格子上の非可換微分形式とそれを用いたフェルミオン

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1 Motivation

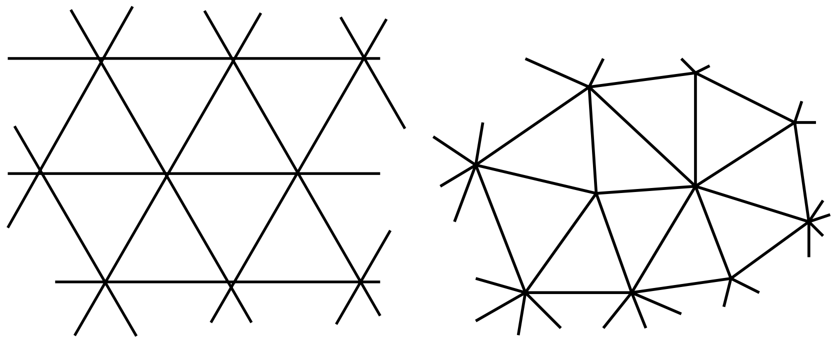
なぜ格子か

素粒子物理学の最終目標： 重力を含め全ての相互作用を統一的に記述する

候補
- 弦理論
- 格子を用いた記述

格子の特徴
- QCD、２次元重力で成功
- 発散が出ない
- 摂動展開によらずに定義できる
  (cf. Matrix Model)
- 微分形式と対応関係がある

例: ２次元空間の三角分割（格子重力）



平らな空間　　曲がった空間

本物の理論は正則化されて、かつ、非摂動的に記述されているはず (構成論的見方)

格子 (のような理論) で記述できなければ本物ではない!

連続 対称性

様々な対称性に基づいて様々な議論がなされている $\Longrightarrow$

- ローレンツ対称性
- ゲージ対称性
- カイラル対称性
- etc.

格子 対称性？

破れるもの (連続極限で回復)

- ローレンツ対称性
- SUSY(?)
- (ライプニッツ則)

残せるもの (破れると困るもの)

- ゲージ対称性
- カイラル対称性？(たぶん G-W で OK)

ゲージ対称性は微分のライプニッツ則を利用

例: $U(1)$ の場合 $\quad \psi \longrightarrow G\psi,\ A \longrightarrow A-(dG)G^{-1}$

$$\begin{aligned}(d+A)\psi &\longrightarrow (d+A-(dG)G^{-1})G\psi \\ &= d(G\psi)+AG\psi-(dG)\psi = G(d+A)\psi \\ &= (dG)\psi + G\,d\psi\end{aligned}$$

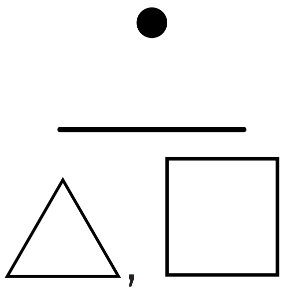
通常、格子ではリンク変数 $U_\mu(x) \longrightarrow G(x)U_\mu(x)G^{-1}(x+\hat{\mu})$ を利用:

$$\begin{aligned}& U_\mu(x+\hat{\mu})\psi(x+\hat{\mu})-\psi(x) \\ &\longrightarrow G(x)U_\mu(x+\hat{\mu})G^{-1}(x+\hat{\mu})G(x+\hat{\mu})\psi(x+\hat{\mu})+G(x+\hat{\mu})\psi(x+\hat{\mu}) \\ &= G(x)\left[U_\mu(x+\hat{\mu})\psi(x+\hat{\mu})-\psi(x)\right]\end{aligned}$$

格子に乗せる際のガイドラインとして非可換微分形式を利用

微分形式(form)

微分幾何学の道具 ——— 曲がった空間でも使える



点		$\Leftrightarrow$	0-form	1
線		$\Leftrightarrow$	1-form	dx^μ
面	,	$\Leftrightarrow$	2-form	$dx^\mu \wedge dx^\nu$
⋮			⋮	

$$\underbrace{\qquad\qquad\qquad\qquad}_{\boxed{\text{微分形式(form)}}}$$

(代数)

$$\boxed{\begin{aligned} dx^\mu \wedge dx^\nu &= -dx^\nu \wedge dx^\mu \\ df = (\partial_\mu f)dx^\mu &= dx^\mu(\partial_\mu f) \\ d^2 f &= 0 \end{aligned}}$$

