Softening and residual loss modulus of jammed grains under oscillatory shear

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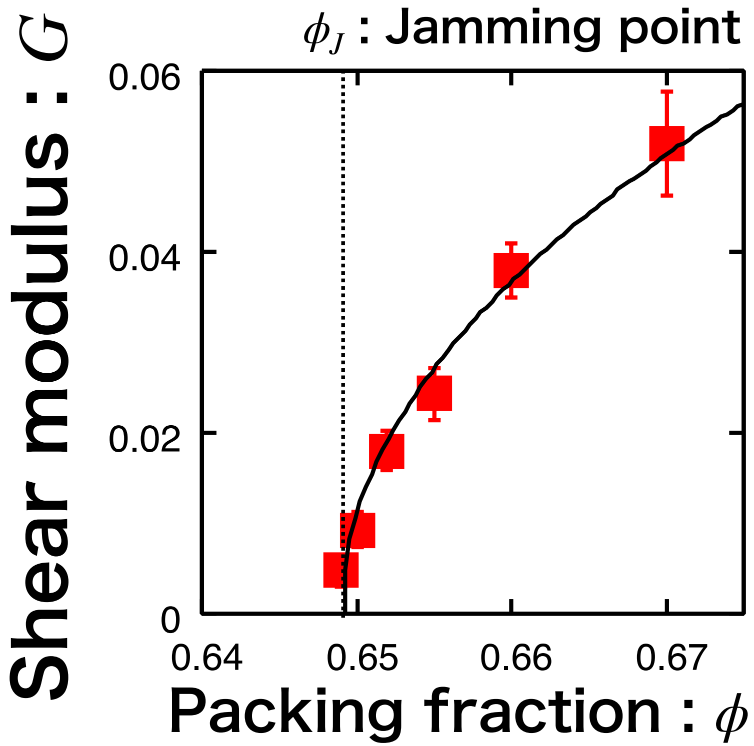
arXiv:2101.07473
Eur. Phys. J. E 44, 106 (2021)

Rheology of jammed grains

Dense amorphous particles : sand, colloidal suspensions

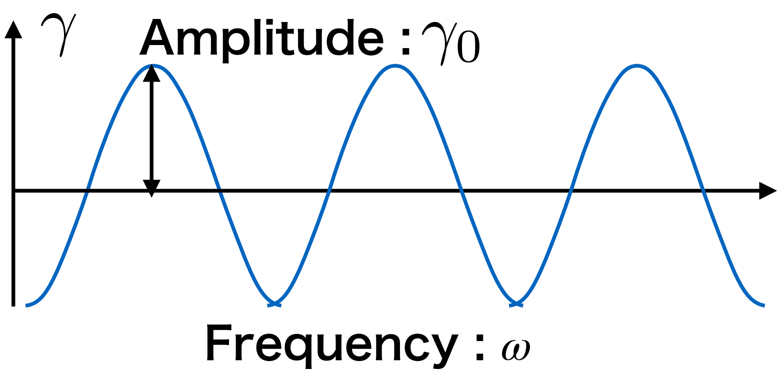


Dense amorphous particles exhibit elasticity.

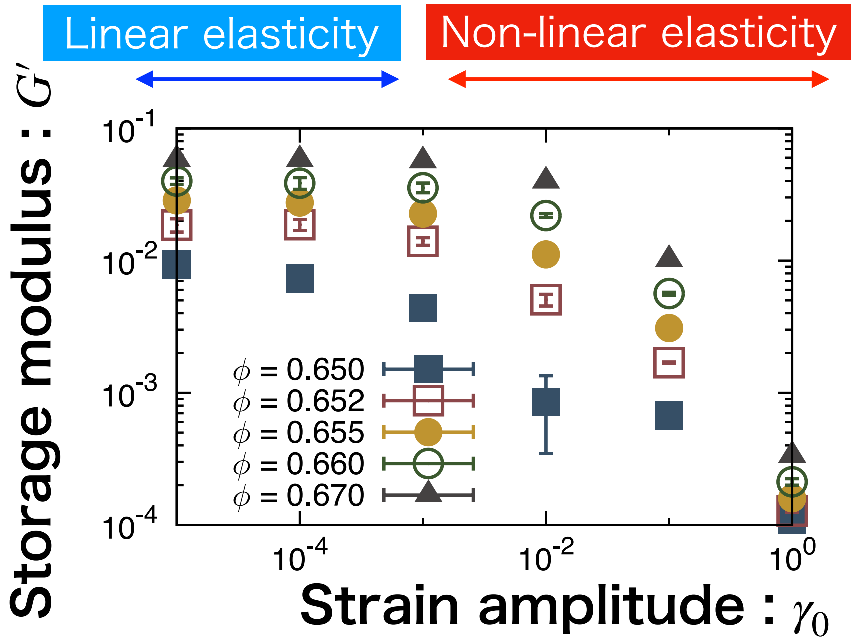
Elasticity of amorphous particles under oscillatory shear

MO and H. Hayakawa, 90, 042202 (2014)

Shear strain : $\gamma(t) = \gamma_0 \sin(\omega t)$



Amplitude dependence of G'



Natural conjecture

Origin of non-linear elasticity : Irreversible plastic deformation?

Recent studies suggest

J. Boschan, et. al.,(2016), S. Dagois-Bohy, et. al., (2017), T. Kawasaki and K. Miyazaki, (2020)

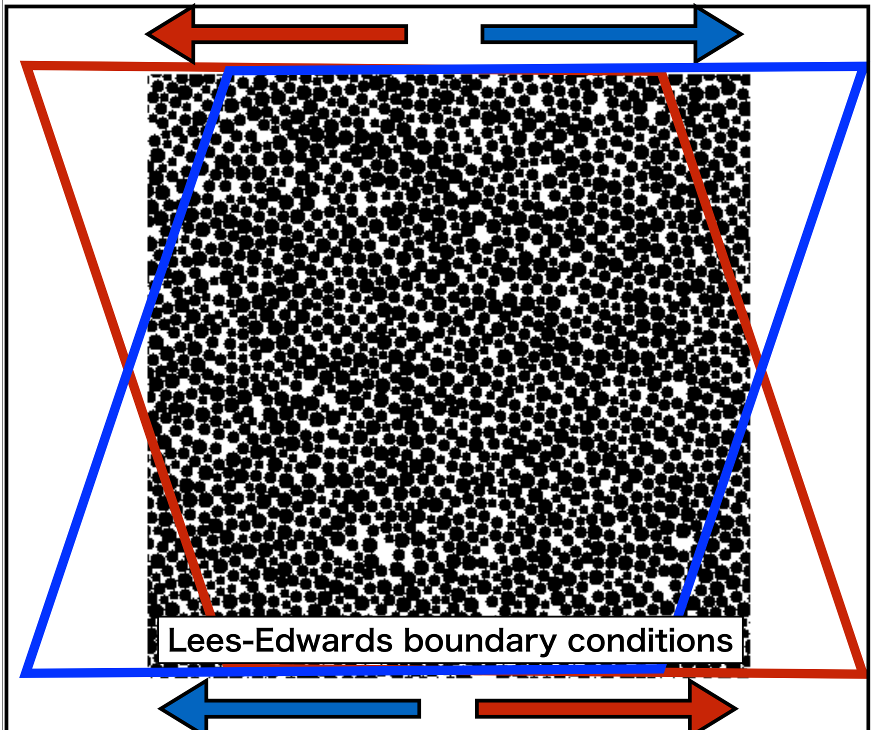
Nonlinear elasticity appears without plastic deformation.

Origin of nonlinear elasticity ?

