Chiral Spin Matter

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DISCLAIMER

- **Lecture is limited to an elementary level.**
- **Textbook knowledge is *only partially* needed.**
  - □ Quantization / Gauge Theories / Chiral Anomaly
- **Not much "Knowns" but much "unknowns"**
  - □ Knowns can be all found in published papers.
  - □ If unknowns are solved, new works can be published!
- **Expressions (coefficients/signs) might have typos.**
  - □ Matter of convention…

Feedbacks are always welcome (taken into a forthcoming textbook)!

Symmetry

Noether Current

$$Z \sim \int \mathcal{D}\phi \, e^{iS[\phi]}$$

Theory has a symmetry under $\phi \to \phi' + \epsilon\delta\phi$ if Z unchanged.

$$S[\phi'] = S[\phi] + \int dx \, \epsilon(x)\partial_\mu j^\mu(x) \quad \text{and} \quad \mathcal{D}\phi = \mathcal{D}\phi'$$

$$\int \mathcal{D}\phi' \, e^{iS[\phi']} - \int \mathcal{D}\phi \, e^{iS[\phi]}$$

$$\simeq \int \mathcal{D}\phi \, e^{iS[\phi]} \, i \int dx \, \epsilon(x)\partial_\mu j^\mu(x) = 0$$

➡ **Conserved Current**

$$\langle \partial_\mu j^\mu(x) \rangle = 0$$

Symmetry

Example

Translational symmetry: $x^\mu \to x'^\mu - \epsilon^\mu$

$$\phi(x) \;\to\; \phi'(x) + \epsilon^\mu \partial_\mu \phi(x)$$

$$S[\phi'] - S[\phi] = \int dx\, \mathcal{L}[\phi'(x)] - \int dx\, \mathcal{L}[\phi(x)]$$

$$\simeq \int dx \left\{ -\frac{\partial \mathcal{L}}{\partial \phi} \epsilon^\nu \partial_\nu \phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu [\epsilon^\nu \partial_\nu \phi] \right\}$$

$$= \int dx \left[-\epsilon^\nu \partial_\nu \mathcal{L} - (\partial_\mu \epsilon^\nu) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi \right]$$

$$= \int dx\, \epsilon^\nu \partial_\mu \left[\underline{-\delta^\mu_\nu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi} \right]$$

Canonical Energy-Momentum Tensor

Symmetry

Fermionic Action in the presence of *A*

$$Z[A] \sim \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS[\bar{\psi},\psi,A]} \qquad \mathcal{L} = \bar{\psi}[i\gamma^\mu(\partial_\mu + ieA_\mu) - m]\psi$$

U(1) symmetry

$$\psi \to e^{i\epsilon}\psi' \simeq \psi' + i\epsilon\psi$$

$\mathcal{L}$ is invariant under this transf.

$$S[\bar{\psi}',\psi'] - S[\bar{\psi},\psi] \simeq \int dx \left[i\epsilon\bar{\psi}\frac{\partial \mathcal{L}}{\partial \bar{\psi}} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\partial_\mu(-i\epsilon\psi) \right]$$

$$= \int dx\, i\epsilon\, \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\psi \Big]$$

Number conservation $\boxed{\langle \partial_\mu j^\mu \rangle = 0 \qquad j^\mu = \bar{\psi}\gamma^\mu\psi}$

Symmetry

Fermionic Action in the presence of A

$$Z[A] \sim \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS[\bar{\psi},\psi,A]} \qquad \mathcal{L} = \bar{\psi}[i\gamma^\mu(\partial_\mu + ieA_\mu) - m]\psi$$

$U(1)_A$ symmetry

Explicit Breaking

$$\psi \to e^{i\epsilon\gamma_5}\psi' \simeq \psi' + i\epsilon\gamma_5\psi \qquad \mathcal{L} \to \mathcal{L} - 2i\epsilon m\bar{\psi}\gamma_5\psi$$

$$S[\bar{\psi}',\psi'] - S[\bar{\psi},\psi] \simeq \int dx \left[-i\epsilon\bar{\psi}\gamma_5\frac{\partial\mathcal{L}}{\partial\bar{\psi}} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\mu(-i\epsilon\gamma_5\psi)\right]$$

$$= \int \left[dx\, i\epsilon\,\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\gamma_5\psi\right]$$

Chirality conservation $\boxed{\langle\partial_\mu j^\mu_{\mathrm{A}}\rangle = 2m\bar{\psi}i\gamma_5\psi \quad j^\mu_{\mathrm{A}} = \bar{\psi}\gamma^\mu\gamma_5\psi}$

Chiral Anomaly

Fermionic Action in the presence of *A*

$$Z[A] \sim \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{iS[\bar{\psi},\psi,A]} \qquad \mathcal{L} = \bar{\psi}[i\gamma^\mu(\partial_\mu + ieA_\mu) - m]\psi$$

