Stochastic finite range processes: non-equilibrium steady states and observables

Amit Kumar Chatterjee
YITP, Kyoto University, Japan

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- **OVERVIEW :**

- Introduction to ***NON-EQUILIBRIUM STEADY STATES*** starting from the ***MASTER EQUATION*** .

- Finite Range Processes (FRP): [*Phys. Rev. E*. 92, 032103 (2015)]

- (i) Introduce the zero range process and motivations to study this process

- (ii) Generalize the zero range process to finite range process

- Non-equilibrium steady state probability distributions of finite range process using methods :

(i) Pair wise balance condition

(ii) *h*- balance technique

(iii) Matrix Product Ansatz

→ counterpart of detailed balance in equilibrium

- Interesting features of observables in *finite range processes* :

(a) Negative differential response [*Phys. Rev. E.* 97, 052137 (2018)]

(current decreases with increasing bias)

(b) Current reversal with density [*Phys. Rev. E*. 98, 062134 (2018)]

(fixed dynamics, changing density changes direction of current)

(c) Condensation in 1-D [*Phys. Rev. E*. 92, 032103 (2015)]

(accumulating macroscopic number of particles in localized region of space)

EQUILIBRIUM : stationary state known: Gibbs-Boltzmann distribution $[P(C) \propto e^{-\beta E(C)}]$

One can directly start calculating observables using this distribution

NON-EQUILIBRIUM : No general formula known for steady state probability distribution *P(C)*

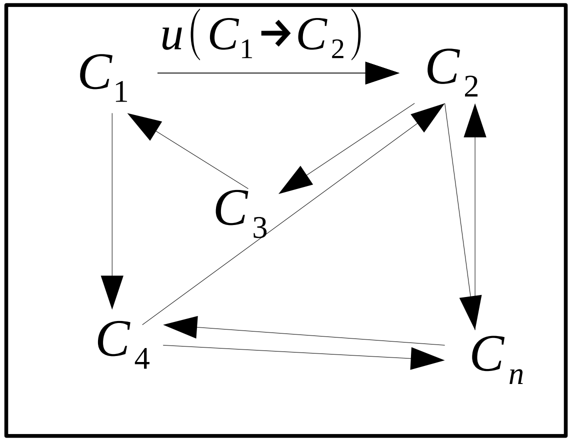
Take one step back, start from time evolution equations for *P(C,t)*

Then use dynamical rates to solve *P(C)*

MASTER EQUATION

Rates: $u(C \to C')$: *how often a system goes from one configuration C to another* C'
P(C,t): probability that system is in configuration *C* at time *t*
P(C): steady state probability that system is in configuration C

- **MASTER EQUATION** :



$$\frac{dP(C,t)}{dt} = \sum_{C'} u(C' \rightarrow C)\, P(C',t) - \sum_{C''} u(C \rightarrow C'')\, P(C,t)$$

In-flux or gain from *C'* to *C*

Out-flux or loss from *C* to *C''*

STEADY STATE : $\frac{dP(C,t)}{dt} = 0$ for all configurations *C* ⇨ $P(C)$

Question : How to achieve the steady state ?

↓

Total in-flux = Total out-flux

↓

many possible ways or ***FLUX CANCELLATION TECHNIQUES***

differentiates EQUILIBRIUM and NON-EQUILIBRIUM

- **EQUILIBRIUM :**

FLUX CANCELLATION TECHNIQUE : **DETAILED BALANCE**

$$u(C' \rightarrow C)\, P(C') = u(C \rightarrow C')\, P(C)$$

for every pair of configurations (*C*, *C'*)

zero flux between every pair (*C*, *C'*)

total current is *zero* in the configuration space

(*ZERO current for any observable in equilibrium*)

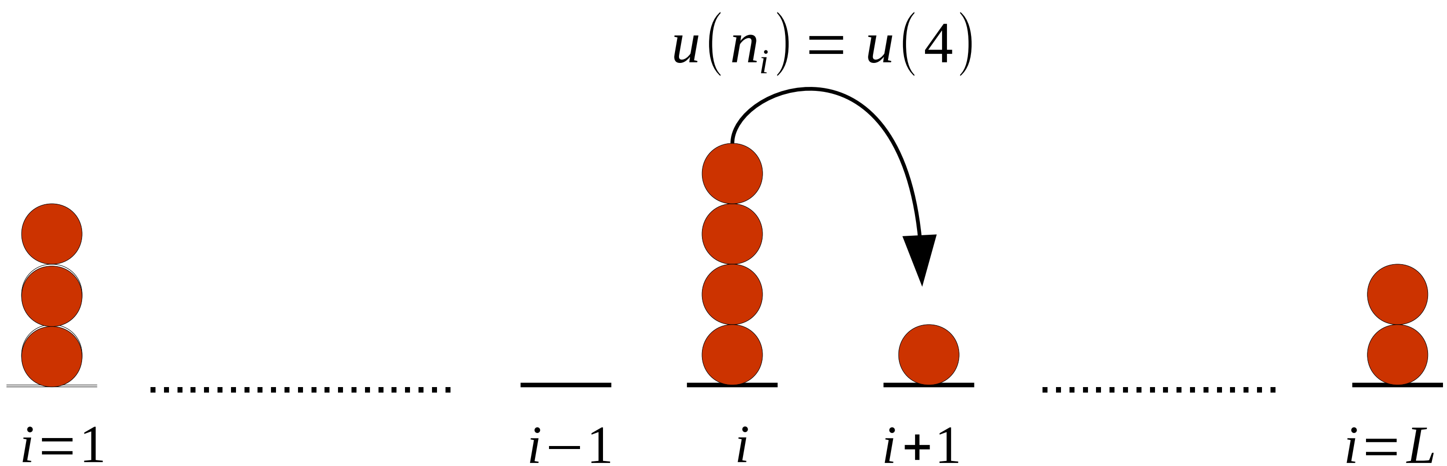
Any other FLUX CANCELLATION TECHNIQUE : **NON-EQUILIBRIUM**

(*NO detailed balance, possibility of NON-ZERO current of observables*)

- **ZERO RANGE PROCESS (ZRP)** : 1-D *periodic* lattice, *L* sites *i=1,2,....L*

NO hard core repulsion : $n_i \geq 0$ $\left(any \quad non-negative \quad integer\right)$

[Spitzer, Adv. Math. 5, 246 (1970)]



Particles interact within a single site ⟶ manifested in rate $u(n_i)$
(i.e. particles at different sites donot interact)

Any configuration: $\{n_1, n_2, \ldots n_{i-1}, n_i, n_{i+1}, \ldots n_L\} = \{n_i\}$

Steady state : **FACTORIZED** $P(\{n_i\}) \propto \prod_{i=1}^{L} g(n_i)$ with $g(n) = \prod_{i=1}^{n} \left(\frac{1}{u(i)}\right)$

Interaction involving no neighboring sites : zero range process
(Interaction range K=0)

ZRP with rate $u(n_i)$ ⟹ FSS with weight function $g(n_i)$

Why study zero range process ?