Dirac-Kähler フェルミオンを記述する

- **微分形式で記述されている**
- staggered **フェルミオン(格子フェルミオンの一種)の連続極限**
- **曲がった空間への拡張?(重力)**
  D-K **フェルミオン** : **曲がった空間へも拡張可能(微分形式)**
  格子フェルミオン : **曲がった空間では(2次元を除けば)ほぼ白紙**
- SUSY **への拡張?**(spinor **と 反対称テンソルをつないでいる**)
  $\cdots$ Topological Filed Theory **の** (Twisted) SUSY **と類似** (**河本** & **月岡**)

2 NC Differential Form on a Lattice

(Dimakis et.al.('93), Aschieri et.al('93), Woronowics('89) etc.)

Leibniz's rule

$$\boxed{d(\psi^{(p)}\psi^{(q)}) = (d\psi^{(p)})\psi^{(q)} + (-)^p\psi^{(p)}\, d\psi^{(q)}}$$

is *not* satisfied if we identified the derivative as a difference

The derivative op.(with forward differences):

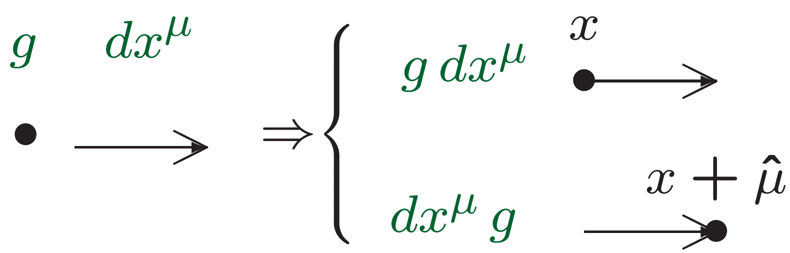
$$df(x) = \sum_\mu [f(x+\hat{\mu}) - f(x)]dx^\mu = \sum_\mu \partial_{+\mu} f(x) dx^\mu$$

$\Longrightarrow$

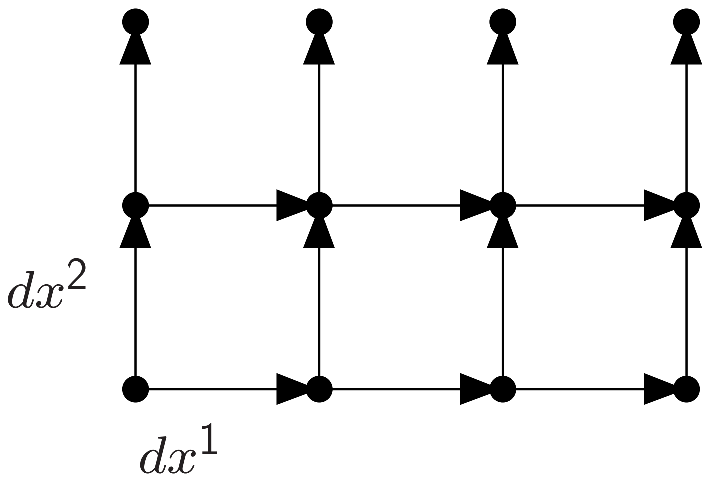
$$\begin{aligned} & d(f(x)g(x)) \\ & = [f(x+\hat{\mu})g(x+\hat{\mu}) - f(x)g(x)]dx^\mu \\ & = (\partial_{+\mu}f(x))g(x+\hat{\mu})\, dx^\mu + f(x)\partial_{+\mu}g(x)\, dx^\mu \end{aligned}$$

$$\boxed{dx^\mu g(x) = g(x+\hat{\mu})dx^\mu}$$ *Noncommutativity*

$$= (\partial_{+\mu}f(x)\, dx^\mu)g(x) + f(x)\partial_{+\mu}g(x)\, dx^\mu$$



NC cares the differences of a start site and an end site of the link



3 Dirac-Kähler fermion(cont.)

(Kähler '62, Becher & Joos '82)