***U*(1)A symmetry**

Explicit Breaking

$$\psi \to e^{i\epsilon\gamma_5}\psi' \simeq \psi' + i\epsilon\gamma_5\psi \qquad \mathcal{L} \to \mathcal{L} - 2i\epsilon m\bar{\psi}\gamma_5\psi$$

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \to \underline{e^{i\delta\Gamma[A]}}\mathcal{D}\bar{\psi}'\mathcal{D}\psi' \qquad \delta\Gamma[A] = -\int dx\, \epsilon a[A]$$

Anomaly

Chiral Anomaly $\boxed{\langle \partial_\mu j^\mu_{\rm A} \rangle = a + 2m\bar{\psi}i\gamma_5\psi}$

Anomaly is "topological" $\displaystyle\int dx\, a = (\text{integer}) = Q_W$

Chiral Anomaly

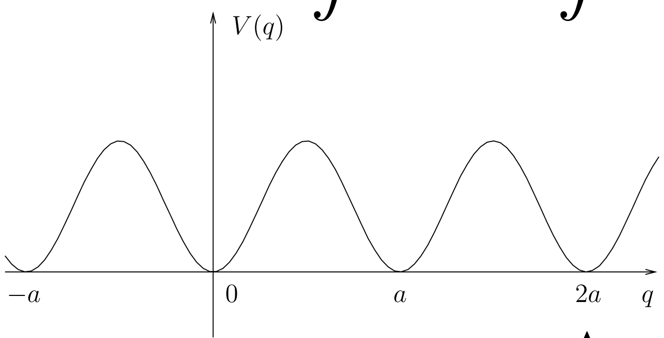
Winding number and Chern-Simons current

$$Q_W[A] = \int d^dx\, \partial_\mu K^\mu$$

Winding number is unchanged under continuous *A* changes (changed under large *A* transf.)

To make the energy finite, *A* should go to a pure gauge at infinite distance, i.e., *F* ~ 0.
In some gauges, spatial surface terms are dropped.

$$Q_W[A] = \int d\tau\, \partial_\tau \int d^{d-1}x\, K^0 = \nu(\tau = +\infty) - \nu(\tau = -\infty)$$



See: Classical Theory of Gauge Fields by V. Rubakov

Chiral Anomaly

Topological susceptibility and the θ angle

If the theory has massless fermions, we can choose axially rotated fields without affecting the classical action.

$$\psi \to e^{i\epsilon\gamma_5}\psi' \simeq \psi' + i\epsilon\gamma_5\psi$$

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \to e^{i\delta\Gamma[A]}\mathcal{D}\bar{\psi}'\mathcal{D}\psi'$$

The theory has an ambiguity of adding $\delta\Gamma \sim \varepsilon Q_W$

If the theory had an additional term:

$$Z[A;\theta] \sim \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS+i\theta Q_W} \quad \Rightarrow \quad \partial_\theta^n Z[A;\theta]\Big|_{\theta=0} = 0$$

Topological susceptibility is zero: $\chi_Q \sim \langle Q_W^2 \rangle \sim \partial_\theta^2 \ln Z$

Chiral Anomaly

For massless fermions

$$\partial_\mu j^\mu_{\mathrm{A}} = \partial_\mu K^\mu$$

It is said that $U(1)_{\mathrm{A}}$ is "explicitly" broken, and η' meson is not a Nambu-Goldsone boson anymore…

$$\partial_\mu \mathcal{J}^\mu = \partial_\mu (j^\mu_{\mathrm{A}} - K^\mu) = 0$$

Conserved current???

If so, another (unobserved) massless boson cannot be avoided?

Let's see a more concrete form of the CS current in the Abelian and the non-Abelian theories.

CS Current

Triangle (ABJ) Anomalies

QED:

$$\partial_\mu j^\mu_{\mathrm{A}} = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \sim \boldsymbol{E}\cdot\boldsymbol{B}$$

$$K^\mu = -\frac{e^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}A_\nu\partial_\alpha A_\beta = -\frac{e^2}{8\pi^2}\epsilon^{\mu\nu\alpha\beta}A_\nu F_{\alpha\beta}$$

QCD:

$$\partial_\mu j^{\mu\,a}_{\mathrm{A}} = -\frac{g^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{tr}\,F_{\mu\nu}F_{\alpha\beta}\,\mathrm{tr}t^a$$

Usually only the flavor-singlet is anomalous.

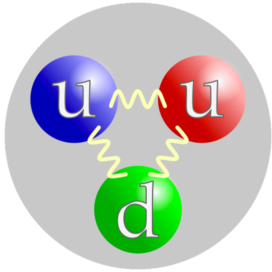
$$K^\mu = -\frac{e^2 N_f}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{tr}A_\nu\left(F_{\alpha\beta} - \frac{2}{3}A_\alpha A_\beta\right)$$

CS currents are NOT gauge invariant … unphysical?