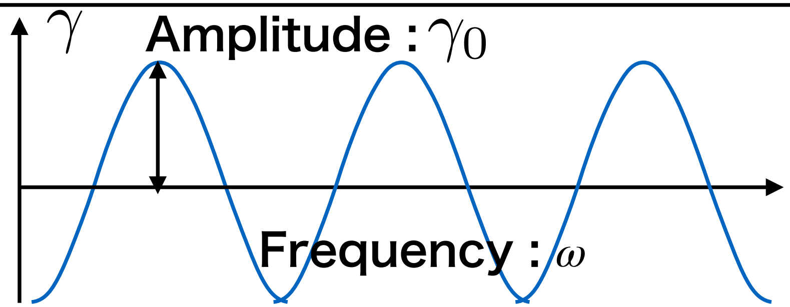
Outline

1. Introduction : Elasticity of jammed grains
2. **Frictionless grains** **arXiv:2101.07473**
3. Frictional grains **Eur. Phys. J. E 44, 106 (2021)**
4. Summary

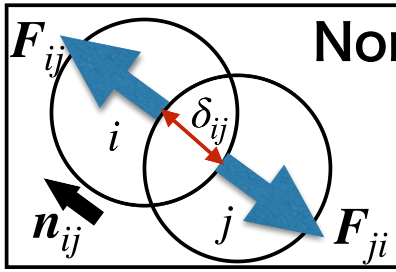
Model : 2D frictionless grains



Oscillatory strain



Strain : $\gamma(t) = \gamma_0 \sin(\omega t)$



Normal repulsive force :

$$\boldsymbol{F}_{ij} = \left(k_n \delta_{ij} + \eta \dot{\delta}_{ij}\right) \boldsymbol{n}_{ij}$$

Contact length : δ_{ij}

Position : $\boldsymbol{r}_i$

Over-damped eq. :

$$\frac{d}{dt} r_i = \dot{\gamma}(t) y_i \boldsymbol{e}_x + \sum_{j \neq i} \boldsymbol{F}_{ij} / \eta$$

c.f. The same results for under-damped systems

Shear stress : $\sigma(t)$

Storage modulus : elasticity

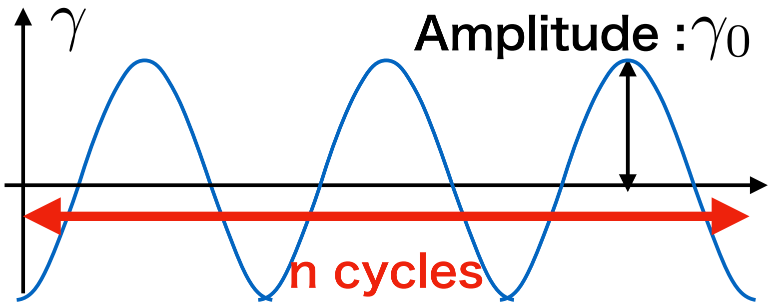
$$G' = \frac{\omega}{\pi} \int_0^{2\pi/\omega} d\theta \, \frac{\sigma(t) \sin(\omega t)}{\gamma_0}$$

Loss modulus : dissipation

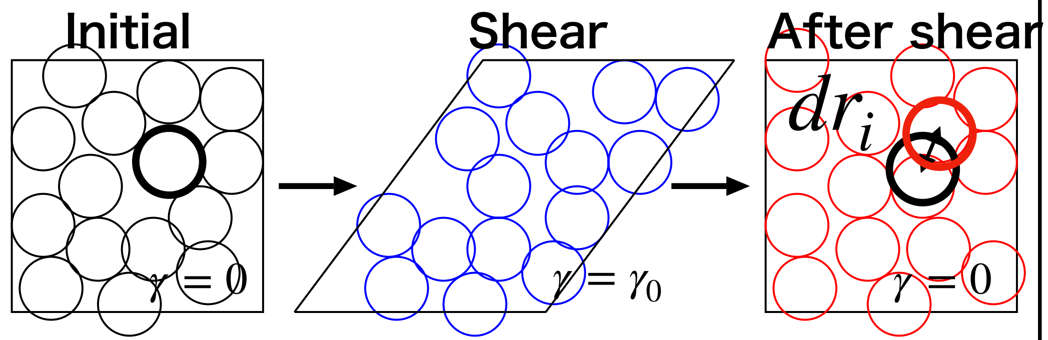
$$G'' = \frac{\omega}{\pi} \int_0^{2\pi/\omega} d\theta \, \frac{\sigma(t) \cos(\omega t)}{\gamma_0}$$

Detecting plastic deformation

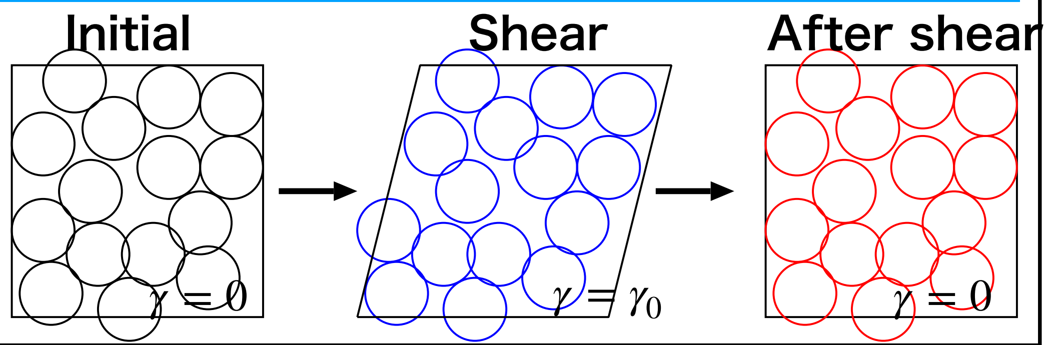
c.f. T. Kawasaki and L. Breathier, Phys. Rev. E 94, 022615 (2016)



Large strain amplitude : finite displacement

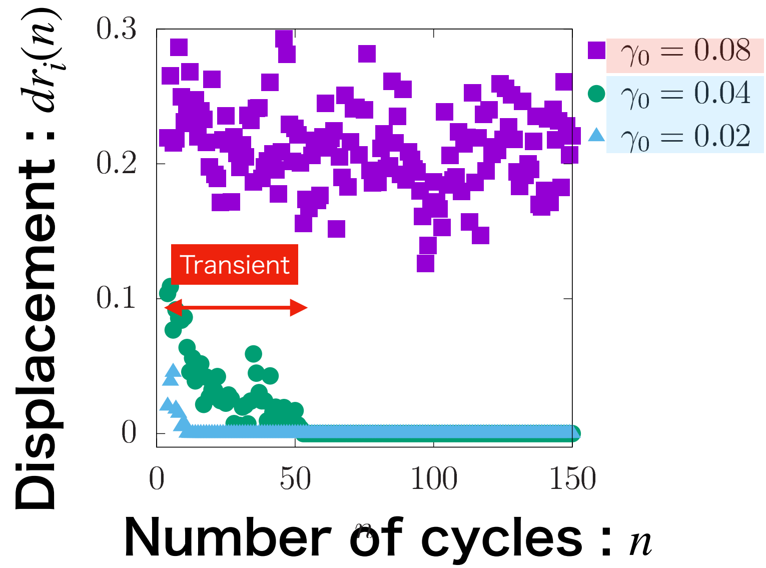


Small strain amplitude : no displacement



Position after n cycles : $\boldsymbol{r}_i(n)$

Displacement : $dr_i(n) = |\boldsymbol{r}_i(n) - \boldsymbol{r}_i(n-1)|$



Plastic state

$dr(n) > 0$: **Irreversible deformation**

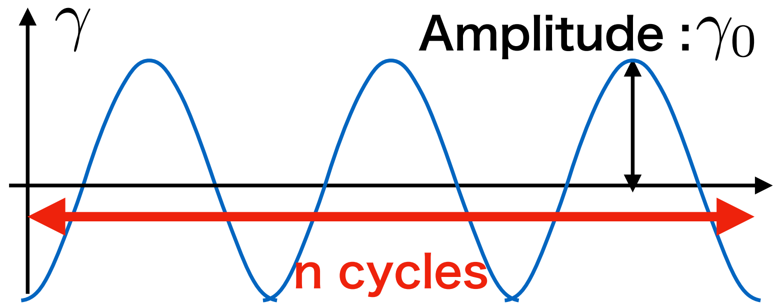
Absorbing state

$dr(n) = 0$ **after a few trainsient cycles**

No plastic deformation in the final state (after the transient).