- **Condensation:**

particle density in grand canonical ensemble:

$$\rho = z\frac{F'(z)}{F(z)} \qquad F(z) = \sum_{n=0}^{\infty} z^n g(n)$$

$$\frac{\partial \rho}{\partial z} = \frac{\langle n^2\rangle - \langle n\rangle^2}{z} > 0 \Longrightarrow \rho(z) \text{ is an increasing function of } z.$$

let the radius of convergence of $F(z)$ be $z = \beta$. $\Longrightarrow$ $\rho_c = \beta\frac{F'(\beta)}{F(\beta)}$ maximum allowed value of density

rate: $u(n) = \beta(1 + b/n)\ \forall n > 0$ $\Longrightarrow$ $\rho_c = \frac{1}{b-2}$

For b>2 and $\rho > \rho_c$
macroscopic number $(\rho - \rho_c)L$ of particles accumulate on some particular site

⇓

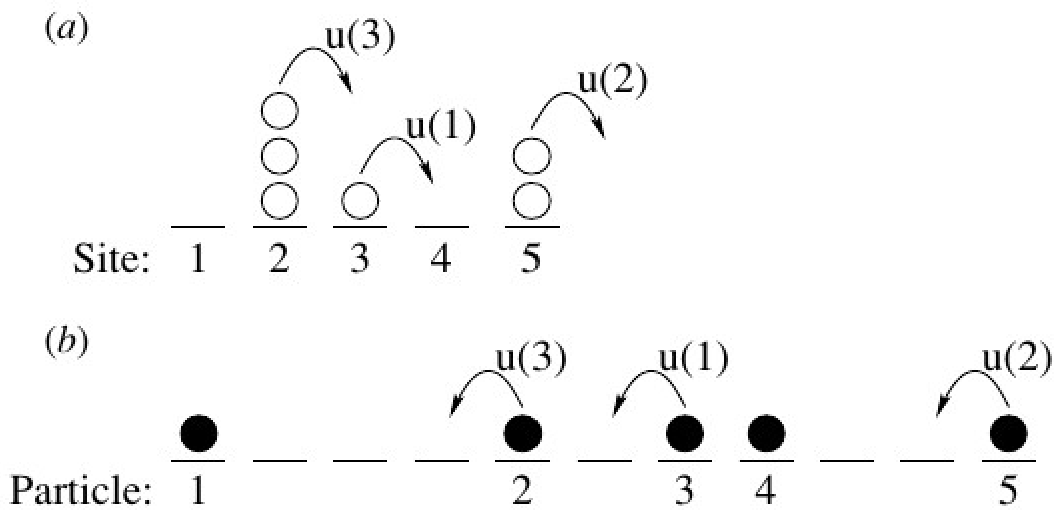
Formation of single site condensate

condensate →
- condensation in macroeconomies
- jamming in traffic
- coalescence in granular systems

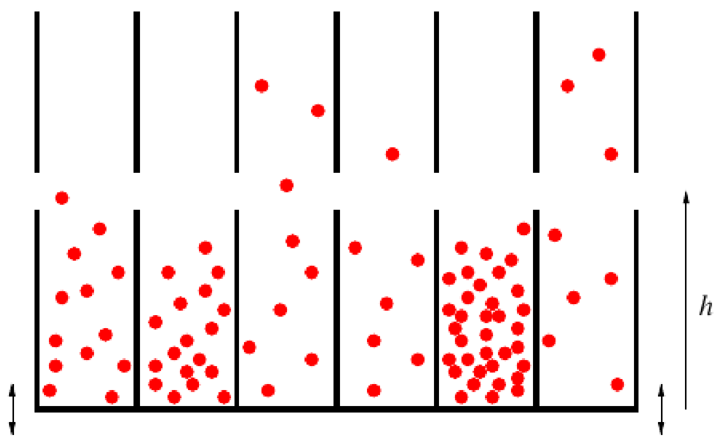
- **Mapping to exclusion process:**

zero range process :

asymmetric exclusion process :



- **Shaken granular gases:**



[Evans et.al., J. Phys. A: Mth. Gen. 38, R195 (2005)]

These models are based on experiments in which a container is divided into L equal compartments by walls, where each wall contains a narrow horizontal slit at height h. The container is then mounted on a shaker and filled with N particles, e.g., plastic balls or sand. When the system is shaken vertically, the particles hop from one compartment to another.

OUR GOAL:

ZRP (K=0) has a trivial factorized steady state where particles at different lattice sites are spatially un-correlated.

We want to generalize the dynamics where particles at different sites interact with each other. So, the interaction range would be finite K>0.

QUESTIONS:

- Can we find non-trivial exact steady states for general K>0 ?
- What kind of flux cancellation techniques to find the steady states ?
- What kind of interesting features do the observables exhibit ?

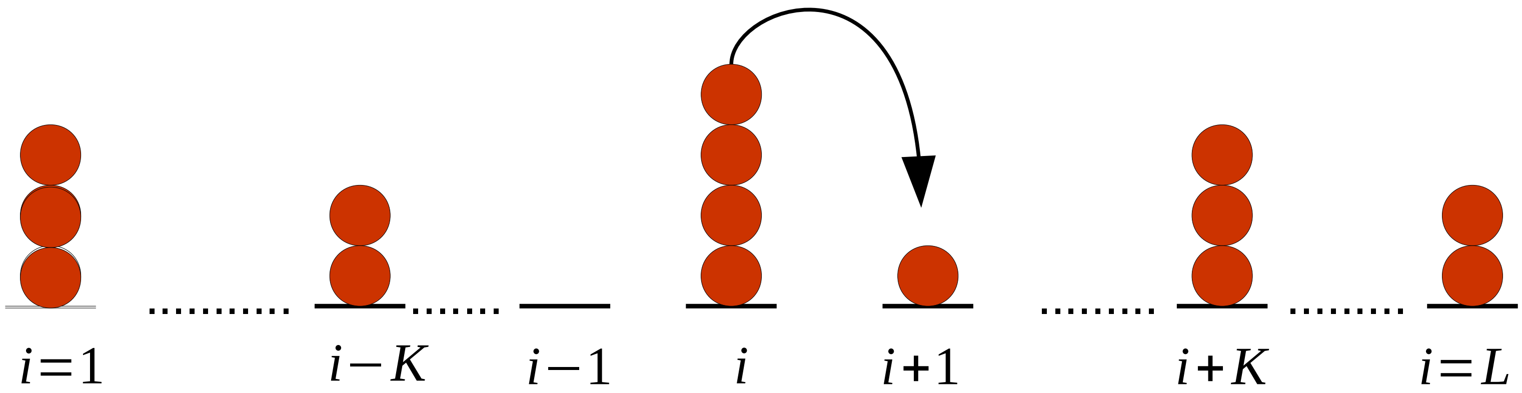
- **FINITE RANGE PROCESS (FRP) :**

particles at different sites interact with each other

hop rates depend on ocuupancy of several sites

Rate : $u\left(n_{i-K},\ldots,n_{i-1},n_i,n_{i+1},\ldots n_{i+K}\right)$ ⟶ interaction range *2K*

$$u\left(n_{i-K},\ldots n_i \ldots n_{i+K}\right) = u\left(2,\ldots,0,4,1,\ldots 3\right)$$