CS Current

Proton Spin Crisis

Proton Spin = 1/2



Each quark has 1/2, and three constitute 1/2

若松さんの1990年の物理学会誌の記事
「陽子のスピンはどこから？」

$$\Delta u = 0.74 \pm 0.08$$

$$\Delta d = -0.51 \pm 0.08$$

$$\Delta s = -0.23 \pm 0.08$$

$$\Rightarrow \quad \Sigma = \Delta u + \Delta d + \Delta s = 0.00 \pm 0.24$$

Quark spins sum up to zero!?

These are *old* values, but qualitatively similar today

CS Current

Proton Spin Crisis

若松さんの1990年の物理学会誌の記事
「陽子のスピンはどこから？」

ラル極限で保存するのに対して，第三項のフレーバー1次元表現の擬ベクトル流（$U_A(1)$ カレント）の保存は QCD のいわゆる三角アノマリーのために破れることが知られている．このことを考慮すると (8) 式の第三項は次のように置き換えるべきであることがいくつかのグループによって指摘された.[8, 9)]

$$\Delta u+\Delta d+\Delta s \to \Delta u+\Delta d+\Delta s-N_f\frac{\alpha_s}{2\pi}\Delta g \qquad (9)$$

Badly criticized by Jaffe-Manohar (1990)

[8] Altarelli-Ross (1988)
[9] Carlitz-Collins-Mueller (1988)

CS Current

Axial Current and Spin

Axial Current is nothing but the spin of fermions.

Rotation: $x^\mu \to (\delta^\mu_\nu + \epsilon^\mu_\nu)x^\nu$

Noether current: $J^{\lambda\mu\nu} = \dfrac{\partial\mathcal{L}}{\partial(\partial_\lambda\psi)}(x^\mu\partial^\nu - x^\nu\partial^\mu - i\Sigma^{\mu\nu})\psi$

$$= \bar{\psi}i\gamma^\lambda(x^\mu\partial^\nu - x^\nu\partial^\mu - i\Sigma^{\mu\nu})\psi$$

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$J^{0\mu\nu}$ is the conserved charge (angular momentum)

$S^{\mu\nu} = \bar{\psi}\gamma^0\Sigma^{\mu\nu}\psi$ represents the spin

$$\boxed{S^i = \frac{1}{2}\epsilon^{ijk}S^{jk} = \frac{1}{2}\bar{\psi}\gamma^i\gamma_5\psi = \frac{1}{2}j^i_{\mathrm{A}}}$$

CS Current

Axial Current and Spin

$$\Sigma(Q^2)s^\mu = \langle N(p,s)|j^\mu_{\mathrm{A}}|\underline{N(p,s)\rangle}_{Q^2}$$

One proton state with momentum *p* and spin *s*

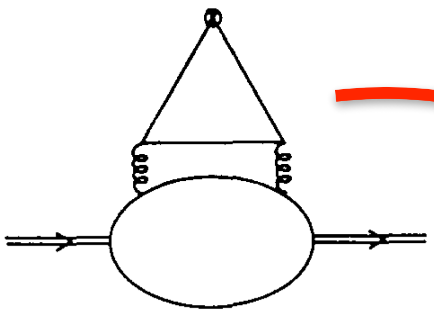


Fig. 3. Coupling of A^0_μ to the proton through the triangle diagram.

[Jaffe-Manohar (1990)]

one might *erroneously* arrive at

$$\Sigma(Q^2) = \underline{\tilde{\Sigma}(Q^2)} - \underline{N_f \frac{\alpha_s(Q^2)}{2\pi}\Delta g(Q^2)}$$

Conclusion made in [8, 9]

Naive value from the quark model?

Gluon contribution???

CS Current

Axial Current and Spin

$$\langle N(p', s)|\bar{\psi}\gamma^\mu\gamma_5\psi|N(p, s)\rangle = \Sigma(t)s^\mu + (l\cdot s)l^\mu h(t)$$

$$l = p' - p, \quad t = l^2$$

$$\langle N(p', s)|N_f\frac{\alpha_s}{2\pi}\mathrm{tr}F\tilde{F}|N(p, s)\rangle = i(l\cdot s)\,\kappa(t)$$

"Accidentally" similar form…

$$\boxed{\Sigma(t) + t\,h(t) = \kappa(t)}$$

$$t \to 0$$

$$-N_f\frac{\alpha_s}{2\pi}\Delta g$$

[Jaffe-Manohar (1990)]

No distinction between Σ and the matrix elements of the anomaly.