Particle trajectory

c.f. T. Kawasaki and L. Breathier, Phys. Rev. E 94, 022615 (2016)



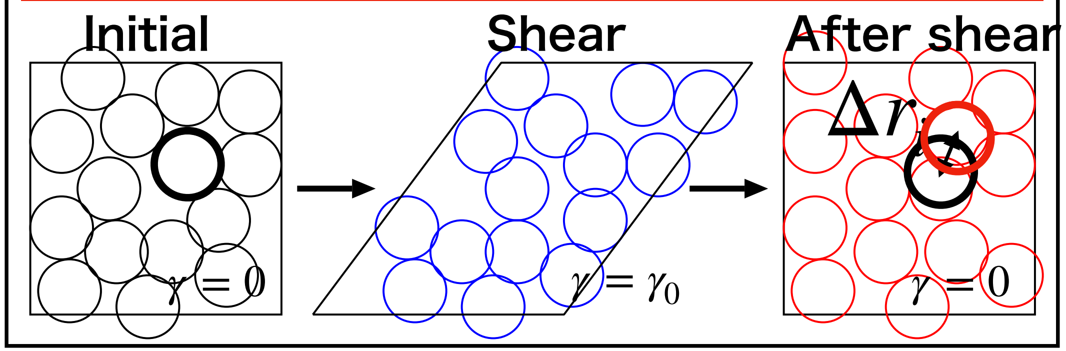
Position : $\boldsymbol{r}_i(t) = \boldsymbol{R}_i + \gamma(t) Y_i \boldsymbol{e}_x + \tilde{\boldsymbol{r}}_i(t)$

Center of trajectory : $\boldsymbol{R}_i = (X_i, Y_i)$

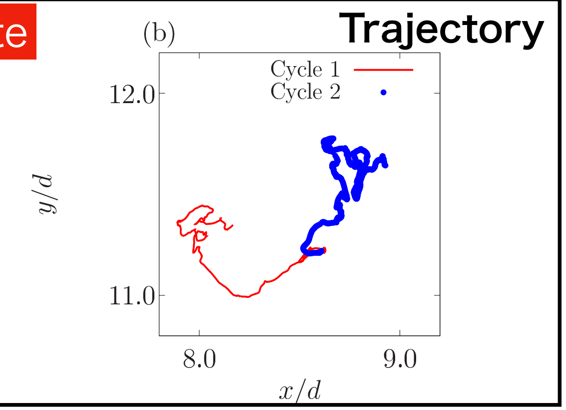
Affine motion : $\gamma(t) Y_i \boldsymbol{e}_x$