CLUSTER FACTORIZED STEADY STATE (CFSS) :

$$P\left(\{n_i\}\right) \propto \prod_{i=1}^{L} g\left(n_i, n_{i+1}, \ldots. n_{i+K}\right)$$

(*K*+*1*)- variable cluster weight function

with $u\left(n_{i-K},\ldots n_i,\ldots n_{i+K}\right) = \prod_{j=0}^{K}\left[\frac{g\left(\bar{n}_{i-K+j},\bar{n}_{i-K+1+j},\ldots.\bar{n}_{i+j}\right)}{g\left(n_{i-K+j},n_{i_K+1+j},\ldots.n_{i+j}\right)}\right]$ where $\bar{n}_l = n_l - \delta_{l\,i}$

K=0 ZRP

[Ref.- A. Chatterjee et.al. *Phys. Rev. E*. 92, 032103 (2015)]

Example: **K=1 : Pair Factorized Steady State (PFSS)**

RATE : $u(n_{i-1}, n_i, n_{i+1})$

Steady State : $P(\{n_i\}) \propto \prod_{i=1}^{L} g(n_i, n_{i+1})$ if $u(n_{i-1}, n_i, n_{i+1}) = \frac{g(n_{i-1}, n_i - 1)\, g(n_i - 1, n_{i+1})}{g(n_{i-1}, n_i)\, g(n_i, n_{i+1})}$

2-variable or pair-weight function ⇨ *Spatially correlated system*

FLUX CANCELLATION TECHNIQUE : **PAIRWISE BALANCE CONDITION**

For any two configurations (C', C) there exists a third configuration C'' such that

flux from C' to C = flux from C to C''

$$u(C' \rightarrow C)\, P(C') = u(C \rightarrow C'')\, P(C)$$

$$u(n_{i-K-1} \ldots n_{i-1}+1, n_i - 1, \ldots n_{i+K-1})\, P(\ldots n_{i-1}+1, n_i - 1, n_{i+1} \ldots) = u(n_{i-K} \ldots n_i\,,\, n_{i+1}, \ldots n_{i+K})\, P(\ldots n_{i-1}, n_i, n_{i+1} \ldots)$$

➢ FINITE RANGE PROCESS with ASYMMETRIC RATE FUNCTIONS (AFRP) :

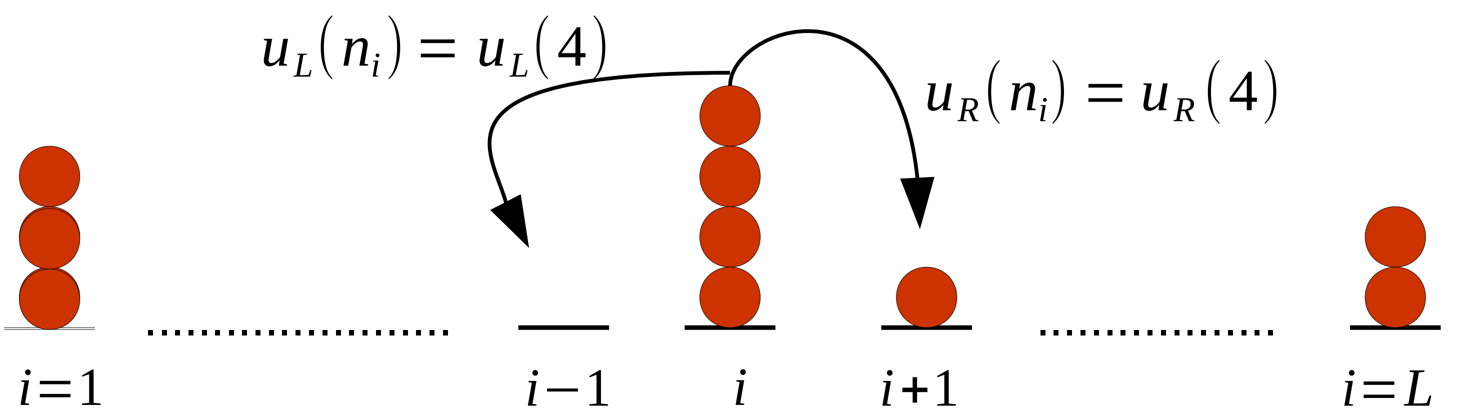
FRP : unidirectional motion BUT, AFRP : particles hop both right and left

Right hop rate : $u_R(n_{i-K},\dots n_i \,,\, n_{i+1},\dots n_{i+K})$

Left hop rate : $u_L(n_{i-K},\dots n_{i-1} \,,\, n_i,\dots n_{i+K})$

→ **same variables, different functions**

Simplest case: **K=0 (AZRP) :** right hop rate $u_R(n_i)$ and left hop rate $u_L(n_i)$



FACTORIZED STEADY STATE : $P(\{n_i\}) \propto \prod_{i=1}^{L} g(n_i)$ with $g(n) = \prod_{i=1}^{n} \left(\frac{1}{u_R(i)+u_L(i)}\right)$

But, unlike simple ZRP, the RATE FUNCTIONS have to satisfy a CONSTRAINT

$$\frac{u_L(n+1)\,u_R(1) - u_R(n+1)\,u_L(1)}{[u_R(n)+u_L(n)]\,[u_R(n+1)+u_L(n+1)]} = C$$

C is a constant

[Ref.- A. K. Chatterjee et. al. *J. Stat. Mech.* 093201, (2017)]

FLUX CANCELLATION TECHNIQUE : **K=0: AZRP**

- Steady state Master Equation: $$\sum_{i=1}^{L} F(n_{i-1},n_i) = 0$$

where $$F(n_{i-1},n_i) = u_R(n_i)+u_L(n_i)-u_R(n_{i-1}+1)\frac{g(n_{i-1}+1)g(n_i-1)}{g(n_{i-1})g(n_i)}-u_L(n_i+1)\frac{g(n_{i-1}-1)g(n_i+1)}{g(n_{i-1})g(n_i)}$$

- local flux balance scheme : if possible,

$$F(n_{i-1},n_i) = h(n_{i-1}) - h(n_i)$$ **(*h*- balance technique)**

h(n) has to be obtained *consistently* → $h(n) = h(0) - u_L(1)\dfrac{u_R(n)+u_L(n)}{u_R(1)+u_L(1)}$