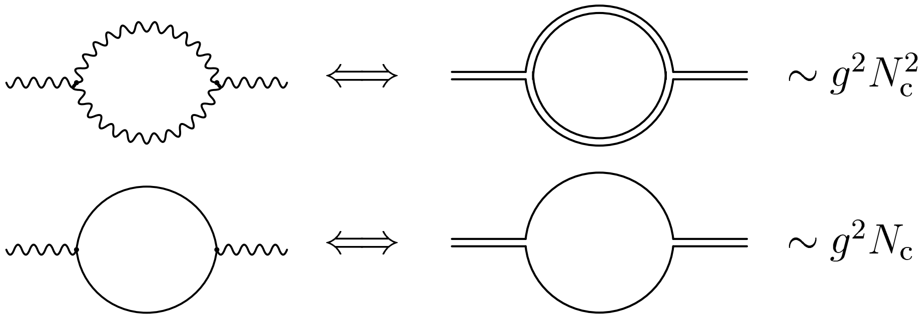
Residue of "massless" η' propagation???

Instantons make η' massive, so this is zero!

A side story about η'

Would-be NG boson killed by anomaly

In the large-*N* limit:

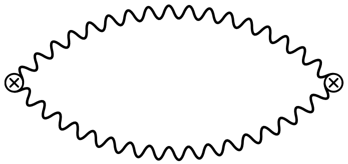


Quark loops are suppressed.

How can Σ and κ be balanced in the large-*N* limit?

More specifically, in a theory with massless fermions the vacuum expectation of the anomaly is always zero (θ derivatives are all zero). How can it be true?

Topological susceptibility: $\int dx\, \langle q_W(x) q_W(0) \rangle$

$\chi_{\rm top}^{\rm (pure)} \sim$  $\sim N_c^2 \quad \Longleftrightarrow \quad \chi_{\rm top}^{\rm (full)} = 0$

Cancellation between the gluon and quark loops!?

A side story about η'

Would-be NG boson killed by anomaly

In the large-*N* limit:

Pure topological susceptibility

$$\chi_Q(k^2) = \sum_{\text{ghosts}} \frac{N_c^2 a_n^2}{k^2 - M_n^2} + \sum_{\text{mesons}} \frac{N_c c_n^2}{k^2 - m_\eta^2}$$

This can be zero at $k = 0$ if $m_{\eta'}^2 \sim 1/N$

$$\sqrt{N_c}\, c_\eta = \langle 0|Q_W|\eta\rangle = \frac{1}{2N_f}\langle 0|\partial_\mu j_A^\mu|\eta\rangle = \frac{1}{2N_f}\sqrt{2N_f}\, m_\eta^2 f_\eta$$

PCAC

$$\boxed{m_\eta^2 = \frac{2N_f}{f_\eta^2}\chi_Q^{(\text{pure})}}$$

Witten (1979)

$$\chi_Q^{(\text{pure})} \sim \langle Q_W^2\rangle \sim \lim_{q\to 0} q_\mu q_\nu \langle j_A^\mu j_A^\nu\rangle \neq 0$$

Massless ghosts hidden here…

Massless (Venetizano) ghosts make
η′ massive (massless mode unobservable).

CS Current

Questions raised by Jaffe-Manohar

(iii) What is the relation between the gluon spin and the anomalous current

$$K_\sigma = \epsilon_{\sigma\alpha\beta\gamma} \,\mathrm{Tr}\, A^\alpha \big(F^{\beta\gamma} - \tfrac{2}{3} A^\beta A^\gamma \big) . \qquad (6.1)$$

We apologize to readers to whom these issues appear elementary and/or well known. We have found some confusion surrounding each of them in recent work on g_1.

I also apologize to the audience to whom these issues appear elementary and/or well known.

Gauge Inv. — Integrated Rels.

Integrated relations

In some gauges, spatial surface terms are dropped.

$$\int_{\tau_1}^{\tau_2} d\tau \partial_\tau \int d^{d-1}x\, K^0 = \nu(\tau_2) - \nu(\tau_1)$$

$$\int_{\tau_1}^{\tau_2} d\tau \partial_\tau \int d^{d-1}x\, j_{\mathrm{A}}^0 = N_5(\tau_2) - N_5(\tau_1)$$

$N_5(\tau) - \nu(\tau)$ is a conserved charge.

These are of course all gauge invariant!

Chirality charge:

$$N_5 = \int d^{d-1}x\, j_{\mathrm{A}}^0$$

Chern-Simons charge:

$$\nu = \int d^{d-1}x\, K^0$$

Gauge Inv. — Integrated Rels.

Chern-Simons charge: $$\nu = \int d^{d-1}x\, K^0$$ **Gauge inv. observable!**

In the case of the *U*(1) theory:

$$\boxed{K^0 = -\frac{e^2}{8\pi^2}\epsilon^{ijk}A_i F_{jk} = \frac{e^2}{4\pi^2}\boldsymbol{A}\cdot\boldsymbol{B}}$$

This itself is NOT gauge invariant, but the integral IS!