Non-affine motion : $\tilde{\boldsymbol{r}}_i(t)$

Large strain amplitude : finite displacement

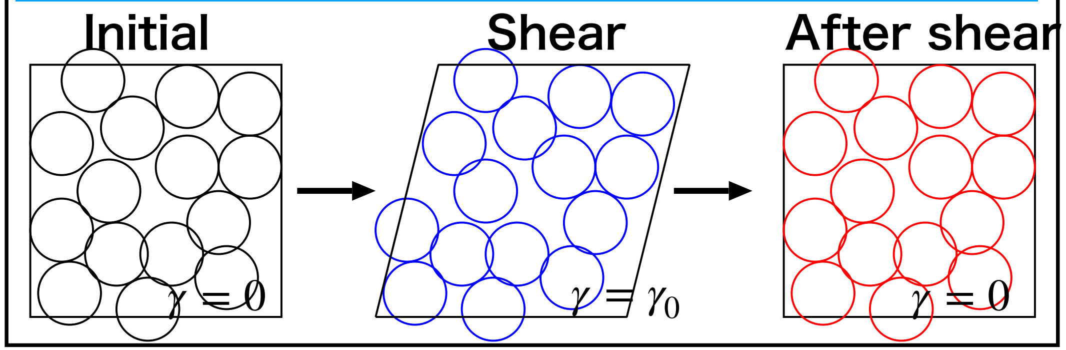


Plastic state

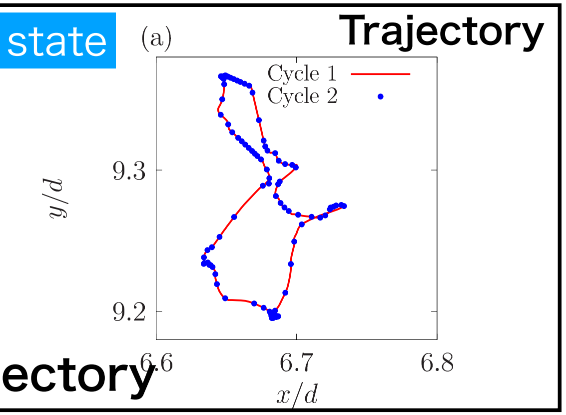


Diffusion

Small strain amplitude : no displacement

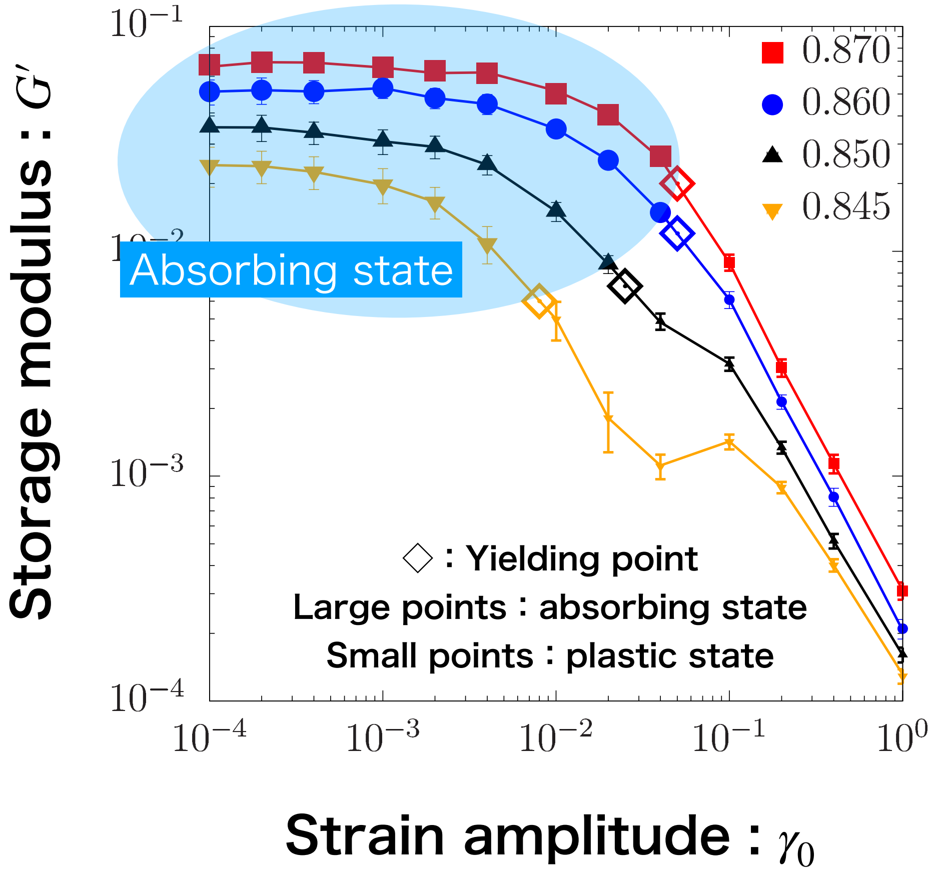


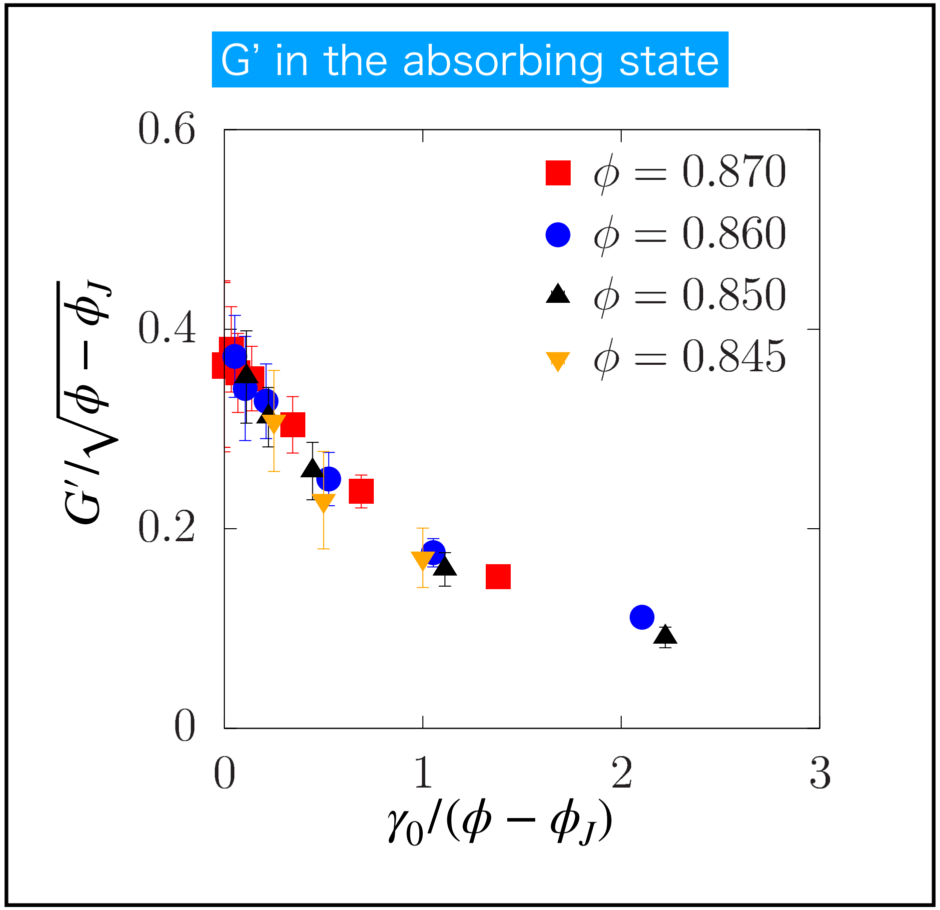
Absorbing state



Loop trajectory

Softening in the absorbing state



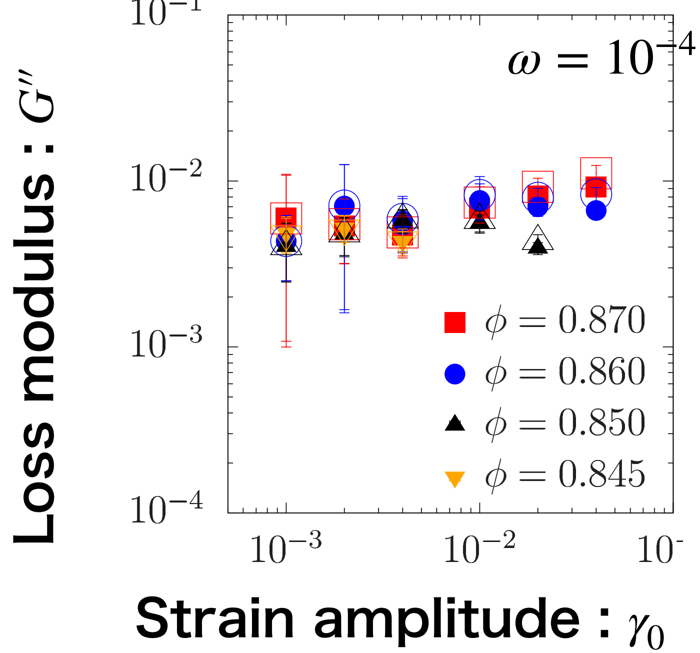


Softening in absorbing state : Decrease of G' without plastic deformation

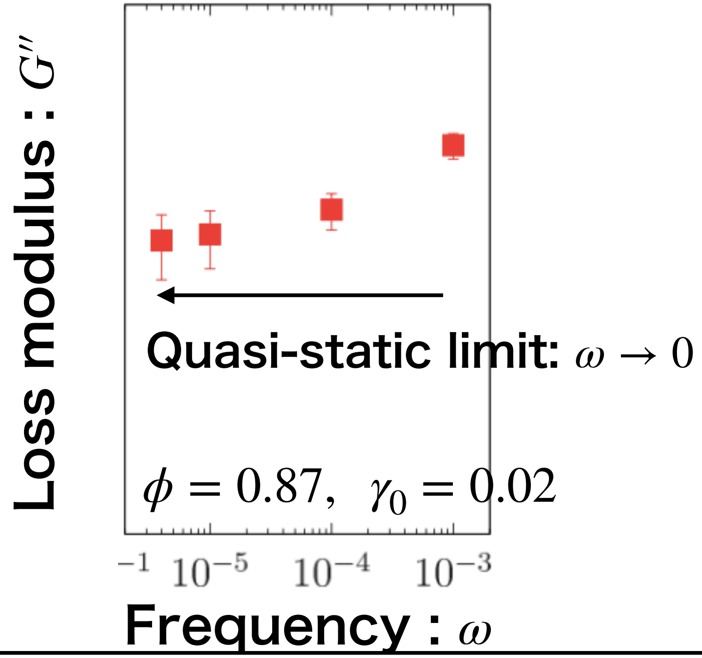
G' decreases to less than half.

Residual loss modulus

Storage modulus in absorbing state



Loss modulus for different ω



G'' is almost independent of ϕ and γ_0.
Residual loss modulus : G'' remains for $\omega \to 0$ without plastic deformation.

cf. Linear visco elasticity : $\sigma = G\gamma + \eta\dot{\gamma}$
(Kelvin-Voigt model) $G' = G, \ G'' = \eta\omega$

Dissipation (G'') disappears for $\omega \to 0$.

Origin of anomalous response: Loop trajectory ?