- A set of rates that satisfy the constraint, $\dfrac{u_L(n+1)u_R(1) - u_R(n+1)u_L(1)}{[u_R(n)+u_L(n)]\,[u_R(n+1)+u_L(n+1)]} = C$ is given by

$$u_R(n) = v(n)\,[\delta - \gamma v(n-1)]\,;\; u_L(n) = v(n)\,[1-\delta+\gamma v(n-1)] \qquad 0 \leqslant \delta \leqslant 1 \text{ and } 0 \leqslant \gamma \leqslant \delta/v(n)|_{\max}$$

Finite *K> 0* : Cluster Factorized Steady State(CFSS) with constrained $u_{R,L}(.)$

[Ref.- A. K. Chatterjee et. al. *J. Stat. Mech.* 093201, (2017)]

❑ CURRENT REVERSAL :

- Generally, change direction of bias

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  direction of current changes

- ***Current reversal*** : *fixed bias*

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  change particle density

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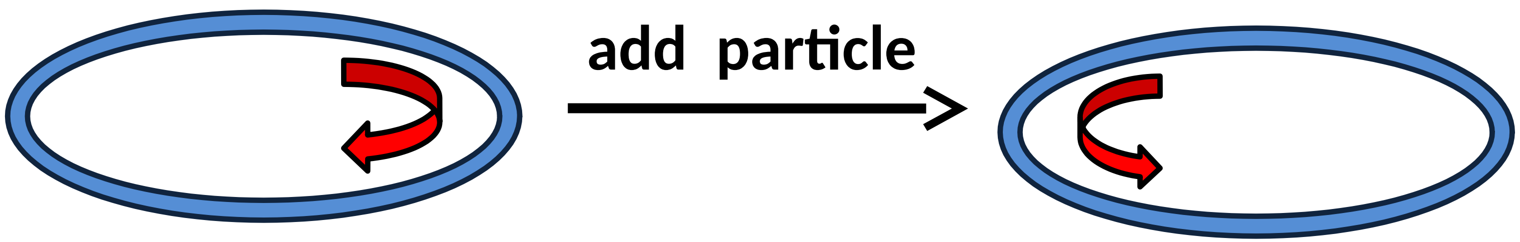
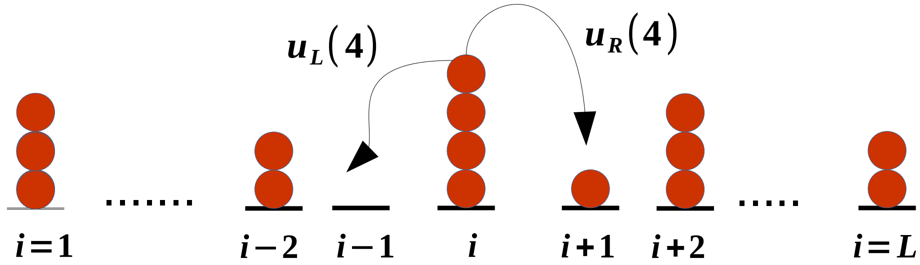
  direction of current changes



- ***Current reversal in Zero Range Process with asymmetric rate functions (AZRP) :***



remember: $u_R(n) = v(n)\,[\delta - \gamma v(n-1)]\,;\; u_L(n) = v(n)\,[1-\delta+\gamma v(n-1)]$

choose: $v(n) = 1 \text{ for } n \geqslant 1$
$\quad = 0 \text{ for } n = 0$
$\gamma = \delta - \alpha$

rates:

	$u_R(n)$	$u_L(n)$
n=1	δ	1-δ
n>1	α	1-α

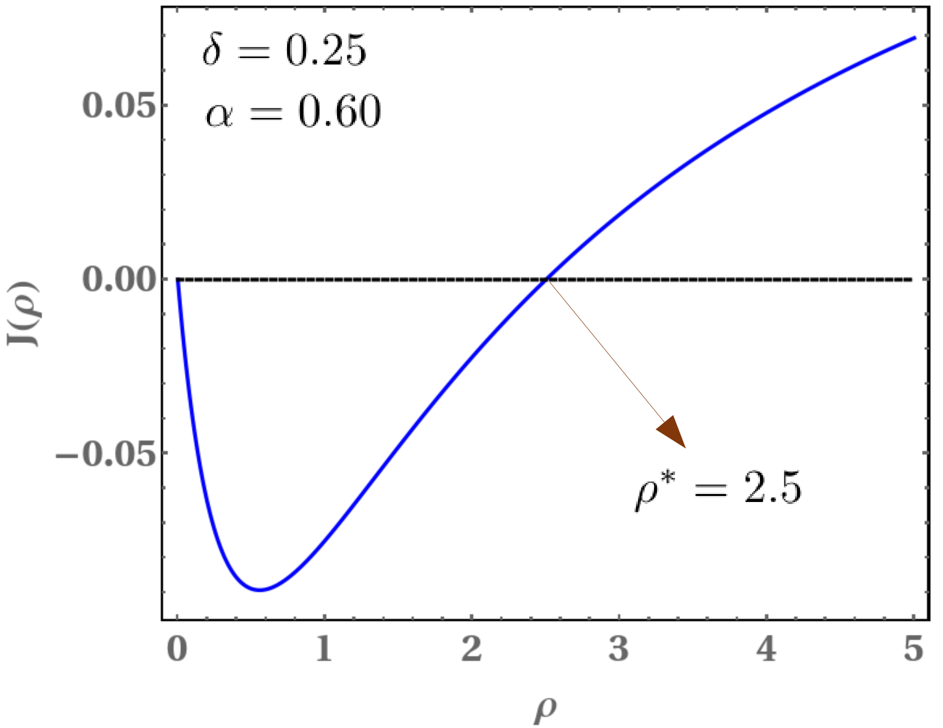
particle current:

$$J = \frac{\rho}{(1+\rho)^2}\,[2\delta - 1 + \rho(2\alpha - 1)]$$

Depending on $\rho > \rho^*$ or $\rho < \rho^*$, $J(\rho)$ changes sign

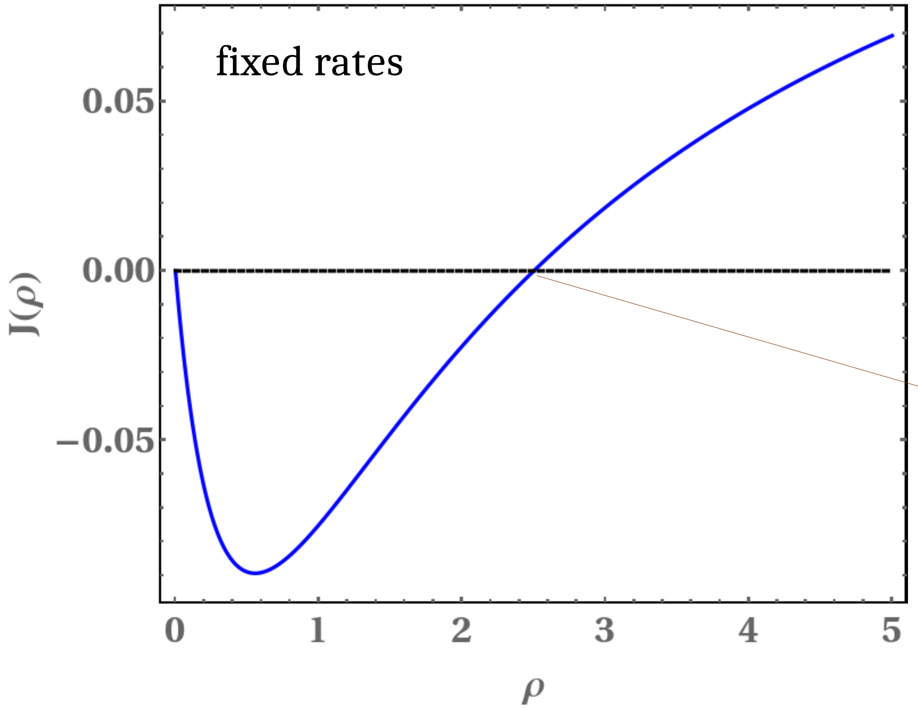
(δ, α) fixed

point of current reversal: $\rho^* = \frac{1-2\delta}{2\alpha-1}$



[Ref.- A. K. Chatterjee et. al. *J. Stat. Mech.* 093201, (2017)]

▪ **Zero current non-equilibrium steady state :**



zero current non-equilibrium steady state corresponding to

$$J(\rho^*) = 0$$

At the point of reversal, net particle current = 0

Zero current non-equilibrium states

every equilibrium state : zero current

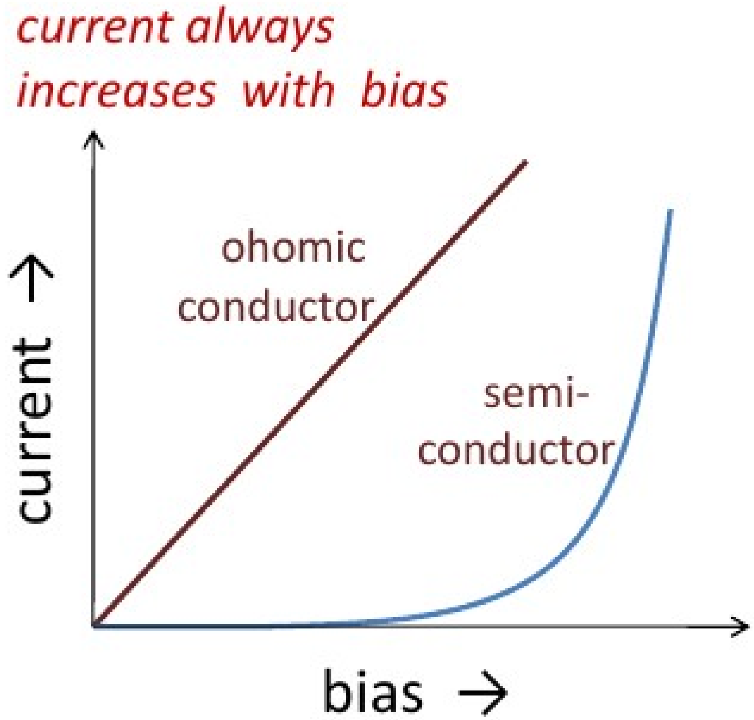
how to differentiate these states??

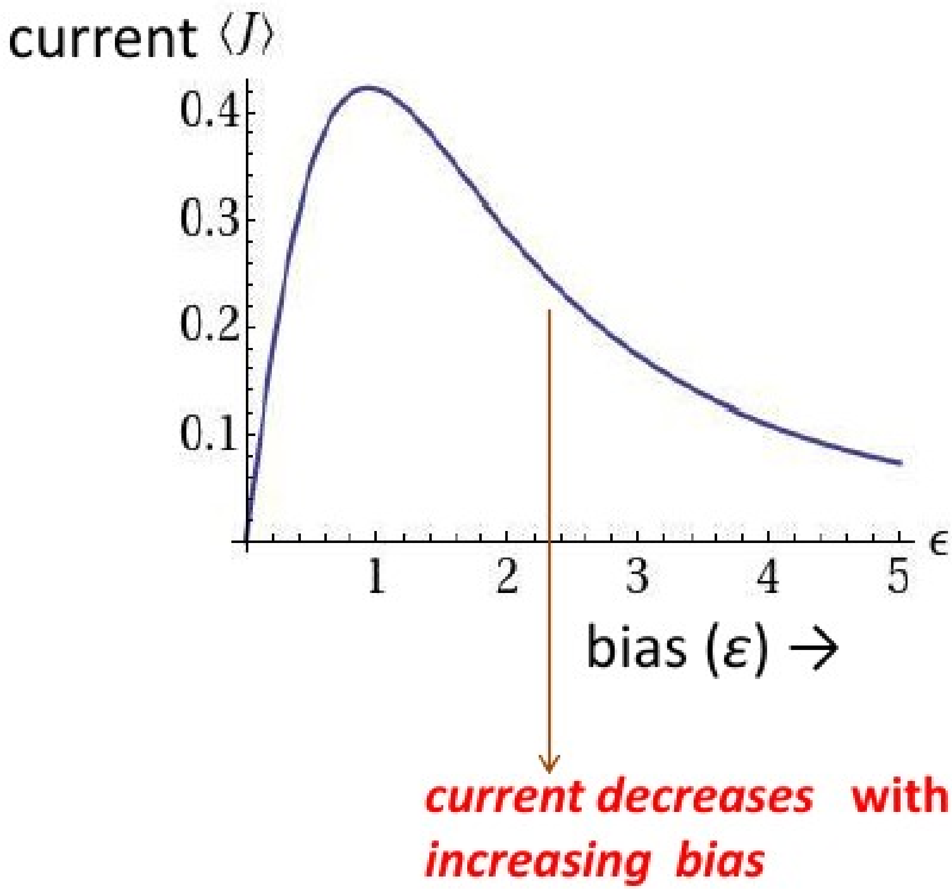
How to characterize *zero current non-equilibrium steady states*
and
how to distinguish them from *zero current equilibrium states* ?