Magnetic Helicity $$\nu = \frac{e^2}{4\pi^2}\int d^3x\, \boldsymbol{A}\cdot\boldsymbol{B}$$

Chiral anomaly dictates a new conservation law of the chiral charge and the magnetic helicity!

N. Yamamoto

Gauge Inv. — External A

***A*'s are just background fields, not necessarily dynamical?**

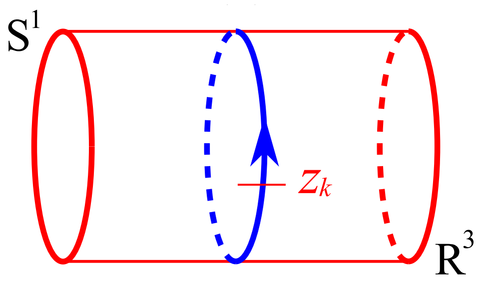
In the case of the $U(1)$ theory:

$$K^\mu = -\frac{e^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}A_\nu\partial_\alpha A_\beta = -\frac{e^2}{8\pi^2}\epsilon^{\mu\nu\alpha\beta}\boxed{A_\nu}\underline{F_{\alpha\beta}}$$

Gauge Inv.

If replaced by external "something" gauge invariance is okay!

If the theory is S^1 compactified along the Euclidean time, A_0 would become gauge invariant!



A_0 cannot be gauged away in finite-T field theory.

Gauge Inv. — External A

***A*'s are just background fields, not necessarily dynamical?**

A_0 cannot be gauged away in finite-T field theory.

$$d\omega = A_0$$

$$B^1(M) = \{d\omega | \omega \in \Omega^0(M)\}$$
$$Z^1(M) = \{\omega \in \Omega^1(M) | d\omega = 0\}$$
$$H^1_{\mathrm{dR}}(M) = Z^1(M)/B^1(M)$$

$$H^1_{\mathrm{dR}}(\mathbb{R}) = 0$$

$$H^1_{\mathrm{dR}}(S^1) \simeq \mathbb{R}$$

This d.o.f. is the gauge invariant A_0 (zero mode)

Gauge Inv. — External A

A's are just background fields, not necessarily dynamical?

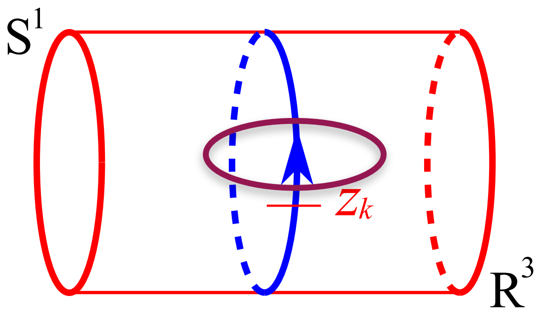
A_0 cannot be gauged away in finite-T field theory.

In the case of the $U(1)$ theory:

In the physical term it is nothing but a *chemical potential*.

In the case of the $SU(N)$ theory:

This corresponds to the *Polyakov loop*.



Z_N (center) symmetry

1-form symmetry (spatial 0-form sym.)

Gauge Inv. — External A

***A*'s are just background fields, not necessarily dynamical?**

$$K^\mu = -\frac{e^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}A_\nu\partial_\alpha A_\beta = -\frac{e^2}{8\pi^2}\epsilon^{\mu\nu\alpha\beta}\boxed{A_\nu}\underline{F_{\alpha\beta}}$$

Gauge Inv.

***A_0* is provided by the chemical potential.**

$$K^i = \frac{e^2}{2\pi^2}\bar{A}_0 B^i \qquad \boldsymbol{\nabla}\cdot(\boldsymbol{j}_\mathrm{A} - \boldsymbol{K}) = 0$$

$$\boxed{\boldsymbol{j}_\mathrm{A} = \frac{e^2}{2\pi^2}\mu\boldsymbol{B}} + \boldsymbol{\nabla}\times\boldsymbol{C}$$

Chiral Separation Effect (CSE)

Spin expectation is induced by a finite density + *B*

Gauge Inv. — External A

If a vector current is wanted, the same argument leads to:

$$\boxed{\boldsymbol{j} = \frac{e^2}{2\pi^2}\mu_{\rm A}\boldsymbol{B}} + \boldsymbol{\nabla}\times\boldsymbol{C}$$

Chiral Magnetic Effect (CME)

$\mu_{\rm A}$ is a chiral chemical potential coupled with γ^5

$$\sim \mu_{\rm A}\bar{\psi}\gamma^0\gamma_5\psi$$

$$\sim (\bar{\psi}e^{-i\gamma_5\mu_{\rm A}t})\gamma^0 i\partial_0(e^{-i\gamma_5\underbrace{\mu_{\rm A}t}_{\theta(x)}}\psi)$$

$U(1)_{\rm A}$ transf.