Fourier analysis of loop trajectory

Phase : $\theta = \omega t$ **(quasi-static limit)**

Shear strain : $\gamma(\theta) = \gamma_0 \sin\theta$

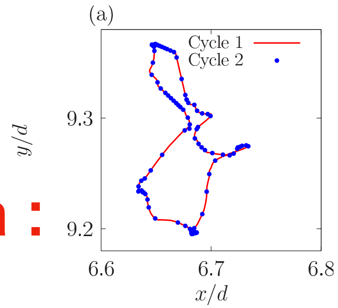
Absorbing state (periodic)

Position :

$$\boldsymbol{r}_i(\theta) = \boldsymbol{r}_i(\theta + 2\pi)$$

Non-affine motion :

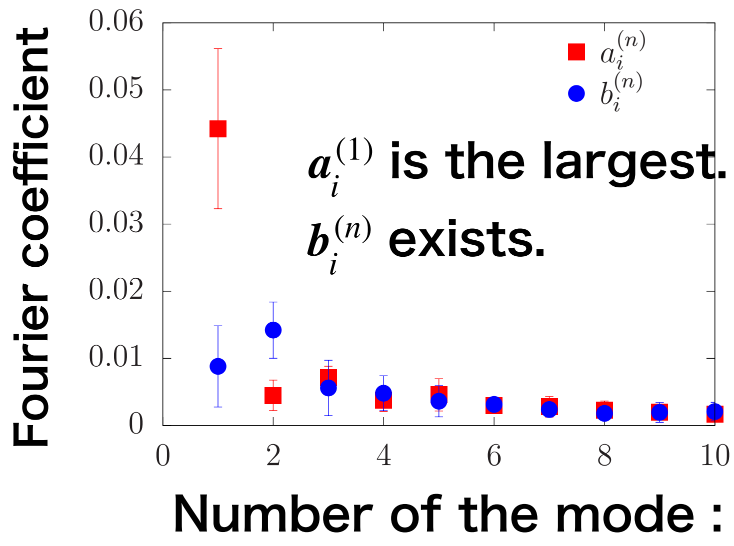
$$\tilde{\boldsymbol{r}}_i(\theta) = \tilde{\boldsymbol{r}}_i(\theta + 2\pi)$$



Fourier series of non-affine motion

$$\tilde{\boldsymbol{r}}_i(\theta) = \sum_n \left\{ \boldsymbol{a}_i^{(n)} \sin n\theta + \boldsymbol{b}_i^{(n)} \cos n\theta \right\}$$

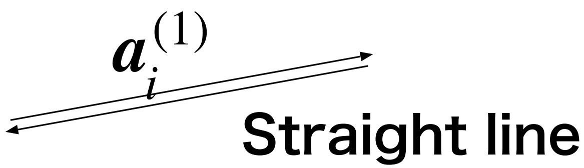
Fourier coefficient



We also find $\boldsymbol{a}_i^{(n)}, \boldsymbol{b}_i^{(n)} \sim \gamma_0$.

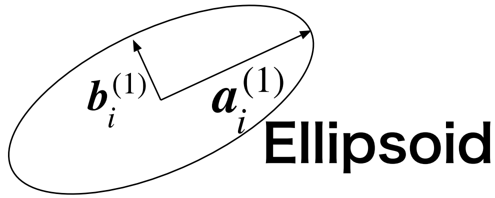
Example 1: $\tilde{\boldsymbol{r}}_i(\theta) = \boldsymbol{a}_i^{(1)} \sin\theta$

Only $\boldsymbol{a}_i^{(1)}$ is finite.



Example 2: $\tilde{\boldsymbol{r}}_i(\theta) = \boldsymbol{a}_i^{(1)} \sin\theta + \boldsymbol{b}_i^{(1)} \cos\theta$

Only $\boldsymbol{a}_i^{(1)}$ and $\boldsymbol{b}_i^{(1)}$ are finite.



Analysis : Residual loss modulus

Shear strain : $\gamma(\theta) = \gamma_0 \sin\theta$
(quasi-static limit)

Loss modulus :

$$G'' = \frac{1}{\pi}\left\langle \int_0^{2\pi} d\theta \, \frac{\sigma(t)\cos\theta}{\gamma_0} \right\rangle$$

Stress : $\sigma(\theta) = -\sum_{i,j} \frac{x_{ij}(\theta)y_{ij}(\theta)}{L^2 r_{ij}(\theta)} F(r_{ij}(\theta))$

Normal repulsive force : $F(r)$

Position : $\boldsymbol{r}_i(\theta) = \boldsymbol{R}_i + \gamma(\theta) Y_i \boldsymbol{e}_x + \tilde{\boldsymbol{r}}_i(\theta)$

Center of trajectory : $\boldsymbol{R}_i = (X_i, Y_i)$

Non-affine motion : Loop

$$\tilde{\boldsymbol{r}}_i(\theta) = \sum_n \left\{ \boldsymbol{a}_i^{(n)} \sin n\theta + \boldsymbol{b}_i^{(n)} \cos n\theta \right\}$$

↓ **First order approximation of** γ_0, $\boldsymbol{a}_i^{(n)}$, **and** $\boldsymbol{b}_i^{(n)}$

$$G'' \simeq G''_1 = \sum_i \left\langle \boldsymbol{B}''_i \cdot \frac{\boldsymbol{b}_i^{(1)}}{\gamma_0} \right\rangle$$

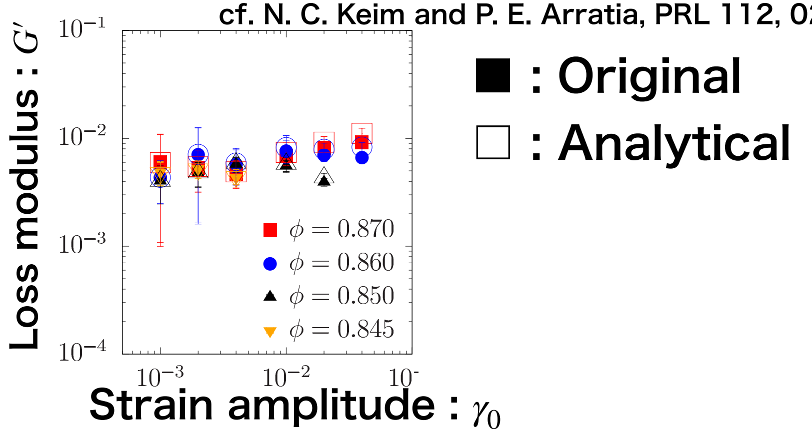
$\boldsymbol{B}''_i$: function of $\boldsymbol{R}_i$

Only $b_i^{(1)}$ remains. $\boldsymbol{a}_i^{(n)}, \boldsymbol{b}_i^{(n)} \sim \gamma_0$.

Detail :

$$G''_1 = -\left\langle \frac{1}{L^2}\sum_{i,j}\left(\frac{b^{(1)}_{ij,x}}{\gamma_0} Y_{ij} + X_{ij}\frac{b^{(1)}_{ij,y}}{\gamma_0}\right) F(R_{ij}) \right\rangle - \left\langle \frac{1}{L^2}\sum_{i,j} X_{ij}Y_{ij}F'(R_{ij})R_{ij}\frac{\boldsymbol{R}_{ij}\cdot \boldsymbol{b}^{(1)}_{ij}}{\gamma_0 R_{ij}^2} \right\rangle$$

cf. N. C. Keim and P. E. Arratia, PRL 112, 028302 (2014).



G''$_1$ reproduces the original results.

Loop trajectory is the origin of the residual loss modulus.

Analysis : Softening in absorbing state

Shear strain : $\gamma(\theta) = \gamma_0 \sin\theta$
(quasi-static limit)

Storage modulus :

$$G' = \frac{1}{\pi}\int_0^{2\pi} d\theta \, \frac{\sigma(t)\sin\theta}{\gamma_0}$$

Stress : $\sigma(\theta) = -\sum_{i,j} \frac{x_{ij}(\theta)y_{ij}(\theta)}{L^2 r_{ij}(\theta)} F(r_{ij}(\theta))$

Normal repulsive force : $F(r)$

Position : $\boldsymbol{r}_i(\theta) = \boldsymbol{R}_i + \gamma(\theta) Y_i \boldsymbol{e}_x + \tilde{\boldsymbol{r}}_i(\theta)$

Center of trajectory : $\boldsymbol{R}_i = (X_i, Y_i)$

Non-affine motion : Loop

$$\tilde{\boldsymbol{r}}_i(\theta) = \sum_n \left\{ \boldsymbol{a}_i^{(n)} \sin n\theta + \boldsymbol{b}_i^{(n)} \cos n\theta \right\}$$

