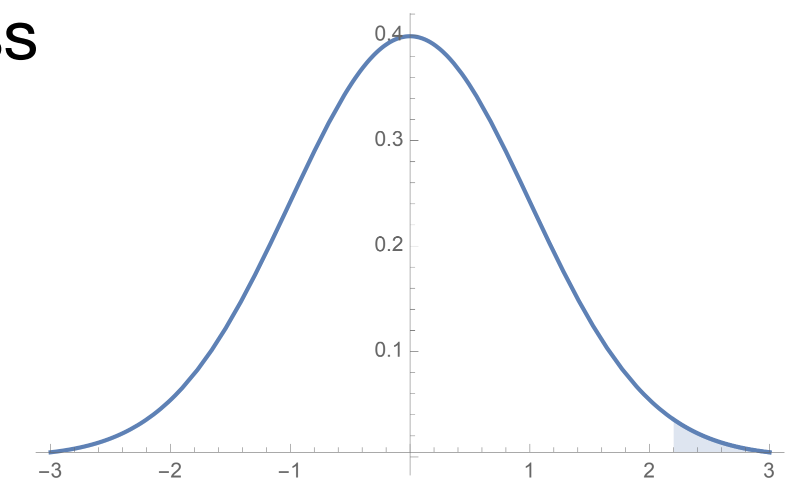
$$\sim \frac{2(1+w_{\mathrm{eos}})^2}{(5+3w_{\mathrm{eos}})^2} P_\zeta(k)$$

Formation probability in RD

Pressure prevents the formation → rare process

$$\beta(\sigma) = \int_{\delta_c}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{\delta}{2\sigma^2}\right) \mathrm{d}\delta$$

[Press, Schechter, 1974]



(exaggerated figure for illustration)

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Window function

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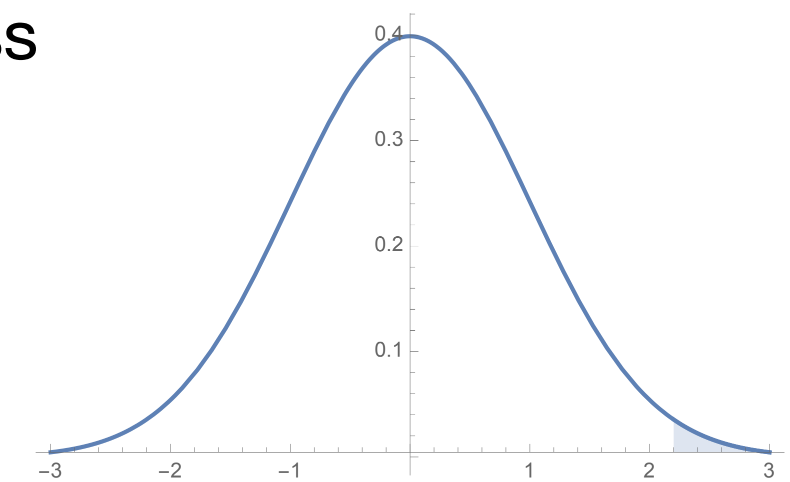

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Transfer function

$$\sim \frac{2(1+w_{\mathrm{eos}})^2}{(5+3w_{\mathrm{eos}})^2} P_\zeta(k)$$

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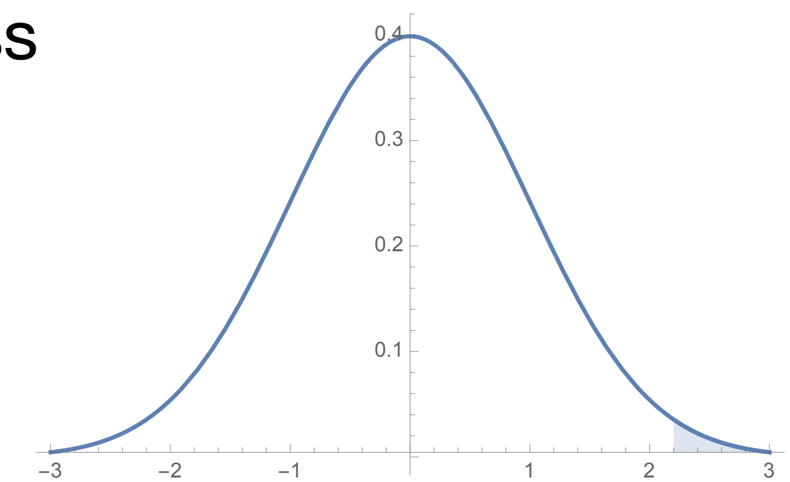