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \to e^{i\delta\Gamma[A]}\mathcal{D}\bar{\psi}'\mathcal{D}\psi'$$

Chiral chemical potential = *t*-dep. *θ* angle!

Gauge Inv. — External A

Equation of motion read from axion QED:

$$\mathcal{L}_{\mathrm{CS-Maxwell}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \theta(x)\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$$

The θ term would not affect the EOM if θ is constant but…

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho + \underbrace{\frac{e^2}{2\pi^2}(\boldsymbol{\nabla}\theta)\cdot\boldsymbol{B}}_{\text{Induced charge}}$$

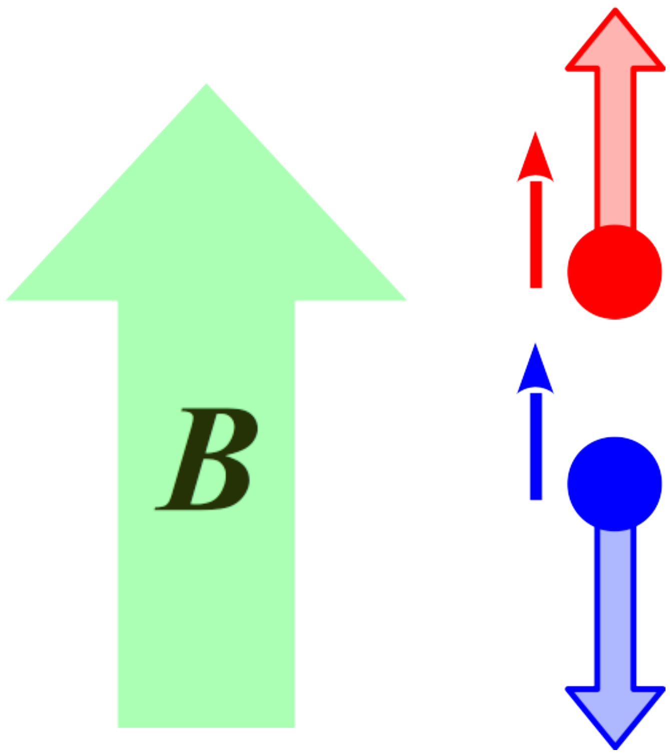
$\boldsymbol{\nabla}\times\boldsymbol{C}$ if static

$$\boldsymbol{\nabla}\times\boldsymbol{B} - \partial_0\boldsymbol{E} = \boldsymbol{j} + \underbrace{\frac{e^2}{2\pi^2}\left[(\partial_0\theta)\boldsymbol{B} - (\boldsymbol{\nabla}\theta)\times\boldsymbol{E}\right]}_{\text{Induced currents}}$$

Chiral Magnetic Effect

Intuitive Picture

"*The Chiral Magnetic Effect*" by Fukushima, Kharzeev, Warringa (2008)



Right-handed particles
Momentum parallel to Spin

Left-handed particles
Momentum anti-parallel to Spin

Rotation Induced Current

Chiral Anomaly on Geometry

$$\boxed{\nabla_\mu j^\mu_{\mathrm{A}} = \frac{1}{384\pi^2}\epsilon^{\mu\nu\rho\lambda}R_{\mu\nu}{}^{\alpha\beta}R_{\rho\lambda\alpha\beta}}$$

For details, see a textbook by Parker-Toms.

Chern-Simons current

$$\boxed{K^\mu = \frac{1}{96\pi^2}\epsilon^{\mu\nu\rho\lambda}\Gamma^\alpha{}_{\nu\beta}\left(\partial_\rho\Gamma^\beta{}_{\alpha\lambda} + \frac{2}{3}\Gamma^\beta{}_{\rho\sigma}\Gamma^\sigma{}_{\alpha\lambda}\right)}$$

Not gauge invariant — not written in terms of tensors only.