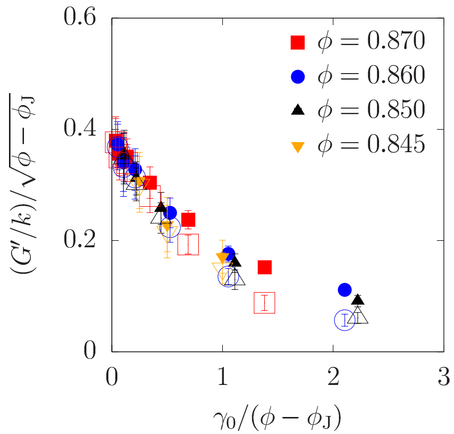
↓ First order approximation of γ_0, $\boldsymbol{a}_i^{(n)}$, and $\boldsymbol{b}_i^{(n)}$

$$G' \simeq G'_1 = \langle A'(\{\boldsymbol{R}_i\})\rangle - \sum_i \left\langle \boldsymbol{B}'_i(\{\boldsymbol{R}_i\}) \cdot \frac{\boldsymbol{a}_i^{(1)}}{\gamma_0} \right\rangle$$

$A', \boldsymbol{B}'_i$: functions of $\boldsymbol{R}_i$ $\quad \boldsymbol{a}_i^{(n)}, \boldsymbol{b}_i^{(n)} \sim \gamma_0$.

Detail :

$$G'_1 = -\left\langle \frac{1}{L^2}\sum_{i,j}\left\{ \frac{X_{ij}^2 Y_{ij}^2}{R_{ij}} F'(R_{ij}) + Y_{ij}^2 F(R_{ij}) \right\} \right\rangle - \left\langle \frac{1}{L^2}\sum_{i,j}\left(\frac{a_{ij,x}^{(1)}}{\gamma_0} Y_{ij} + X_{ij}\frac{a_{ij,y}^{(1)}}{\gamma_0} \right) F(R_{ij}) \right\rangle + \cdots$$



■ : Original
□ : Analytical

G'₁ reproduces the original results.

What is the origin of softening?

Origin of softening in the absorbing state

Center of trajectory : $\boldsymbol{R}_i = (X_i, Y_i)$

Non-affine motion : Loop

$$\tilde{\boldsymbol{r}}_i(\theta) = \sum_n \left\{ \boldsymbol{a}_i^{(n)} \sin n\theta + \boldsymbol{b}_i^{(n)} \cos n\theta \right\}$$

$$G' \simeq G'_1 = \langle A'(\{\boldsymbol{R}_i\}) \rangle - \sum_i \left\langle \boldsymbol{B}'_i(\{\boldsymbol{R}_i\}) \cdot \frac{\boldsymbol{a}_i^{(1)}}{\gamma_0} \right\rangle$$

$A', \boldsymbol{B}'_i$ **: functions of** $\boldsymbol{R}_i$

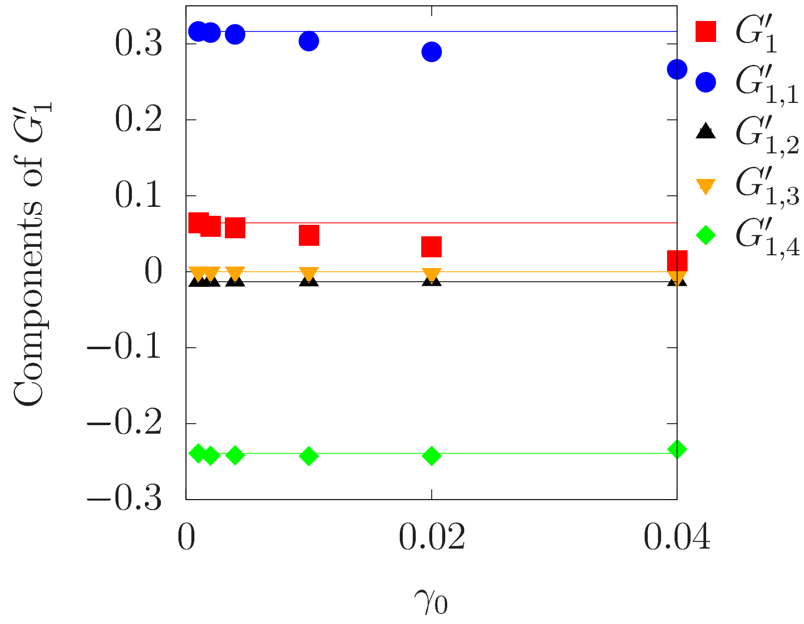
Detail :

$$G'_1 = -\left\langle \frac{1}{L^2} \sum_{i,j} \left\{ \frac{X_{ij}^2 Y_{ij}^2}{R_{ij}} \Psi'(R_{ij}) \right\} \right\rangle \quad G'_{1,1} \text{ (Affine)}$$

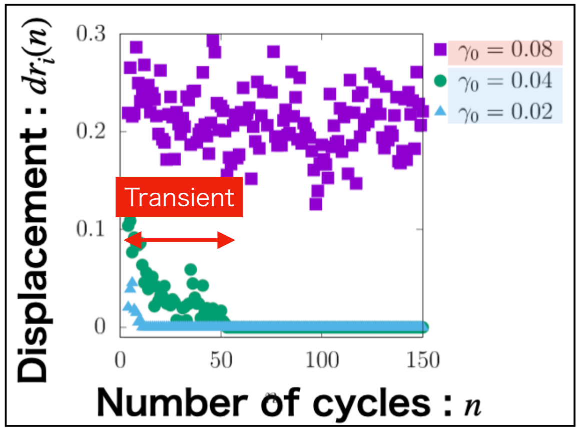
$$-\left\langle \frac{1}{L^2} \sum_{i,j} \left\{ Y_{ij}^2 \Psi(R_{ij}) \right\} \right\rangle \quad G'_{1,2} \text{ (Affine)}$$

$$-\left\langle \frac{1}{L^2} \sum_{i,j} \left(\frac{a_{ij,x}^{(1)}}{\gamma_0} Y_{ij} + X_{ij} \frac{a_{ij,y}^{(1)}}{\gamma_0} \right) \Psi(R_{ij}) \right\rangle \quad G'_{1,3} \text{ (Non-affine)}$$

$$-\left\langle \frac{1}{L^2} \sum_{i,j} X_{ij} Y_{ij} \Psi'(R_{ij}) \frac{\boldsymbol{R}_{ij} \cdot \boldsymbol{a}_{ij}^{(1)}}{\gamma_0 R_{ij}} \right\rangle, \quad G'_{1,4} \text{ (Non-affine)}$$



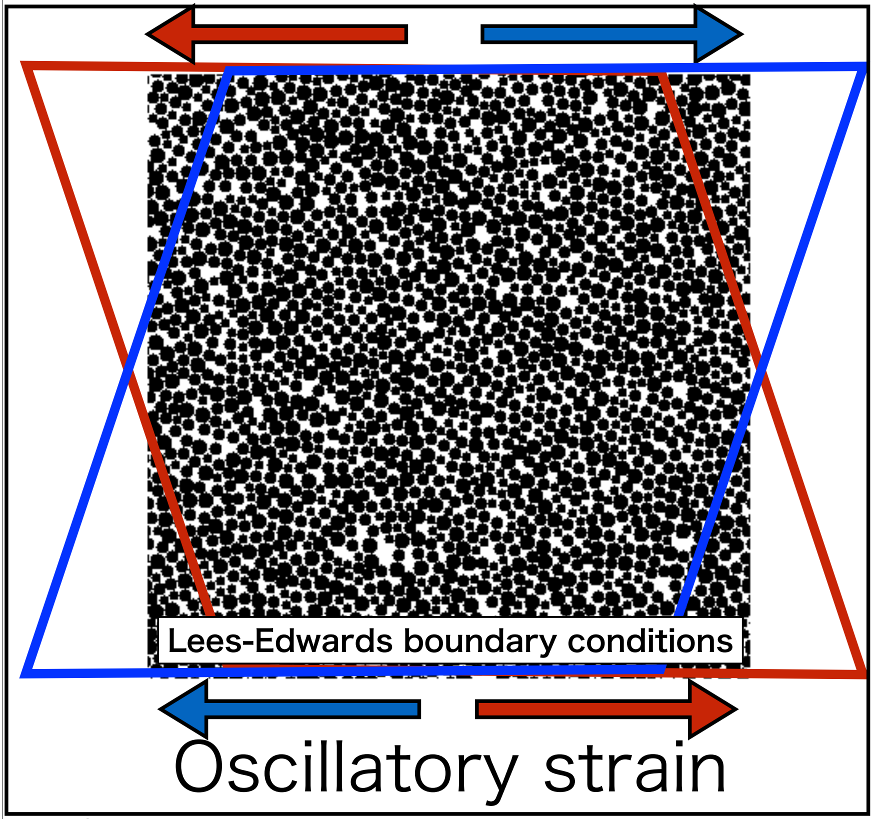
- **Softening results from G'$_{1,1}$ through the change of R_i in the transient.**