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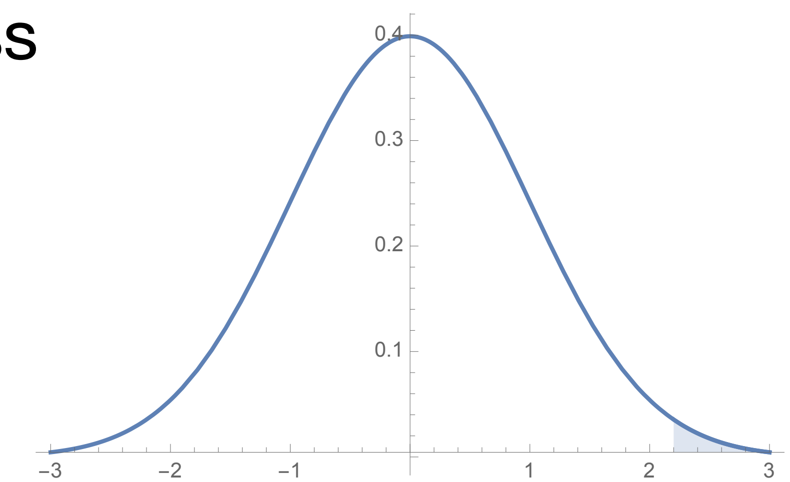

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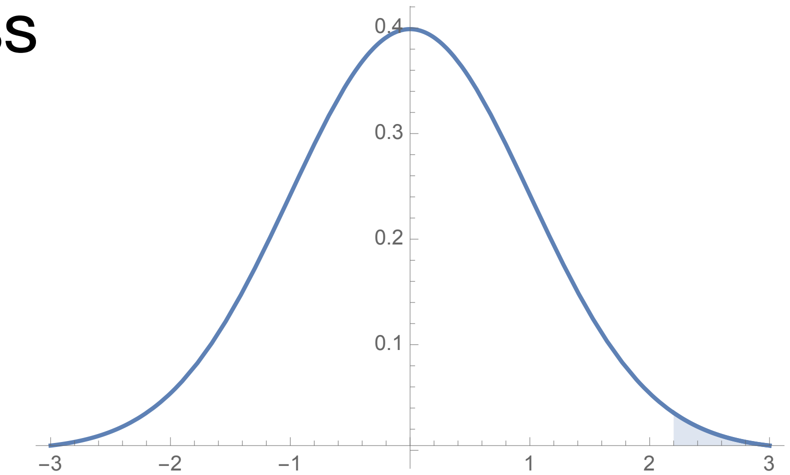

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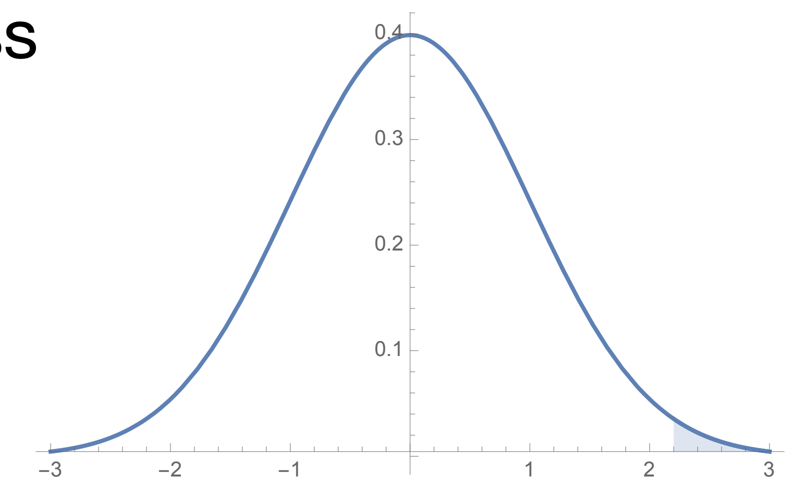

(exaggerated figure for illustration)

Formation probability in MD

[Khlopov, Polnarev, 1980, 1985]

$$\frac{\delta\rho}{\rho} \propto a(t)$$

Perturbation grows in the MD era → low threshold

Anisotropy & angular momentum affects the formation rate.