Is there any "something" that makes it gauge invariant?
Honestly, I do not fully understand this point yet…

Rotation Induced Current

Rotation (along the *z* axis) $\Gamma \to \Gamma + \delta\Gamma$

$$\delta\Gamma^x{}_{0y} = -\delta\Gamma^y{}_{0x} = \omega$$ **Rotation angular velocity**

Expand the CS current in terms of *ω* and the first order gives:

$$K^z = \frac{\omega}{48\pi^2}\left(R^0{}_{x0x} + R^0{}_{y0y} - R^x{}_{yxy} - R^y{}_{xyx}\right)$$

If the theory is put in a finite-*T* and constant curvature space, this expression is simplified as

$$K^z = -\frac{\omega}{96\pi^2}R \xrightarrow{\text{Direct calculation}} j^z_{\mathrm{A}} = \left(\frac{T^2}{12} - \frac{R}{96\pi^2}\right)\omega$$

Chiral Vortical Effect

Rotation Induced Current

Chiral Vortical Effect from Anomalous Hydrodynamics

Son-Suruwka (2009)

Relativistic (covariant) vorticity: $\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}u_\nu\partial_\lambda u_\rho$

Current: $j^\mu = nu^\mu - \sigma T(g^{\mu\nu} + u^\mu u^\nu)\partial_\nu\left(\frac{\mu}{T}\right) + \boxed{\xi\omega^\mu}$

Conclusion:

$$\boxed{\xi = C\left(\mu^2 - \frac{2}{3}\frac{\mu^3 n}{e+p}\right)} \qquad \partial_\mu j^\mu = -\frac{1}{8}C\,\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$$

Strategy: Inserting ***E*** and ***B*** and “define” the entropy current s^μ and require $\partial_\mu s^\mu \geq 0$

Rotation Induced Current

Chiral Vortical Effect from Anomalous Hydrodynamics

Hydrodynamic Equations

= Conservation Laws in terms of Hydro Variables

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} j_\lambda \,, \quad \partial_\mu j^\mu = C E^\mu B_\mu$$

$$E^\mu = F^{\mu\nu} u_\nu \quad B^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}$$

$$T^{\mu\nu} = \underline{(e+p)u^\mu u^\nu - p g^{\mu\nu}} + \pi^{\mu\nu}$$

$$\sim \mathcal{O}(\partial)$$

$$j^\mu = \underline{n u^\mu} + \nu^\mu$$

Ideal Hydro

Dissipative terms

→ Entropy production

Rotation Induced Current

Chiral Vortical Effect from Anomalous Hydrodynamics

Hydrodynamic Equations
= Conservation Laws in terms of Hydro Variables

$$T^{\mu\nu} = (e+p)u^\mu u^\nu - pg^{\mu\nu} + \pi^{\mu\nu}$$

$$j^\mu = nu^\mu + \nu^\mu$$

$$\nu^\mu = -\sigma T(g^{\mu\nu} - u^\mu u^\nu)\partial_\nu\left(\frac{\mu}{T}\right) + \sigma E^\mu + \xi\omega^\mu + \xi_B B^\mu$$

$$s^\mu = su^\mu - \frac{\mu}{T}\nu^\mu + D\omega^\mu + D_B B^\mu$$

$$\partial_\mu s^\mu = (\dots)^2 + (\dots)^2 + \dots \rightarrow$$

$\xi,\ \xi_B,\ D,\ D_B$
All solved!

Coming back to...

Questions raised by Jaffe-Manohar

(iii) What is the relation between the gluon spin and the anomalous current

$$K_\sigma = \epsilon_{\sigma\alpha\beta\gamma} \operatorname{Tr} A^\alpha \left(F^{\beta\gamma} - \tfrac{2}{3} A^\beta A^\gamma \right). \tag{6.1}$$

$$\begin{aligned} J^{\lambda\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \psi)} (x^\mu \partial^\nu - x^\nu \partial^\mu - i\Sigma^{\mu\nu})\psi \\ &= \bar{\psi} i \gamma^\lambda (x^\mu \partial^\nu - x^\nu \partial^\mu - i\Sigma^{\mu\nu})\psi \end{aligned}$$

Similar calculations are possible for the gluon part:

$$\boldsymbol{J} = \underbrace{-\frac{1}{2}\bar{\psi}\gamma_5 \boldsymbol{\gamma}\psi}_{\frac{1}{2}\Delta\Sigma} + \underbrace{\boldsymbol{E} \times \boldsymbol{A}}_{\Delta G} \underbrace{- i\psi^\dagger (\boldsymbol{x} \times \boldsymbol{\nabla})\psi}_{L^q_{\text{can}}} + \underbrace{\boldsymbol{E}(\boldsymbol{x} \times \boldsymbol{\nabla})\boldsymbol{A}}_{L^g_{\text{can}}}$$

Coming back to...