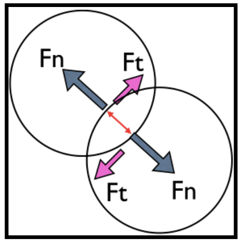


Outline

1. Introduction : Elasticity of jammed grains
2. Frictionless grains arXiv:2101.07473
3. **Frictional grains** Eur. Phys. J. E 44, 106 (2021)
4. Summary

Model : 2D frictional grains(DEM)



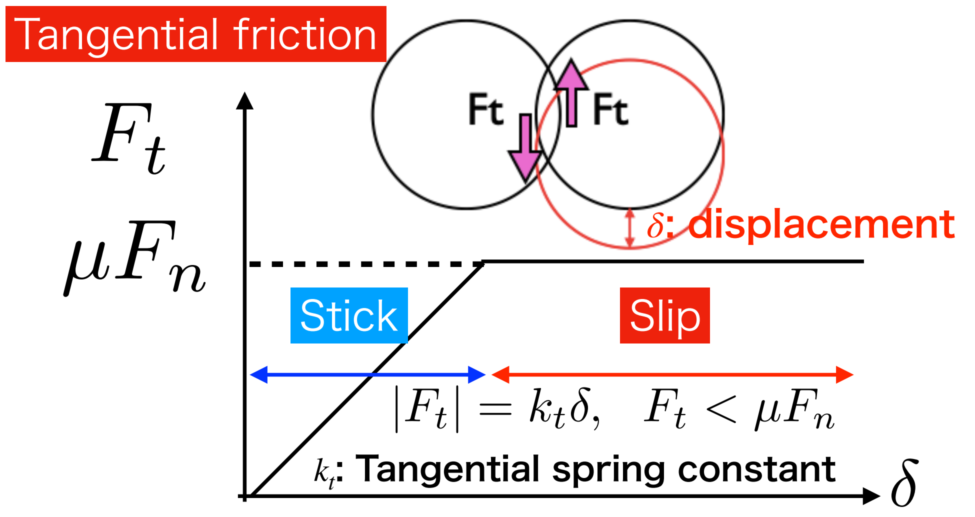


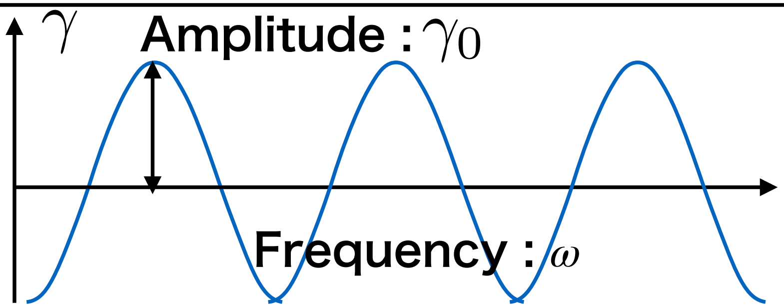
F_n : Normal force

F_t : Tangential friction

Coulomb law : $F_t \leq \mu F_n$

μ : Friction coefficient





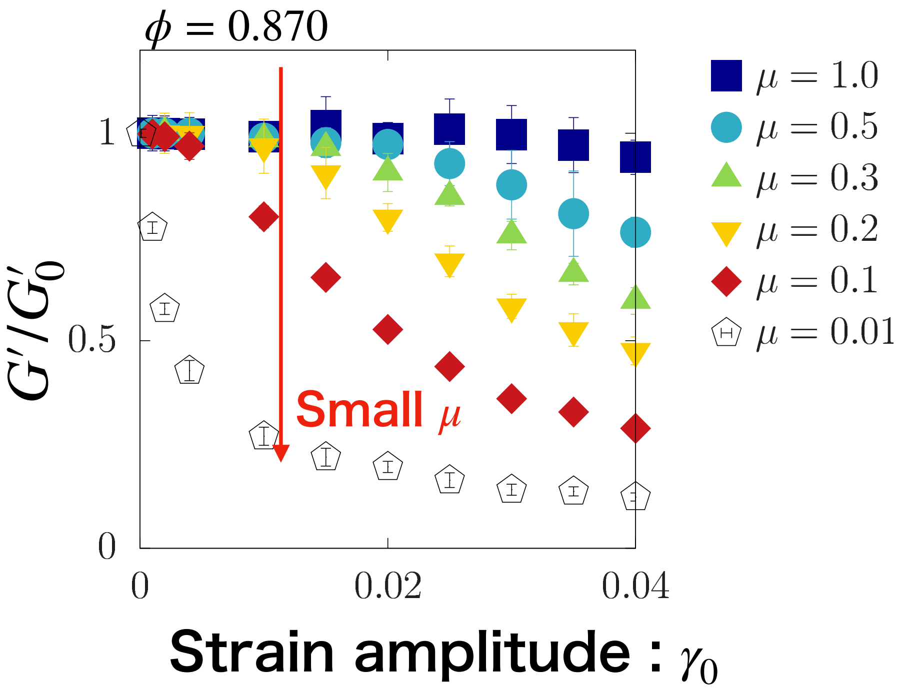
Strain : $\gamma(t) = \gamma_0 \sin(\omega t)$

$$G' = \frac{\omega}{\pi}\int_0^{2\pi/\omega} d\theta \, \frac{\sigma(t)\sin(\omega t)}{\gamma_0}$$

$$G'' = \frac{\omega}{\pi}\int_0^{2\pi/\omega} d\theta \, \frac{\sigma(t)\cos(\omega t)}{\gamma_0}$$

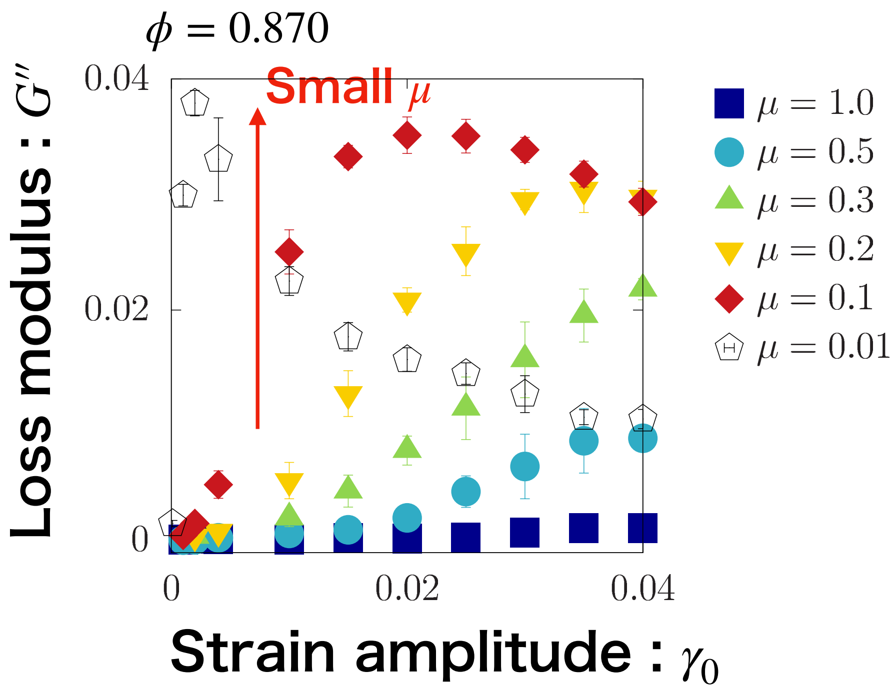
Rheology in the absorbing state

Storage modulus in absorbing state



G'_0 : Storage modulus in the limit of $\gamma_0 \to 0$.