$(d-\delta)^2 = \Box = (\not{\partial})^2 \Rightarrow \ d-\delta \sim \not{\partial}$

We formulate in terms of *Clifford product*

Clifford Product(Cup Product, $\vee$)

- $d-\delta = (\underbrace{dx^\mu \wedge + \frac{\partial}{\partial dx^\mu}})\partial_\mu \sim \gamma^\mu \partial_\mu$
  $$= dx^\mu \vee$$
  Clifford algebra:
  $$\{dx^\mu, dx^\nu\}_\vee = dx^\mu \vee dx^\nu + dx^\nu \vee dx^\mu = 2g^{\mu\nu}$$
  $$\sim \{\gamma^\mu, \gamma^\nu\}$$

- Generalization to higher forms(associative)
  $$(f\, dx^K)\vee(g\, dx^L)$$
  $$\equiv \sum_{p=0}^{D} \frac{1}{p!}(-)^{\frac{p(p-1)}{2}} \left(\eta^p \frac{\partial}{\partial dx^{\mu_1}} \cdots \frac{\partial}{\partial dx^{\mu_p}} f\, dx^K\right)$$
  $$\wedge \left(\frac{\partial}{\partial dx_{\mu_1}} \cdots \frac{\partial}{\partial dx_{\mu_p}} g\, dx^L\right)$$

  $dx^K \equiv dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}$

  η: sign factor, $\eta(p\text{-form}) = (-)^p(p\text{-form})$

Action

$d \vee = (dx^\mu \wedge + \frac{\partial}{\partial dx_\mu})\partial_\mu$ mixes degrees of forms
$\longrightarrow$ We need all degrees of forms.

For

$$\Psi = \varphi + \varphi_\mu \, dx^\mu + \varphi_{\mu\nu} \, dx^\mu \wedge dx^\nu + \cdots ,$$
$$\bar{\Psi} = \bar{\varphi} + \bar{\varphi}_\mu \, dx^\mu + \bar{\varphi}_{\mu\nu} \, dx^\mu \wedge dx^\nu + \cdots ,$$

the action is

$$\begin{aligned} S &= \int \left(\bar{\Psi} \vee (d+m) \vee \Psi \right) \Big|_{\text{0-form}} \; vol \qquad (vol = d^D x) \\ &= 2^{D/2} \int \sum_{(j)} \bar{\psi}_{(j)} (\gamma^\mu \partial_\mu + m) \psi^{(j)} dx^1 \cdots dx^D \\ &= \boxed{\text{(Dirac fermion)} \otimes \text{(flavor)}} \sim \text{K-S fermion} \end{aligned}$$

(j) : flavor $(1 \sim 2^{D/2})$

The components are:

$$\psi_\alpha{}^{(j)} = \left(\varphi \mathbb{1} + \varphi_\mu \gamma^\mu + \varphi_{\mu\nu} \gamma^{[\mu} \gamma^{\nu]} + \cdots \right)_\alpha{}^{(j)}$$