❖ NEGATIVE RESPONSE :

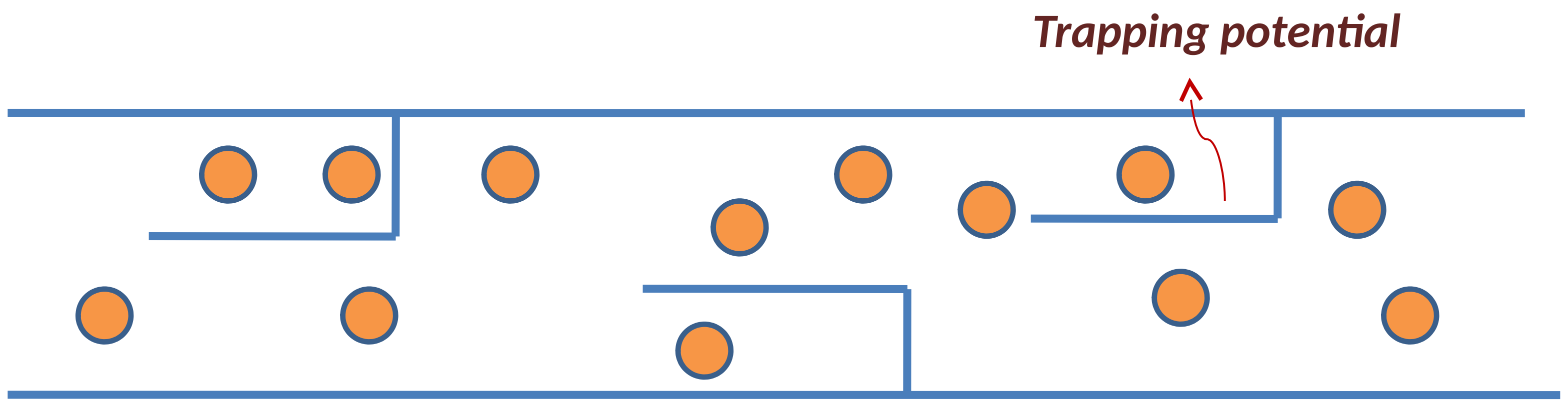
what we observe ***normally***

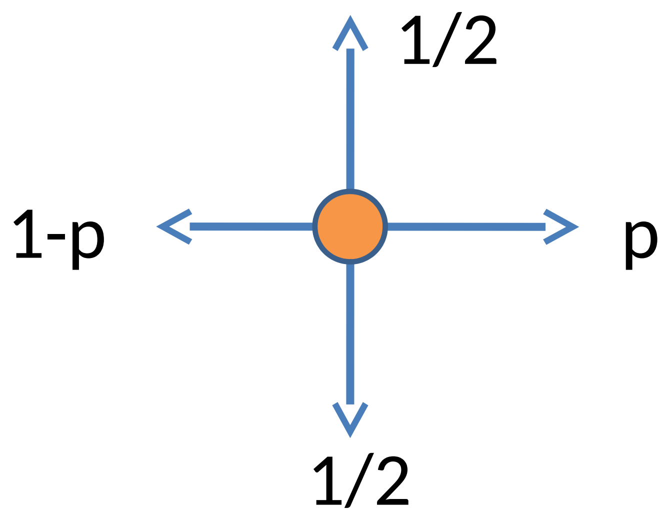
what we discuss in case of ***negative response***





- **Negative Response** : *non-interacting particles*





(a) p=1/2 ***J=0***

(b) p=0.51 , only few particles get trapped , **positive response**

(c) p=0.95 , most of the particles get trapped, **negative response**

Baerts *et.al.* , Phys. Rev. E 88, 052109 (2013)

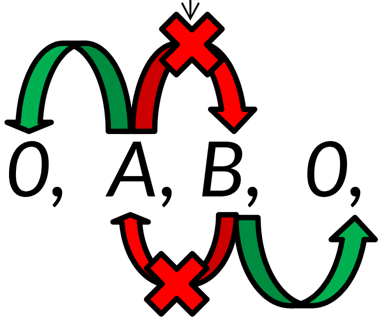
❖ Two random walkers (*A* & *B*): 1-D periodic lattice, *L* sites

$$i = 1, 2, \ldots, L$$

- *Dynamics*: $A0 \underset{q}{\overset{p}{\rightleftharpoons}} 0A \quad , \quad B0 \underset{\alpha}{\overset{\alpha}{\rightleftharpoons}} 0B$

Interaction : hard core exclusion

$$C_{i,i+1} : \{0, \ldots., 0, A, B, 0, \ldots 0\}$$



Question : what is *P*(*C*) for two interacting RW ?

- Matrix Product Ansatz :

i) associate a matrix to each constituent (τ_i)

constituent	matrix
RW 1	D_1
RW 2	D_2
vacancy	E

ii) $$P(C) \equiv P(\{X_i\}) \propto \mathrm{Tr}\,[\prod_{i=1}^{L} X_i]$$

X_i : matrix presenting the state at site *i*

$$\longrightarrow \quad X_i = D_1\delta_{\tau_i,A} + D_2\delta_{\tau_i,B} + E\delta_{\tau_i,0}$$

iii) *Matrix Algebra* :

$p\, D_1 E - q\, E D_1 = x_0 D_1$
$\alpha\,(D_2 E - E D_2) = x_0 D_2$
$D_1^2=0\ ,\quad D_2^2=0$

$$x_0 = \alpha\frac{p-q}{\alpha+q}$$

iv) *Matrix representation* :

$$D_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ D_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ E = \begin{pmatrix} \frac{p+\alpha}{q+\alpha} & 0 \\ 0 & 1 \end{pmatrix}$$

- ***Average particle current*** : as $L \to \infty$

$$\langle J \rangle = \left(p\langle D_1 E \rangle - q\langle E D_1 \rangle \right) + \left(\alpha\langle D_2 E \rangle - \alpha\langle E D_2 \rangle \right) = \frac{2(p-q)\alpha}{(p+\alpha)}$$

current of ***biased*** random walker ***A*** ($\langle J_A \rangle$)

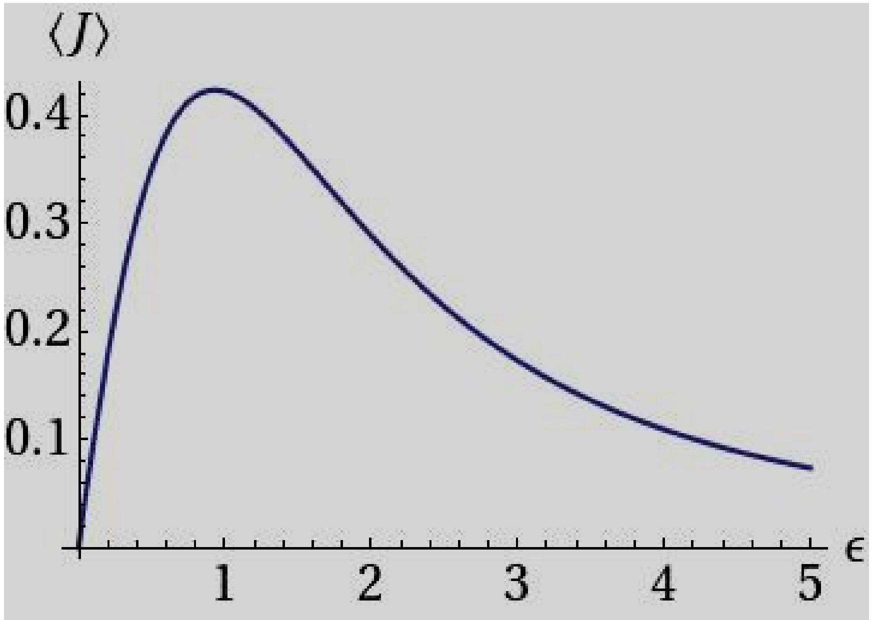
current of ***unbiased*** random walker ***B*** ($\langle J_B \rangle$)