PBH formation probability

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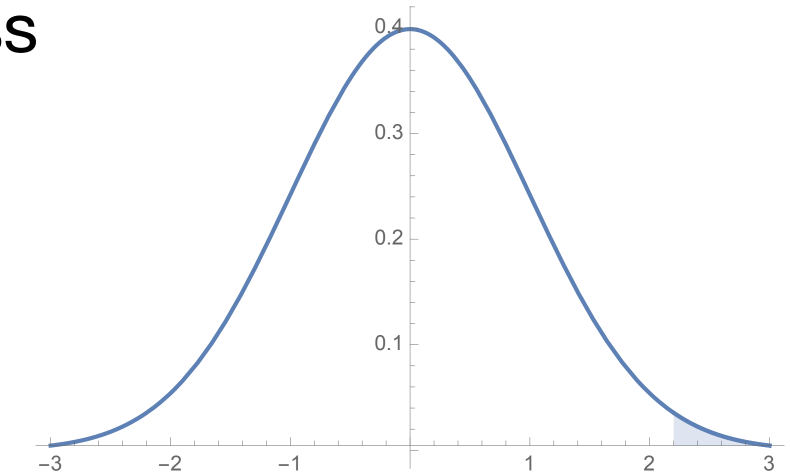

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$$\beta(\sigma) = \begin{cases} 1.894 \times 10^{-6} \times f_q \mathcal{I}^6 \sigma^2 \exp\left(-0.1474 \frac{\mathcal{I}^{4/3}}{\sigma^{2/3}}\right) & (\sigma < 0.005) \\ 0.05556\sigma^5 & (\sigma \geq 0.005) \end{cases}$$

[Harada, Yoo, Kohri, Nakao, Jhingan, 1609.01588]

[Harada, Yoo, Kohri, Nakao, 1707.03595]

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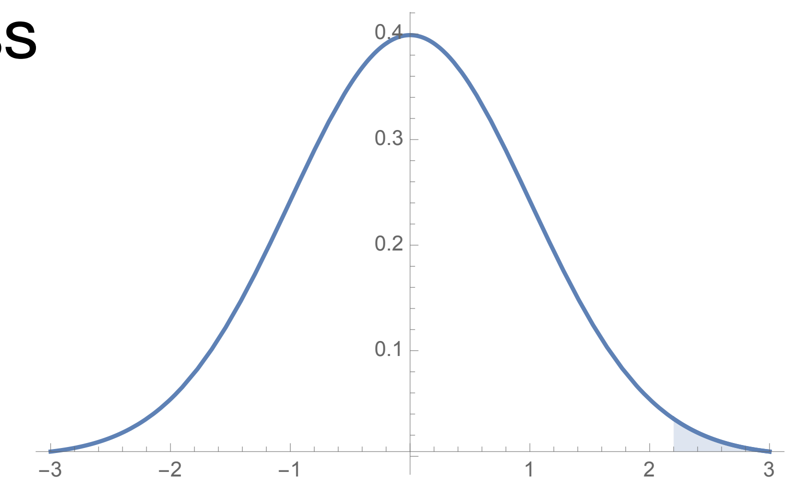

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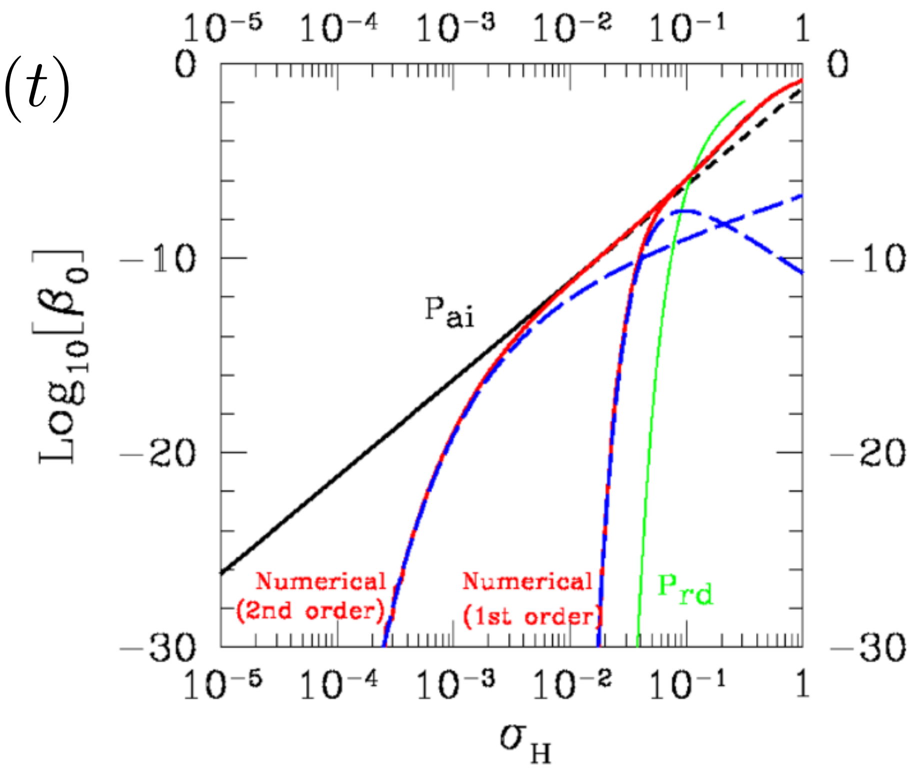