4 Clifford Product(Lattice)

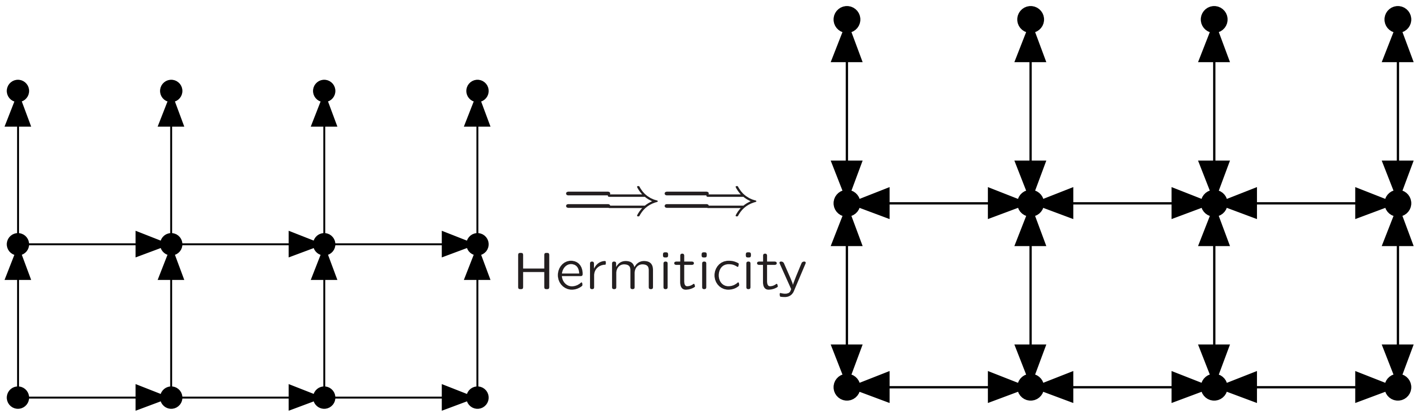
Hermiticity

- forward difference operators
  $\Longrightarrow$ both forward and backward difference op's.
- $dx^{\mu}(\rightarrow) \Longrightarrow \theta^{+\mu}(\rightarrow)$ and $\theta^{-\mu}(\leftarrow)$
  (link variables $U_{\mu}, U_{\mu}^{\dagger}$)

The exterior derivative op. on the lattice:

$$df = \sum_{\mu} \left\{ (\partial_{+\mu} f)\theta^{+\mu} - (\partial_{-\mu} f)\theta^{-\mu} \right\}$$

(Dimakis et.al '94, Dai et.al '01, etc.)



symmetric lattice

Clifford Product

In continuum space:(up to sign factor)

$$(f\,dx^K)\vee(g\,dx^L) = f\,dx^K \wedge g\,dx^L + (\frac{\partial}{\partial dx^\mu} f\,dx^K) \wedge (\frac{\partial}{\partial dx_\mu} g\,dx^L) + \frac{1}{2}(\frac{\partial}{\partial dx^\mu}\frac{\partial}{\partial dx^\nu} f\,dx^K) \wedge (\frac{\partial}{\partial dx_\mu}\frac{\partial}{\partial dx_\nu} g\,dx^L) + \cdots$$

$\downarrow$

Keep associative
Positivity of inner product

Clifford Product on a Lattice

$$f\theta^K \vee g\theta^L = f\theta^K \wedge g\theta^L - (\eta \frac{\partial}{\partial \theta^{-\mu}} T_{-\mu} f\theta^K) \wedge (\frac{\partial}{\partial \theta^{+\mu}} g\theta^L) + \cdots$$

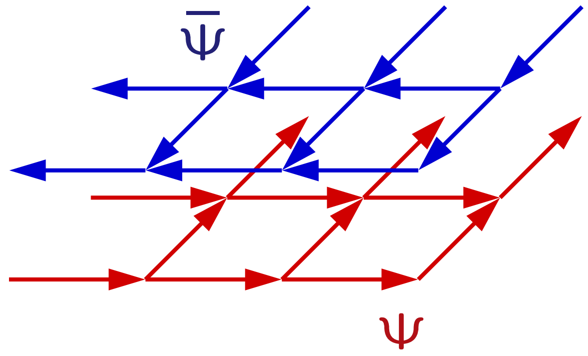
$$= \sum_{p=0}^{D} (-)^{\frac{p(p-1)}{2}} (-)^p \sum_{\{\mu_i\}} \frac{1}{p!} \left\{ \eta^p \left(\prod_{i=1}^{p} \frac{\partial}{\partial \theta^{-\mu_i}} T_{-\mu_i} \right) f\theta^K \right\} \wedge \left\{ \prod_{i=1}^{p} \frac{\partial}{\partial \theta^{+\mu_i}} g\theta^L \right\}$$

