Quantum Brownian motion with nongaussian stochastic forces

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Outline

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I. Quantum Brownian motion

A particle (system) coupled linearly to a set of harmonic oscillators (environment):

$$
\begin{aligned}
S[x] &= \int_0^t ds \left[\frac{1}{2} M\dot{x}^2 - V(x)\right] \\
S_e[q_n] &= \int_0^t ds \sum_n \left[\frac{1}{2} m_n \dot{q}_n^2 - \frac{1}{2} m_n \omega_n^2 q_n^2\right] \\
S_{int}[x, \{q_n\}] &= \int_0^t ds \sum_n \left(-C_n x\, q_n\right)
\end{aligned}
$$

(Schwinger, Feynman-Vernon, Caldeira-Leggett, Hu-Paz-Zhang, ...)

The dynamics of the particle is governed by the CTP effective action

$$
\begin{aligned}
e^{i\Gamma[x_+,x_-]} &= e^{iS[x_+]-iS[x_-]} \times \\
&\int_{CTP} \prod_n Dq_{n+} Dq_{n-} \left(e^{iS_e[\{q_{n+}\}]-iS_e[\{q_{n-}\}]} \right. \\
&\qquad \left. e^{iS_{int}[x_+,\{q_{n+}\}]-iS_{int}[x_-,\{q_{n-}\}]} \right) \\
&= e^{iS[x_+]-iS[x_-]+iS_{IF}[x_+,x_-]}
\end{aligned}
$$

where S_{IF} is the influence action due to the quantum harmonic oscillators.

The influence action S_{IF} can be expressed in terms of the Schwinger-Keldysh propagators

$$\begin{aligned} & S_{IF}[x_+, x_-] \\ = & \sum_n \frac{1}{2} \int ds\, ds' \\ & \quad \big[x_+(s) G_{n++}(s,s') x_+(s') - x_+(s) G_{n+-}(s,s') x_-(s') \\ & \quad\quad - x_-(s) G_{n-+}(s,s') x_+(s') + x_-(s) G_{n--}(s,s') x_-(s')\big] \end{aligned}$$

due to the corresponding boundary conditions.

The influence action S_{IF} can be written as

$$
\begin{aligned}
e^{iS_{IF}} &= e^{-i\int_0^t ds \int_0^s ds' [\Delta x(s)\eta(s-s')\Sigma x(s')]} \\
&\quad e^{-\frac{1}{2}\int_0^t ds \int_0^t ds' \, [\Delta x(s)\nu(s-s')\Delta x(s')]}
\end{aligned}
$$

where $\Delta x(s) = x_+(s) - x_-(s)$ and $\Sigma x(s) = x_+(s) + x_-(s)$, and

$$
\begin{aligned}
\eta(s-s') &= \sum_n \eta_n(s-s') = -\sum_n \frac{C_n^2}{2m_n\omega_n} \sin\omega_n(s-s') \\
\nu(s-s') &= \sum_n \nu_n(s-s') = \sum_n \frac{C_n^2}{2m_n\omega_n} \cos\omega_n(s-s')
\end{aligned}
$$

S_{IF} is basically separated into its real and imaginary parts.

Rewriting the imaginary part of S_{IF} as

$$\begin{aligned} & e^{-\frac{1}{2}\int \Delta x\,\nu\,\Delta x} \\ = \;& N\int D\xi e^{-\frac{1}{2}\int \xi\nu^{-1}\xi} e^{-\frac{1}{2}\int \Delta x\nu\Delta x} \\ = \;& N\int D\xi e^{-\frac{1}{2}\int (\xi - i\nu\Delta x)\nu^{-1}(\xi - i\nu\Delta x)} e^{-\frac{1}{2}\int \Delta x\nu\Delta x} \\ = \;& N\int D\xi P[\xi] e^{i\int \xi\Delta x} \end{aligned}$$

where $P[\xi] = e^{-\frac{1}{2}\int \xi\nu^{-1}\xi}$ is the Gaussian probability density of the stochastic force ξ.