[Harada, Yoo, Kohri, Nakao, 1707.03595]

Inflation with running spectrum

$$\mathcal{P}_{\mathcal{R}}(k) = A_s \left(\frac{k}{k_*}\right)^{n_s - 1 + \frac{\alpha_s}{2} \ln \frac{k}{k_*} + \frac{\beta_s}{6}\left(\ln \frac{k}{k_*}\right)^2 + \cdots}$$

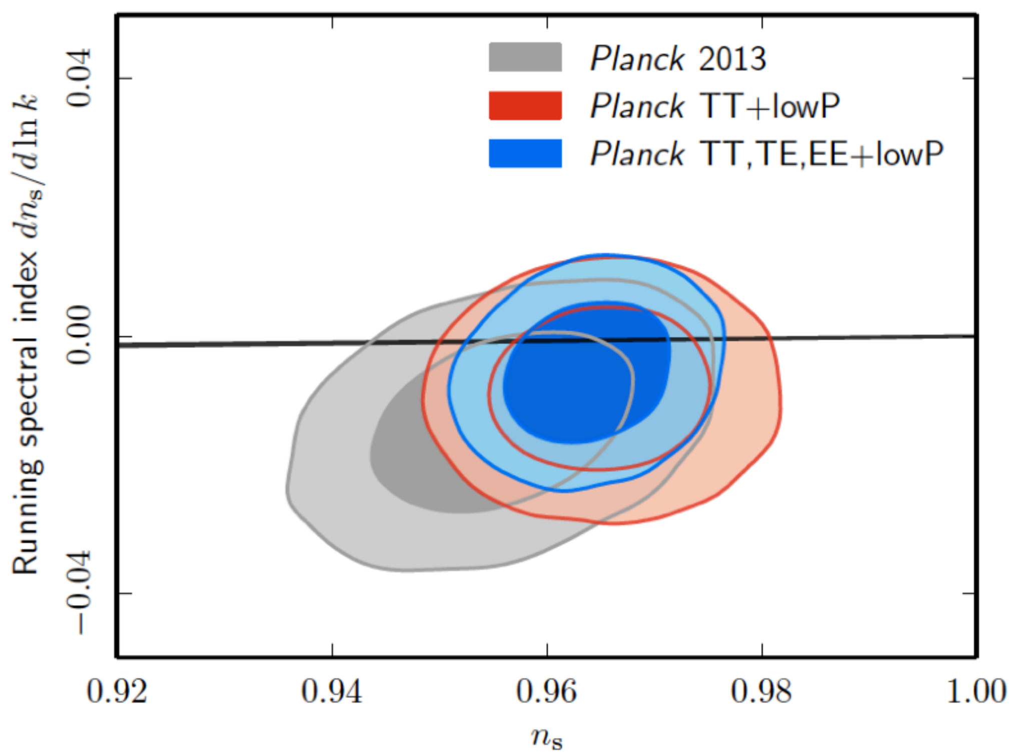


Fig. 4. Marginalized joint 68 % and 95 % CL for (n_s , $dn_s/d\ln k$) using *Planck* TT+lowP and *Planck* TT,TE,EE+lowP. For comparison, the thin black stripe shows the prediction for single-field monomial chaotic inflationary models with $50 < N_* < 60$.