Shift op. $T_{\pm\mu} f(x) = f(x \pm \hat{\mu})$

5 Dirac-Kähler $\to$ staggered

Expand Ψ and $\bar{\Psi}$ with $dx^\mu = \theta^{+\mu} - \theta^{-\mu}$:

$$\Psi = \varphi + \varphi_\mu + \varphi_{\mu\nu}\, dx^\mu dx^\nu + \cdots$$
$$\bar{\Psi} = \bar{\varphi} + dx^\mu \bar{\varphi}_\mu + dx^\mu\, dx^\nu \bar{\varphi}_{\mu\nu} + \cdots \simeq \Psi^\dagger$$



$$\theta^{+\mu} \vee \theta^{-\mu} = \theta^{+\mu}\theta^{-\mu} \Rightarrow 0$$
$$\theta^{-\mu} \vee \theta^{+\mu} = \theta^{-\mu}\theta^{+\mu} - 1 \Rightarrow -1$$

Action

$$\int \bar{\Psi} \vee (d+m) \vee \Psi \Big|_{\text{0-form}} vol$$
$$= 2^{D/2} \sum_x \Big[\frac{1}{2} \sum_\mu \Big\{ \bar{\psi}_{(j)}(x)\gamma_\mu \Big(\psi(x+\mu) - \psi(x-\mu)\Big)^{(j)}$$
$$+ \bar{\psi}_{(j)}(x)\gamma_5^\dagger \Big(\psi(x+\mu) + \psi(x-\mu) - 2\psi(x)\Big)^{(l)} (\gamma_5\gamma_\mu)_{(l)}{}^{(j)} \Big\}$$
$$+ m\bar{\psi}_{(j)}(x)\, \psi^{(j)}(x) \Big]$$

$$= \sum_y \Big[\frac{1}{2} \sum_\mu \eta_\mu(y) \Big\{ \bar{\chi}(y)\chi(y+\hat{\mu}) - \bar{\chi}(y+\hat{\mu})\chi(y) \Big\} + m\bar{\chi}(y)\chi(y) \Big]$$

with $x = 2y$ (Gliozzi '82, Kluberg-Stern et.al '83)

This is the staggered fermion action written in spinors, exactly.

Dirac-Kähler fermion $\to$ staggered fermion !!!

form の数の減らし方

$\theta^{+\mu}$ と $\theta^{-\mu}$ で通常より form の数が多い(1-from が $2D$ 個、全体で 2^{2D} 個の form)

1. $\boxed{\text{Cliffor 積の定義で減らす} \rightarrow \text{fermion は小細工なし}}$
2. Clifford 積では減らさない $\rightarrow$ fermion の定義で減らす

どちらも staggered フェルミオンを与える

$$
\begin{aligned}
df &= \sum_\mu \left[\partial_{+\mu} f \theta^{+\mu} - \partial_{-\mu} f \theta^{-\mu}\right] \\
&= \sum_\mu \left[\frac{1}{2}(\partial_{+\mu} + \partial_{-\mu}) f \underbrace{(\theta^{+\mu} - \theta^{-\mu})}_{= dx^\mu} + \frac{1}{2}(\partial_{+\mu} - \partial_{-\mu}) f \underbrace{(\theta^{+\mu} + \theta^{-\mu})}_{\equiv \tau^\mu}\right] \\
&\rightarrow \partial_\mu f\, dx^\mu + a \partial_\mu \partial^\mu f\, \tau^\mu \qquad (a \rightarrow 0)
\end{aligned}
$$

6 Path Integral

$$\Psi = \sum_{x} \sum_{\alpha,(j)} \psi_\alpha{}^{(j)}(x)\ \mathrm{e}^x Z^\alpha{}_{(j)}$$

$$\uparrow \qquad\qquad \uparrow$$

可換　　**非可換**

成分 $\psi_\alpha{}^{(j)}(x)$ **の積分として経路積分を定義すれば良いだろう**

$$Z^\alpha{}_{(j)} \equiv \sum_{p=0}^{D} \frac{1}{p!} \left(\gamma_{\mu_1}{}^T \ldots \gamma_{\mu_p}{}^T\right)^i{}_{(j)}\ dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