Due to this probability density one has the stochastic average $\langle\xi(s)\xi(s')\rangle_s = \nu(s - s')$ which is called the noise kernel.

After this procedure the effective action

$$\begin{aligned}\Gamma[x_+, x_-] &= S[x_+] - S[x_-] \\ &\quad - \int_0^t ds \int_0^s ds' \Delta x(s) \eta(s - s') \Sigma x(s') \\ &\quad + \int_0^t ds \Delta x(s) \xi(s)\end{aligned}$$

The equation of motion for the particle is then given by

$$\left.\frac{\delta \Gamma[x_+, x_-]}{\delta x_+}\right|_{x_+ = x_- = x} = 0$$

The equation of motion is a Langevin equation with the stochastic force $\xi(t)$,

$$M\ddot{x} + V'(x) + \int_0^t ds\, \eta(t-s)x(s) = \xi(t)$$

The integral term is related to dissipation as one can write

$$\eta(t) = \frac{d}{dt}\gamma(t) \Rightarrow \gamma(t) = \sum_n \frac{C_n^2}{2m_n\omega_n^2}\ cos\omega_n t$$

and we have

$$M\ddot{x} + V'(x) + \int_0^t ds\, \gamma(t-s)\dot{x}(s) = \xi(t)$$

$\eta(s-s')$ is called the dissipation kernel.

The dissipation kernel and the noise kernel are respectively the real and the imaginary parts of the same Green's function.

They are related by the fluctuation-dissipation relation (FDR)

$$\nu(s) = \int_{-\infty}^{\infty} ds'\, K(s - s')\gamma(s')$$

where in this simple case

$$K(s) = \int_0^{\infty} \frac{d\omega}{\pi}\omega\, \cos\omega s$$

II. Nongaussian stochastic forces

The Brownian motion model with a different interaction term,

$$
\begin{aligned}
S[x] &= \int_0^t ds \left[\frac{1}{2} M\dot{x}^2 - V(x) \right] \\
S_e[q_n] &= \int_0^t ds \sum_n \left[\frac{1}{2} m_n \dot{q}_n^2 - \frac{1}{2} m_n \omega_n^2 q_n^2 \right] \\
S_{int}[x, \{q_n\}] &= \int_0^t ds \sum_n \left(-\lambda C_n \, x q_n^2 \right)
\end{aligned}
$$

The influence action can be expanded as a power series in λ.

$$S_{IF}[x_+, x_-] = \sum_i \delta A^{(i)}[x_+, x_-]$$

The first term gives

$$\delta A^{(1)} = \int_0^t ds \left(- \sum_n \delta V_n(x_+) \right) - \int_0^t ds \left(- \sum_n \delta V_n(x_-) \right)$$

where

$$\delta V_n(x) = \frac{\lambda\, C_n x}{2 m_n \omega_n}$$

This term can be interpreted as a renormalization of the potential.

The second term reads

$$\delta A^{(2)} = -\int_0^t ds \int_0^s ds' \, \Delta(s)\Sigma(s')\eta(s-s') + i\int_0^t ds \int_0^t ds' \, \Delta(s)\Delta(s')\nu(s-s')$$

with $\Delta(s) \equiv x_+(s) - x_-(s)$ and $\Sigma(s) \equiv x_+(s) + x_-(s)$.

Similarly to the bilinear interaction case, η and ν are related to the dissipation and the noise kernels respectively.