Planck TT+lowP (TT,TE,EE+lowP)

$$n_s = 0.9569 \pm 0.0077 \quad (0.9586 \pm 0.0056)\,,$$
$$dn_s/d\ln k = 0.011^{+0.014}_{-0.013} \quad (0.009 \pm 0.010)\,,$$
$$d^2 n_s/d\ln k^2 = 0.029^{+0.015}_{-0.016} \quad (0.025 \pm 0.013)\,,$$

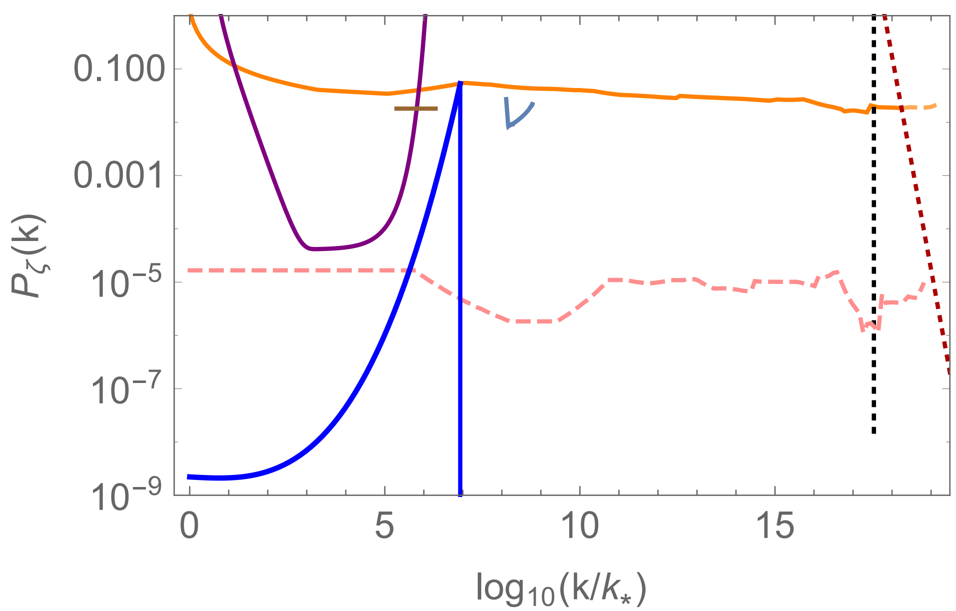
at 68% CL at the pivot scale k*=0.05/Mpc.

[Planck 2015, “Constraints on Inflation”]