Let $p = 1,\ q = e^{-\varepsilon},\ \alpha = \dfrac{1}{(1+\varepsilon^2)}$ → bias: $\ln\left(\dfrac{p}{q}\right) = \varepsilon$

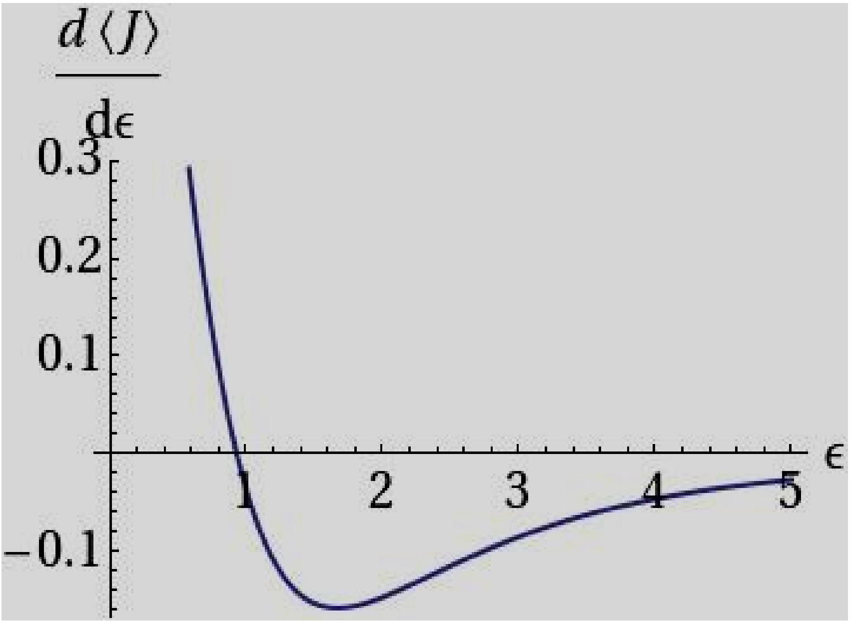
$$\langle J \rangle(\varepsilon) = \frac{2(1-e^{-\varepsilon})}{(2+\varepsilon^2)}$$

QUESTION:
does current (<J>) always increases with bias (ε) ?

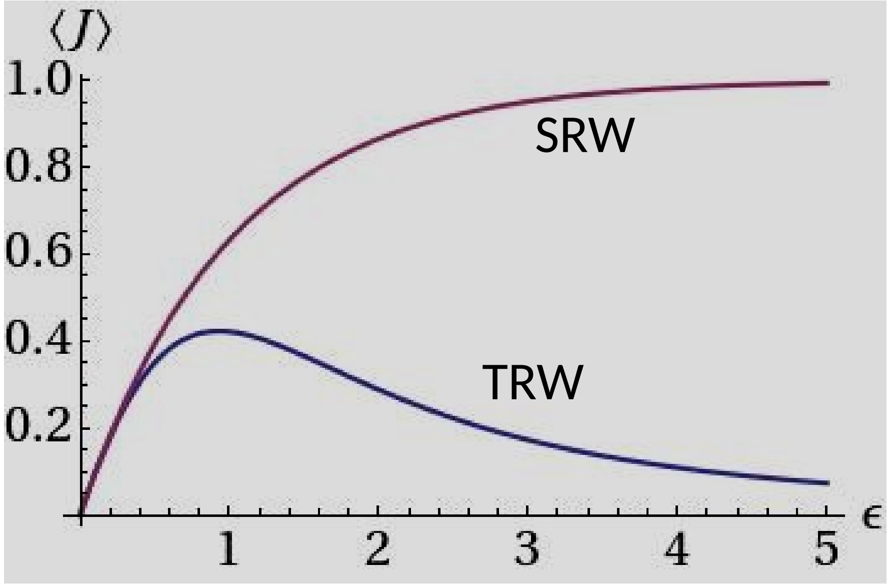
- Plot : *current vs bias* : two interacting RW