The third term is

$$
\begin{aligned}
& \delta A^{(3)} \\
= \ & \int_0^t ds \int_0^s ds' \int_0^{s'} ds'' \ \Delta(s)\Sigma(s')\Sigma(s'') \times \\
& \quad \sum_n 8\lambda^3 C_n^3 \, \eta_n(s-s') \left[\nu_n(s'-s'')\eta_n(s''-s) - \eta_n(s'-s'')\nu_n(s''-s) \right] \\
& + i \int_0^t ds \int_0^s ds' \int_0^{s'} ds'' \ \Delta(s)\Sigma(s')\Delta(s'') \times \\
& \quad \sum_n 8\lambda^3 C_n^3 \, \eta_n(s-s') \left[\nu_n(s'-s'')\nu_n(s''-s) + \eta_n(s'-s'')\eta_n(s''-s) \right] \\
& - i \int_0^t ds \int_0^s ds' \int_0^{s'} ds'' \ \Delta(s)\Delta(s')\Sigma(s'') \times \\
& \quad \sum_n 8\lambda^3 C_n^3 \, \nu_n(s-s') \left[\nu_n(s'-s'')\eta_n(s''-s) - \eta_n(s'-s'')\nu_n(s''-s) \right] \\
& + \int_0^t ds \int_0^s ds' \int_0^{s'} ds'' \ \Delta(s)\Delta(s')\Delta(s'') \times \\
& \quad \sum_n 8\lambda^3 C_n^3 \, \nu_n(s-s') \left[\nu_n(s'-s'')\nu_n(s''-s) + \eta_n(s'-s'')\eta_n(s''-s) \right]
\end{aligned}
$$

To the next order with terms $\Delta\Sigma\Sigma\Sigma$, $i\Delta\Delta\Sigma\Sigma$, $\Delta\Delta\Delta\Sigma$, $i\Delta\Delta\Delta\Delta$, and so on.

$$
\begin{aligned}
& \Gamma[x_+, x_-] \\
= \; & S[x_+] - S[x_-] - \int_0^t ds\, \delta V(x_+) + \int_0^t ds\, \delta V(x_-) \\
& - \int_0^t ds\, \Delta(s) H(s;\Sigma) + \frac{i}{2} \int_0^t ds \int_0^t ds'\, \Delta(s)\Delta(s') N_2(s,s';\Sigma) \\
& - \frac{1}{3!} \int_0^t ds \int_0^t ds' \int_0^t ds''\, \Delta(s)\Delta(s')\Delta(s'') N_3(s,s',s'') + \cdots
\end{aligned}
$$

where the kernels

$$
\begin{aligned}
H(s;\Sigma) &= H^{(0)}(s;\Sigma) + H^{(1)}(s;\Sigma) + \cdots \\
N_2(s,s';\Sigma) &= N_2^{(0)}(s,s') + N_2^{(1)}(s,s';\Sigma) + \cdots \\
N_3(s,s',s'') &= N_3^{(1)}(s,s',s'') + \cdots
\end{aligned}
$$

Main idea: Terms quadratic or higher in powers of $\Delta(s)$ can be interpreted as the effect of a single stochastic force $\xi(s)$.

$$
\begin{aligned}
& e^{i\left[\frac{i}{2}\int_0^t ds\int_0^t ds'\,\Delta(s)\Delta(s')N_2(s,s';\Sigma)-\frac{1}{3!}\int_0^t ds\int_0^t ds'\int_0^t ds''\,\Delta(s)\Delta(s')\Delta(s'')N_3(s,s',s'')\right]} \\
= & \int D\xi\, P[\xi] e^{i\int_0^t ds\,\Delta(s)\xi(s)}
\end{aligned}
$$

where $P[\xi]$ is the probability density of the stochastic force $\xi(s)$.

The two-point correlation

$$\left\langle \xi(s)\xi(s')\right\rangle = N_2(s,s';\Sigma)$$

$N_2(s,s';\Sigma)$ is the new noise kernel which is history dependent.

The stochastic force is nongaussian.

$$\left\langle \xi(s)\xi(s')\xi(s'')\right\rangle = N_3(s,s',s'')$$

The probability density $P[\xi]$ is also not gaussian.
Possible application to the nongaussianity of the CMB anisotropy spectrum.