Interplay between PBH & GW

PBH-for-LIGO scenario

Curvature perturbations



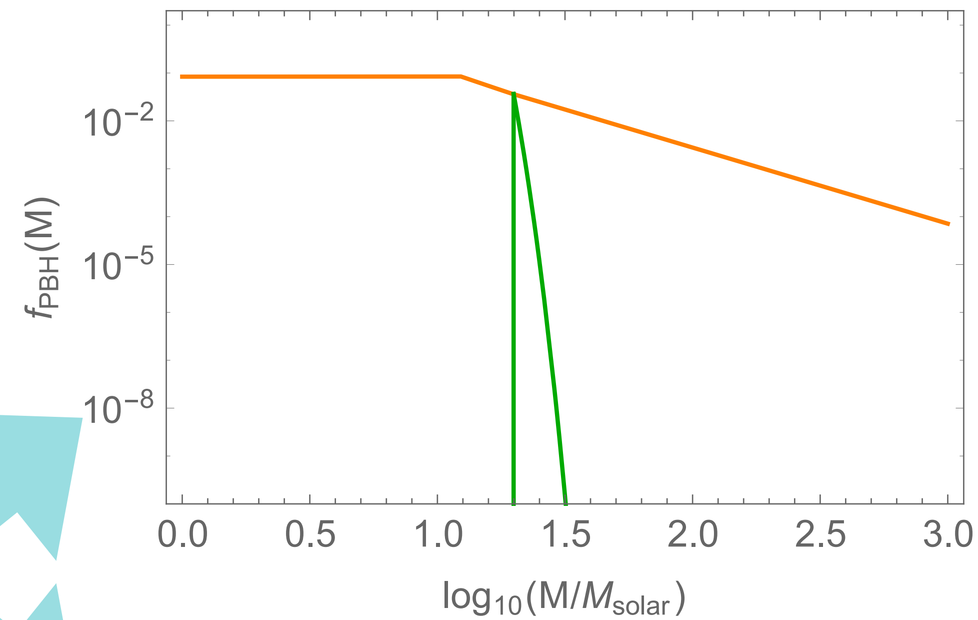
$n_{\rm s} = 0.96$

$\alpha_{\rm s} = 0$

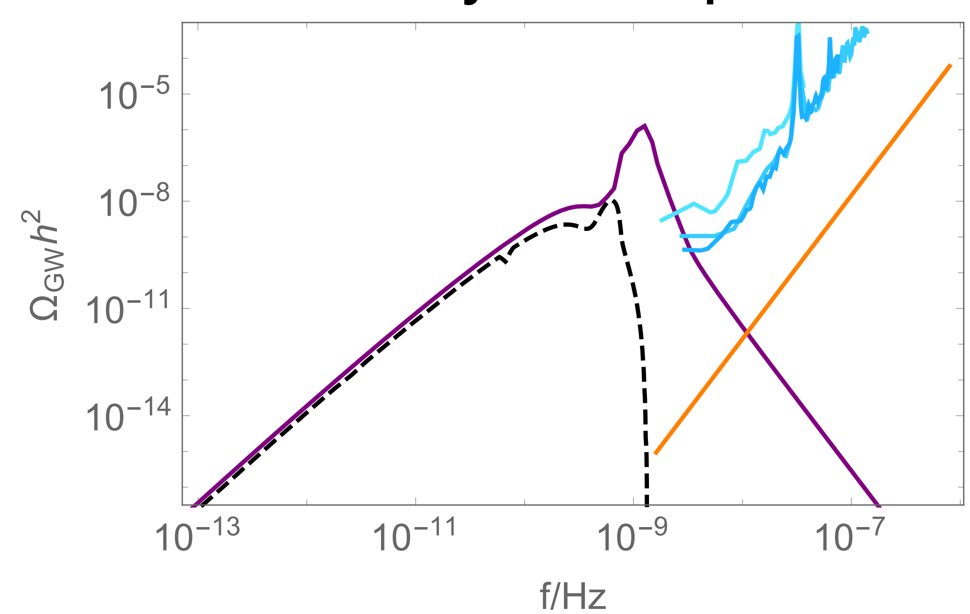
$\beta_{\rm s} = 0.026$

$T_{\rm R} = 10^9\ {\rm GeV}$

PBH mass spectrum



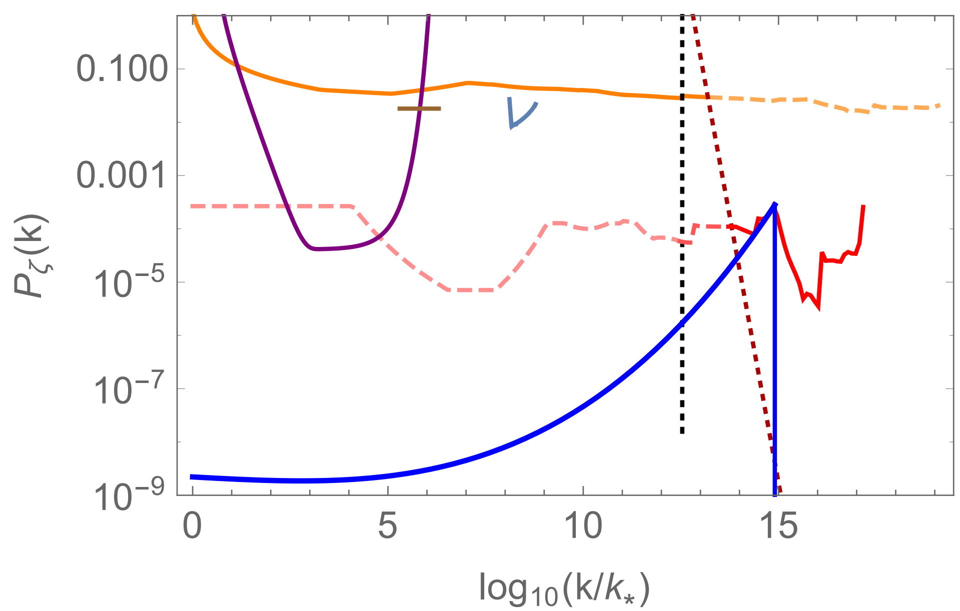
Secondary GW spectrum