Questions raised by Jaffe-Manohar

(iii) What is the relation between the gluon spin and the anomalous current

$$K_\sigma = \epsilon_{\sigma\alpha\beta\gamma} \operatorname{Tr} A^\alpha \left(F^{\beta\gamma} - \tfrac{2}{3} A^\beta A^\gamma \right). \qquad (6.1)$$

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$$= \bar{\psi} i\gamma^\lambda (x^\mu \partial^\nu - x^\nu \partial^\mu - i\Sigma^{\mu\nu})\psi$$

Similar calculations are possible for the gluon part:

Gauge Spin? **Gauge Orbital?**

$$\boldsymbol{J} = \underbrace{-\frac{1}{2}\bar{\psi}\gamma_5 \boldsymbol{\gamma}\psi}_{\frac{1}{2}\Delta\Sigma} + \underbrace{\boldsymbol{E} \times \boldsymbol{A}}_{\Delta G} \underbrace{- i\psi^\dagger (\boldsymbol{x} \times \boldsymbol{\nabla})\psi}_{L^q_{\text{can}}} + \underbrace{\boldsymbol{E}(\boldsymbol{x} \times \boldsymbol{\nabla})\boldsymbol{A}}_{L^g_{\text{can}}}$$

Coming back to...

Questions raised by Jaffe-Manohar

$$S^{\lambda\mu\nu}_{\text{gauge}} = A^\mu F^{\lambda\nu} - A^\nu F^{\lambda\mu}$$

$$\epsilon^{\lambda\mu\nu\sigma} K_\sigma = A^\mu F^{\lambda\nu} - A^\nu F^{\lambda\mu} - A^\lambda F^{\mu\nu}$$

CS current looks like a gauge spin, and
the anomaly looks like a spin conservation…???

They are DIFFERENT by a term corresponding to the CME!?

constructs the rotation generators. The spin contribution to $\boldsymbol{J}_{(g)}$ is $\int d^3x\,(\boldsymbol{A}\times\boldsymbol{E})$. The analogous integral over $\epsilon^{\mu\nu\lambda\sigma}K_\sigma$ contains an additional term: $\int d^3x\, A^0 \boldsymbol{B}$. In $A^0=0$ gauge the two expressions agree. The same is true for the analogous

Any deep physics???

Serious Problem in Spin Hydro

How to construct the hydrodynamics with *J* ?

Hattori-Hongo-Huang-Matsuo-Taya (2019)

$$J^{\lambda\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} + \Sigma^{\lambda\mu\nu}$$

$$\partial_\lambda J^{\lambda\mu\nu} = 0$$

$$\partial_\lambda T^{\lambda\mu} = 0$$

$$\Rightarrow \quad \boxed{\partial_\lambda \Sigma^{\lambda\mu\nu} = -2T^{\mu\nu}_{(\mathrm{a})}} \qquad T^{\mu\nu}_{(\mathrm{a})} = -T^{\nu\mu}_{(\mathrm{a})}$$

Spin tensor ~ Anti-symmetric part of the EMT

$$\Sigma^{\lambda\mu\nu} = u^\lambda S^{\mu\nu} + \cdots \qquad p = \cdots + \omega_{\mu\nu} S^{\mu\nu}$$

The dynamics of spin $S^{\mu\nu}$ is constrained by $\partial_\mu s^\mu \geq 0$

Looks good… but not good in gauge theories… ?

Serious Problem in Spin Hydro

How to construct the hydrodynamics with *J* ?

An exercise in Peskin-Schroeder

2.1 Classical electromagnetism (with no sources) follows from the action

$$S = \int d^4x \left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \qquad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

(a) Derive Maxwell's equations as the Euler-Lagrange equations of this action, treating the components $A_\mu(x)$ as the dynamical variables. Write the equations in standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$.

(b) Construct the energy-momentum tensor for this theory. Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\widehat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

Pseudo-gauge transformation

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu,$$

leads to an energy-momentum tensor $\widehat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \tfrac{1}{2}(E^2 + B^2); \qquad \mathbf{S} = \mathbf{E} \times \mathbf{B}.$$

Serious Problem in Spin Hydro

How to construct the hydrodynamics with *J* ?

In gauge theories only symmetrized EMT is gauge invariant!

$$\partial_\lambda \Sigma^{\lambda\mu\nu} = -2T^{\mu\nu}_{(\mathrm{a})}$$ is missing in a symmetric formulation?

$$J^{\lambda\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} + \Sigma^{\lambda\mu\nu}$$

can be "redefined" to be a conserved angular momentum
in terms of symmetrized EMT as

$$\widehat{J}^{\lambda\mu\nu} = x^\mu \widehat{T}^{\lambda\nu} - x^\nu \widehat{T}^{\lambda\mu}$$ **Spin apparently missing!?**

I wrote a paper (Fukushima-Pu 2020) but I am not satisfied with my own arguments yet…

Summary

- **Chern-Simons current tells us a lot.**
  - CS charge is a gauge invariant observable
    = Magnetic Helicity in QED.
  - "Something" replacing A makes it gauge invariant.
- **Rotation effects also induces a current.**
  - Curvature / T / μ + w = (axial) current
  - Anomalous hydrodynamics
- **Relation between CS current and Spin?**
  - Spin (helicity) of gauge field well separated?
  - Spin hydrodynamics with gauge invariant EMT?