$dx^\mu \vee Z^\alpha{}_{(j)} = (\gamma_\mu^T Z)^\alpha{}_{(j)}$: $\vee$-**積を** γ **行列の積で表現する基底**

e^x: **格子点** x **で定義された** δ-**関数**

$$\mathrm{e}^x(y) = \delta^{x,y} \qquad \sum_x f(x)\mathrm{e}^x(y) \sim \int dx f(x)\delta(x-y) = f(y)$$

- Feynman rule
- propagator **は** staggered **フェルミオンと一致**

計算例

$$
\begin{aligned}
&\left\langle 2^{-\frac{D}{2}} \int \bar{\Psi} \vee A \vee \Psi \Big|_{\text{0-form}} vol\ 2^{-\frac{D}{2}} \int \bar{\Psi} \vee B \vee \Psi \Big|_{\text{0-form}} vol \right\rangle \\
&= -\,\mathrm{Tr}\left[A \vee \frac{1}{d+m} \vee B \vee \frac{1}{d+m} \right] \\
&= -2^{\frac{D}{2}} \int \sum_{x} Z^{(j)}{}_{\alpha}\, \mathrm{e}^{x} \vee A \vee \frac{1}{d+m} \vee B \vee \frac{1}{d+m} \vee \mathrm{e}^{x} Z^{\alpha}{}_{(j)} \Big|_{\text{0-form}}
\end{aligned}
$$

- 途中の計算を非可換微分形式でまとめたまま実行できる
- 計算結果も非可換微分形式でまとめて書けると期待

7 Yang-Mills Action

Yang-Mills action(cont.)

$\Downarrow$ NC forms

Plaquette action(lat.)

(Dimakis et.al '94)

Leibniz's rule $\rightarrow d + A$: covariant derivative

$$A = \sum_\mu [A_\mu \theta^{+\mu} + A_{-\mu}\theta^{-\mu}]$$

$$S = -\frac{1}{2g^2}\,\mathrm{tr}\int F \vee F\Big|_{\text{0-form}} vol$$
$$= \frac{1}{2g^2}\sum_x \sum_{\mu,\nu} \mathrm{tr}\left\{1 - U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x)\right\}$$

$$F = dA + A \wedge A = U^2$$
$$U(x) = \sum_\mu [U_\mu(x)\theta^{+\mu} + U_\mu^\dagger(x-\hat{\mu})\theta^{-\mu}]$$

8 Conclusion

Summary

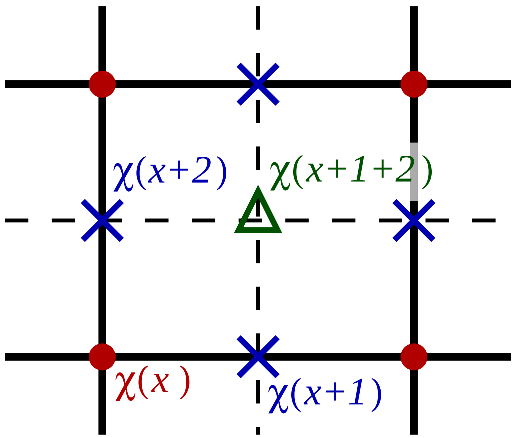
- Defined $\vee$ product on a lattice
- Dirac-Kähler fermion $\longrightarrow$ staggered fermion
- Path integral

Future work

- Differential forms ··· Lattice fermion with gravity?
- Leibniz's rule··· SUSY on a lattice?
- G-W relation, Chiral anomaly

Comment staggerd フェルミオンと Dirac-Kähler フェルミオンとの対応

（従来の見方）



staggered フェルミオンは小さい格子の格子点に乗っている。2倍の大きさの格子で見たときに

点 0-form : $\varphi(x) = \chi(x)$

線 1-form : $\varphi_1(x) = \chi(x+1)$

: $\varphi_2(x) = \chi(x+2)$

面 2-form : $\varphi_{12}(x) = \chi(x+1+2)$

と見なすことに相当（連続極限）

非可換微分形式でも、同じ見方ができる（有限の格子間隔で ）