Interplay between PBH & GW

PBH-DM scenario

Curvature perturbations



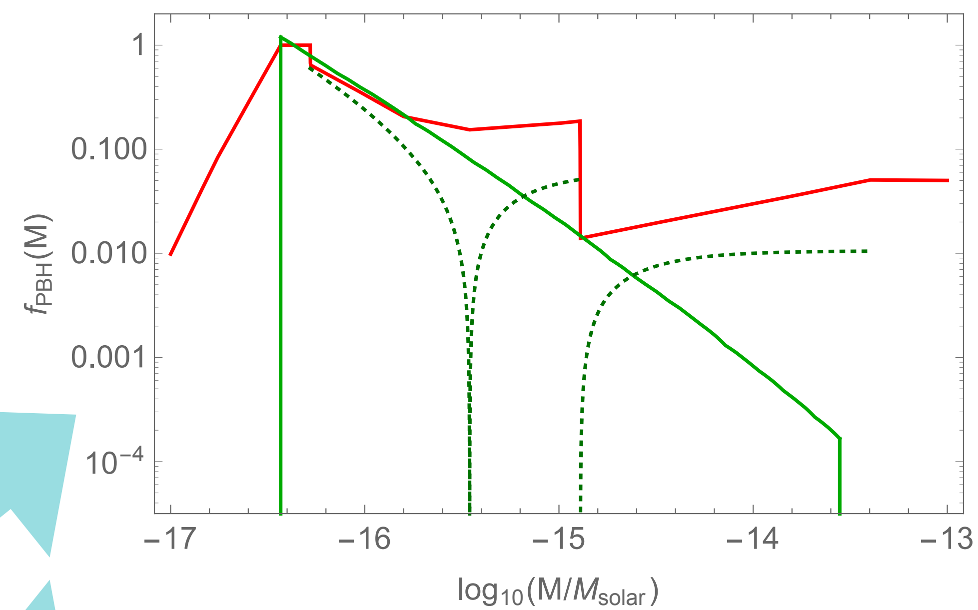
$n_s = 0.96$

$\alpha_s = 0$

$\beta_s = 0.0019485$

$T_R = 10^4\,\mathrm{GeV}$

PBH mass spectrum



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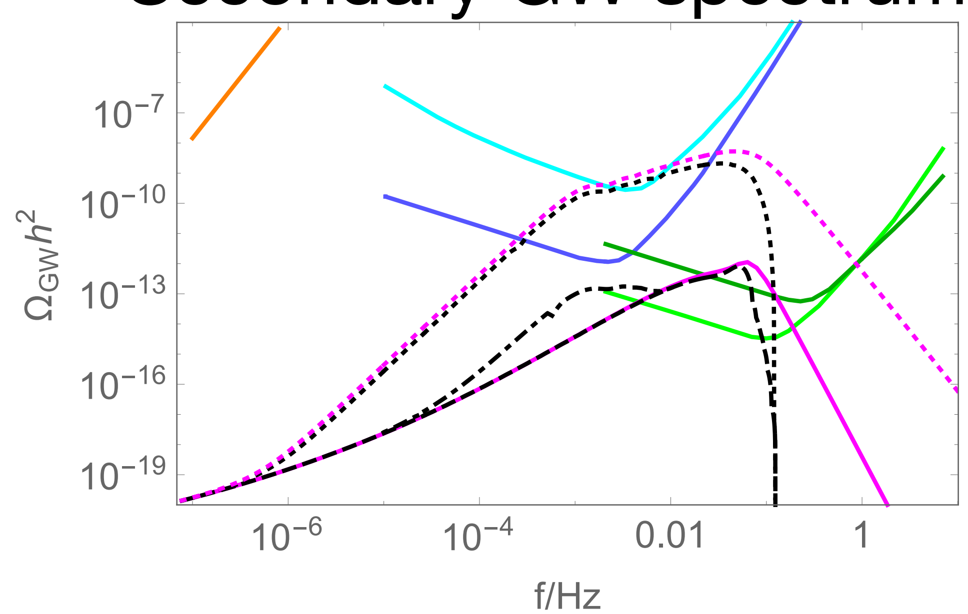
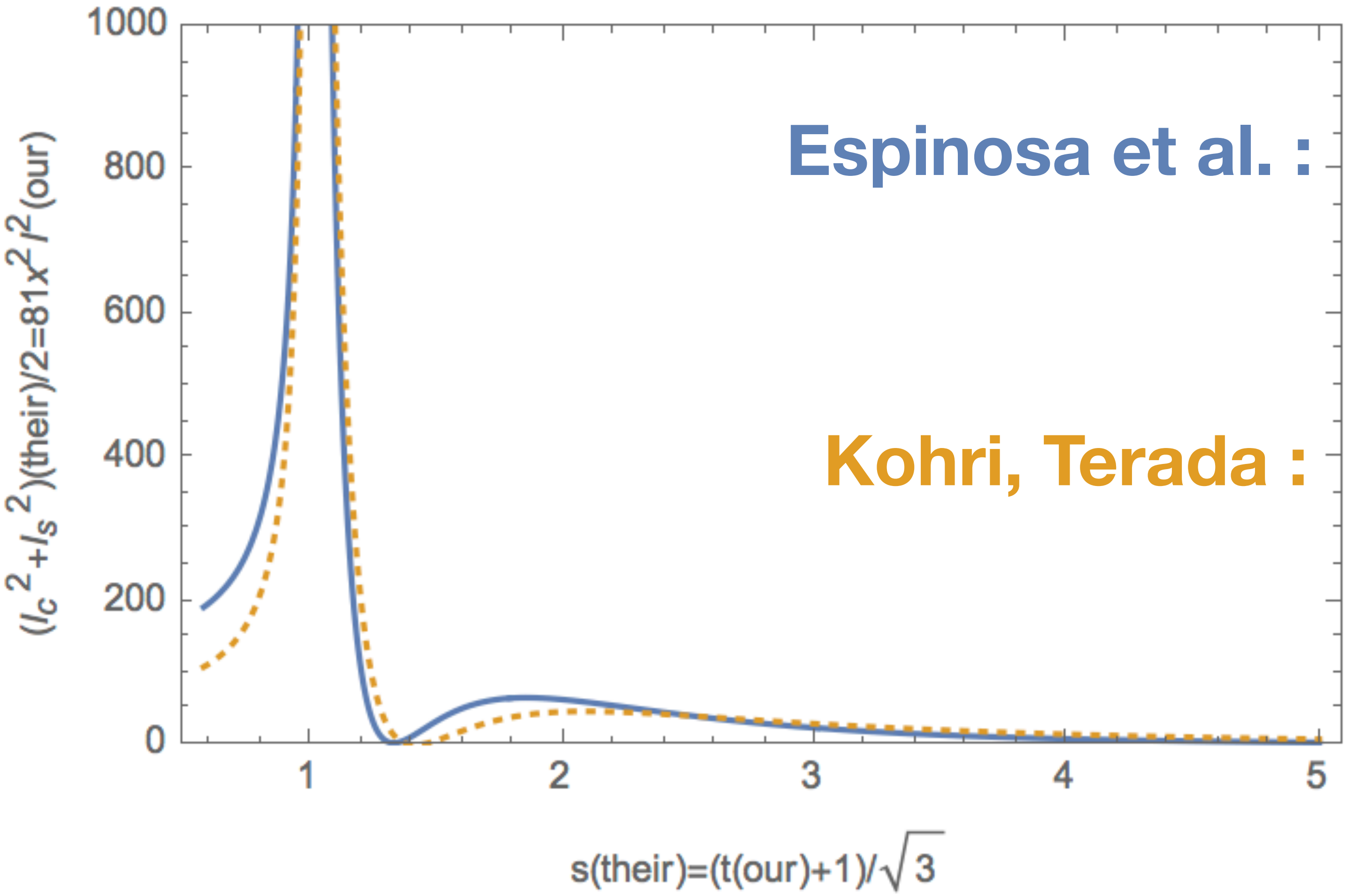


Comparison with Espinosa et al.

$$I(v,u,x) = \int_{x_{\min}}^{x} \mathrm{d}\bar{x} \frac{a(\bar{\eta})}{a(\eta)} k G_k(\eta,\bar{\eta}) f(v,u,\bar{x})$$



$$x_{\min} = k\eta_{\min} = 1$$

GW production right after inflation

$$x_{\min} = k\eta_{\min} = 0$$

inflation → RD/MD → GW production

[Espinosa, Racco, Riotto, 1804.07732]
[Kohri, Terada, 1804.08577]