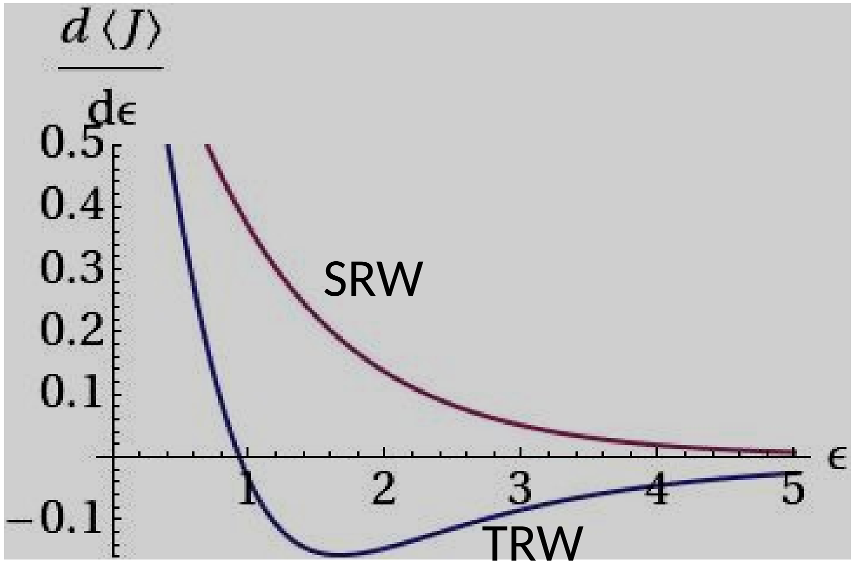


negative response →



- Compare : *single RW (SRW) vs two interacting RW (TRW)*





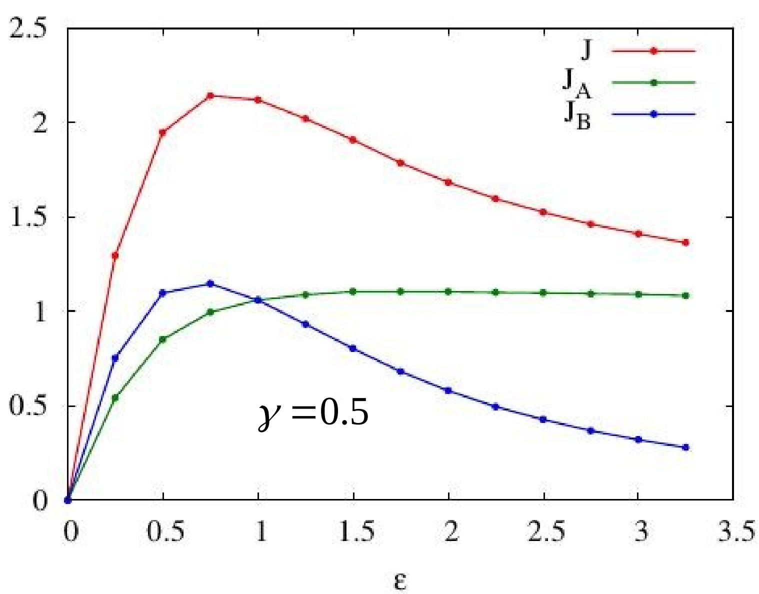
- ***Negative response*** : many particles

Multiple random walkers : (many A's & many B's)

$$A0 \underset{q}{\overset{p}{\rightleftharpoons}} 0A\,, \quad B0 \underset{\alpha}{\overset{\alpha}{\rightleftharpoons}} 0B\,, \quad AB \underset{\gamma}{\overset{\gamma}{\rightleftharpoons}} BA$$

A's & B's interact :

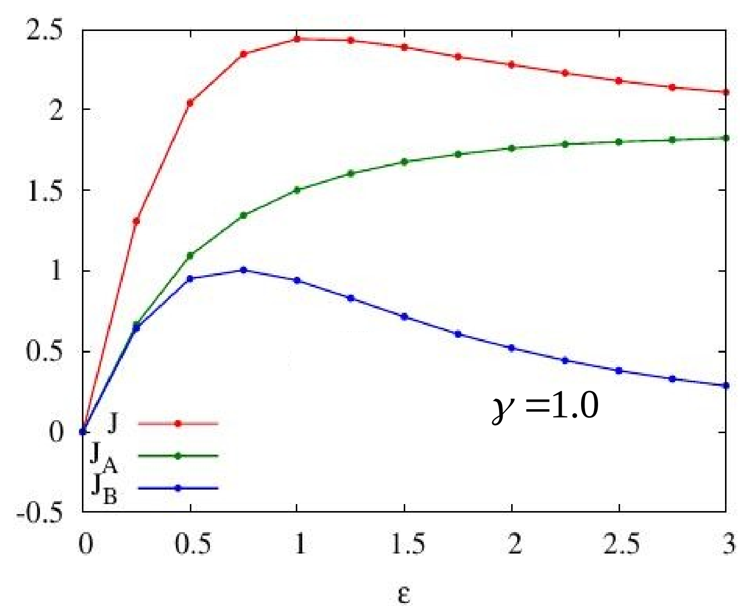
(i) *hard core exclusion* (ii) *exchange dynamics*



$p = 1,$

$q = e^{-\varepsilon},$

$\alpha = \dfrac{1}{(1+\varepsilon^2)}$



- **QUESTION:** ***what gives rise to negative Response ?***

- **Mechanism:** *Multiple current carrying modes*

(e.g. *A* & *B* in case of TRW model)

↓

Slow down at least one mode through biasing other modes

(e.g. bias(ε) for *A* slows down *B* by $\alpha = \frac{1}{(1+\varepsilon^2)}$)

↓

Total current may decrease

❑ **MASTER EQUATION** (time evolution of probabilities in config. space)

STEADY STATE

EQUILIBRIUM

(detailed balance)

zero current

Free energy

All thermodynamic observables

NON-EQUILIBRIUM
(FRP, AFRP, two random walkers)

(e.g. pair-wise balance, *h*-balance, MPA cancellation)

non-zero current

negative response

(multiple current carrying modes + slow down some mode)

current reversal

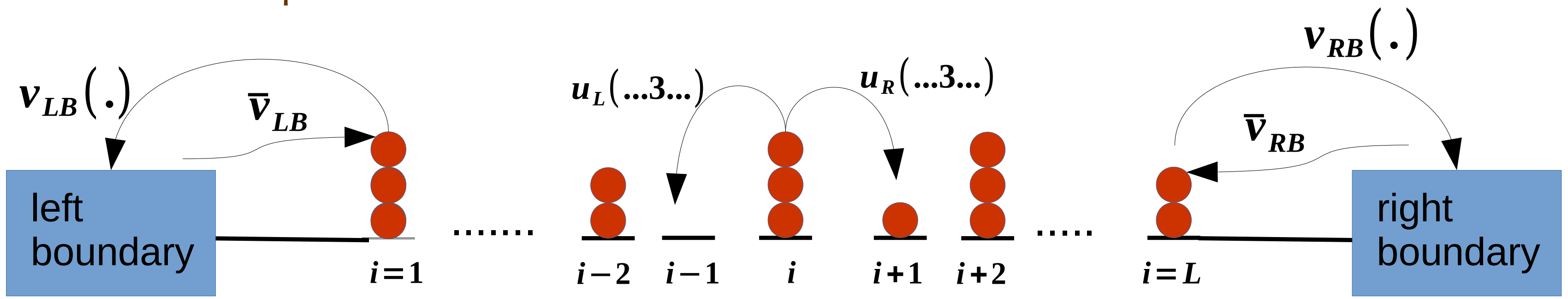
(fixed bias + vary particle density)

condensation

(macroscopic particle accumulation at single site)

A few possible future directions :

- AFRP with open boundaries :



Steady state ?? *Current ??* *Phase transition ??*

Remember, TASEP
- *periodic lattice* → factorized state, no spatial correlation, no phase transition
- *open boundary* → matrix product state, spatial correlation, three different phases

- FRP on networks :

directed network with vertices and links
each link individually execute finite range process with no hard core exclusion
particles are exchanged between links through the vertices (junctions)

How does the connectivity of the network affects the steady state and observables of interest ?

- FRP in higher dimensions : how to obtain steady states (exactly or perturbatively) for two-lane or multi-lane FRP where each lane is executing FRP with some particle exchange dynamics between the lanes

- Periodic time dependent modulation of input and output rates at the boundaries of FRP or even TASEP like models. With zero mean of input and output rates, can one have non-zero current in these kind of models ? Analogues of Thouless pumping process in these models ?

- Can one study stochastic thermodynamics incorporating these exactly solvable models having interesting features of current ?

THANK YOU