Term proportional to $\Delta(s)$ is related to the dissipation kernel γ.

$$H(s, s'; \Sigma) = \int_0^s ds' \, \gamma(s, s'; \Sigma)\dot{\Sigma}(s')$$

where

$$\gamma(s, s'; \Sigma) = \gamma^{(0)}(s, s') + \gamma^{(1)}(s, s'; \Sigma) + \cdots$$

$\gamma(s, s'; \Sigma)$ should be viewed as the new dissipation kernel which is also history dependent.

Fluctuation-dissipation relation

$$N_2(s, s'; \Sigma) = \int_{-\infty}^{\infty} ds_1\, K(s, s_1; \Sigma)\gamma(s_1, s'; \Sigma)$$

where the fluctuation-dissipation kernel $K(s, s'; \Sigma)$ will also be history dependent in general.

III. The Langevin equation and the master equation

Effective action of the particle

$$\Gamma[x_+, x_-] = \int_0^t ds \left(-M\dot{\Sigma}\dot{\Delta} + M\Omega_{ren}^2 \Sigma\right) - \int_0^t ds \int_0^t ds' \, \Delta(s) H(s;\Sigma) + \int_0^t ds \, \Delta(s)\xi(s)$$

Classical equation of motion as a nonlinear Langevin equation

$$M\ddot{\Sigma} + M\Omega_{ren}^2 \Sigma + H(s;\Sigma) = \xi$$

The Langevin equation can be solved perturbatively, $\Sigma(s) = \Sigma^{(0)}(s) + \Sigma^{(1)}(s)$.

$$\begin{aligned}
\Sigma^{(0)}(s) &= \Sigma_h(s) + \int_0^t G_{ret}(s,s')\xi(s') \\
\Sigma^{(1)}(s) &= -\int_0^t G_{ret}(s,s')H^{(1)}(s',\Sigma^{(0)})
\end{aligned}$$

where $\Sigma_h(s)$ is the homogeneous part which depends on the initial conditions, and G_{ret} is the retarded Green's function

$$M\ddot{G}_{ret}(s,s') + M\Omega_{ren}^2 G_{ret}(s,s') + H^{(0)}(s, G_{ret}) = \delta(s-s')$$

The corresponding master equation for the reduced Wigner function W_r can also be derived.

To the lowest order we have

$$\begin{aligned}\frac{\partial W_r}{\partial t} &= -\frac{p}{M}\frac{\partial W_r}{\partial \Sigma} + M\Omega_{ren}^2 \Sigma \frac{\partial W_r}{\partial p} \\ & A(t)\frac{\partial (pW_r)}{\partial p} + B(t)\frac{\partial^2 W_r}{\partial \Sigma \partial p} + C(t)\frac{\partial^2 W_r}{\partial p^2}\end{aligned}$$

which is of the same form as for the linear coupling case.

To the next order the master equation becomes

$$
\begin{aligned}
\frac{\partial W_r}{\partial t} &= \cdots + \frac{\partial}{\partial p}(D(\Sigma, p, t)W_r) + \frac{\partial}{\partial \Sigma}(E(\Sigma, p, t)W_r) \\
&+ \frac{\partial^2}{\partial p^2}(E(\Sigma, p, t)W_r) + \frac{\partial^2}{\partial p \partial \Sigma}(F(\Sigma, p, t)W_r) \\
&+ G(t)\frac{\partial^3 W_r}{\partial p^3} + H(t)\frac{\partial^3 W_r}{\partial p^2 \partial \Sigma} + I(t)\frac{\partial^3 W_r}{\partial p \partial \Sigma^2}
\end{aligned}
$$

where ... represents the lowest order result.

VI. Conclusions

1. We have considered the quantum Brownian motion model with nonlinear coupling. Stochastic force with nongaussian probability density is obtained.
2. Both the dissipation kernel and the probability density for the stochastic force are history dependent.
3. The nonlinear Langevin equation and the corresponding master equation are derived.
4. We shall apply this idea to field theory. In particular, we would like to see how this nongaussian force works in the stochastic gravity setting.