Loss modulus in the absorbing state



Softening : G' decreases in the absorbing state.

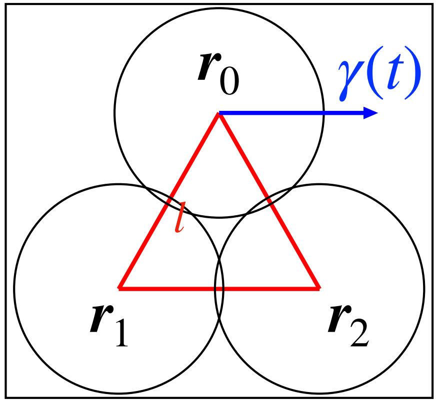
The softening becomes significant for large γ_0 and small μ.

Residual loss modulus : G'' remains in the absorbing state.

The loss modulus becomes significant for large γ_0 and small μ.

Explanation by 3 particle model

3 particle model:



$l = d(1-\epsilon)$: **Initial distance**

ϵ : **compressive strain** $\propto \phi - \phi_J$

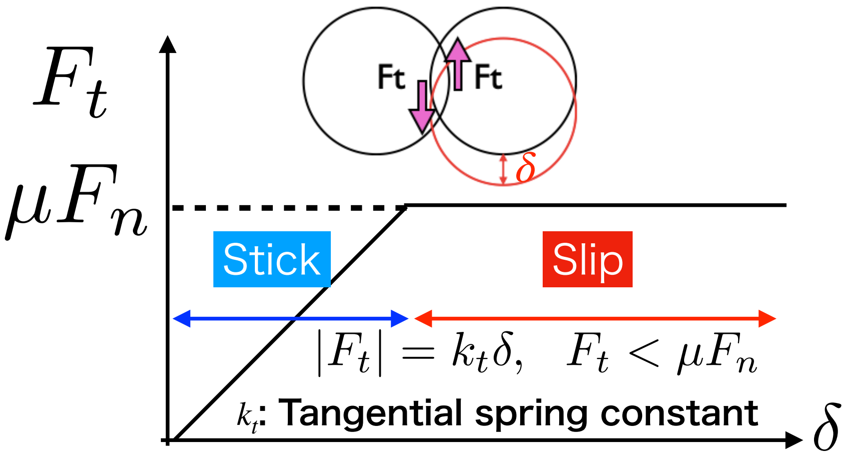
$$\boldsymbol{r}_0 = (\sqrt{3}\gamma(t)l/2,\ \sqrt{3}\gamma(t)l/2)$$

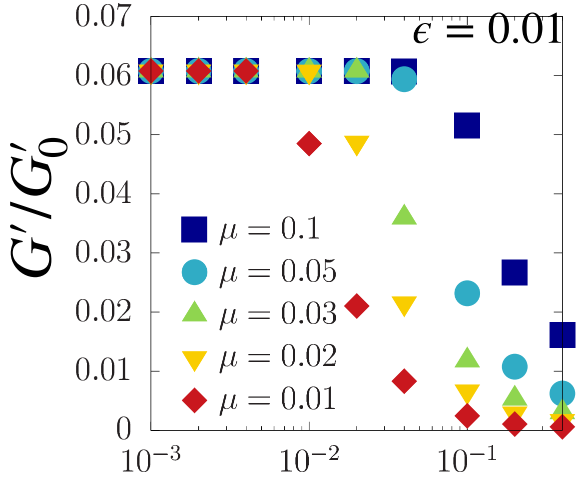
$$\boldsymbol{r}_1 = (-l/2,\ 0) \qquad \boldsymbol{r}_2 = (l/2,\ 0)$$

$$\gamma(t) = \gamma_0 \sin(\omega t)$$

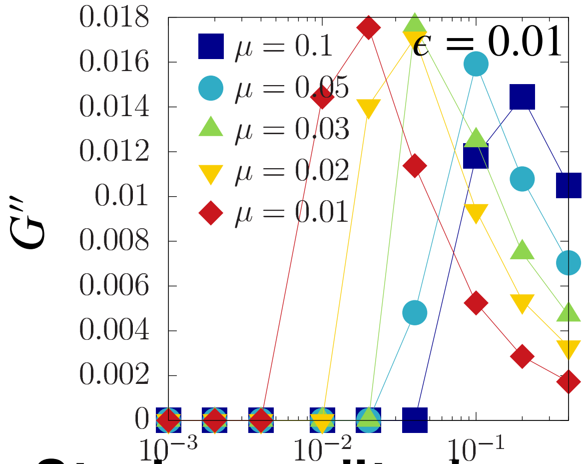
[Interaction force] $= \boldsymbol{F}_{ij}$ **used in DEM.**

Tangential friction: Origin of non-linear response





Strain amplitude : γ_0



Strain amplitude : γ_0

3 particle model reproduces the non-linear response in DEM.

Prediction of 3 particle model: scaling law

Analytical solution of 3 particle model:

Assumption: $\mu \ll \epsilon$

Strain for slip : $\gamma_s = \mu P / k_t$

$$\frac{G'(\gamma_0)}{G'_0} = \begin{cases} 1, & \gamma_0 < \alpha\gamma_s \quad \text{Stick} \\ \dfrac{1}{1+k_t/k_n}\left\{1 + (k_t/k_n) C(\gamma_0/\gamma_s)\right\}, & \gamma_0 > \alpha\gamma_s \quad \text{Slip} \end{cases}$$

$$G''(\gamma_0)/k_n = \begin{cases} 0, & \gamma_0 < \alpha\gamma_s \quad \text{Stick} \\ 3\alpha \dfrac{k_t}{k_n}\dfrac{1}{\pi}\left\{1 - \left(1 - 2\alpha\dfrac{\gamma_s}{\gamma_0}\right)^2\right\}, & \gamma_0 > \alpha\gamma_s \quad \text{Slip} \end{cases}$$

μ: **Friction coefficient**

P: **Pressure**

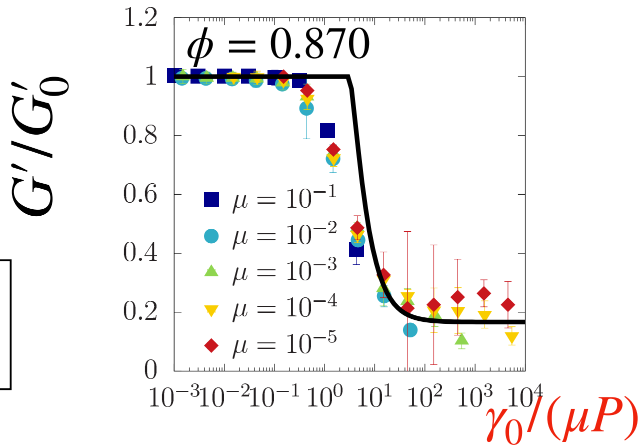
k_t: **Tangential spring constant**

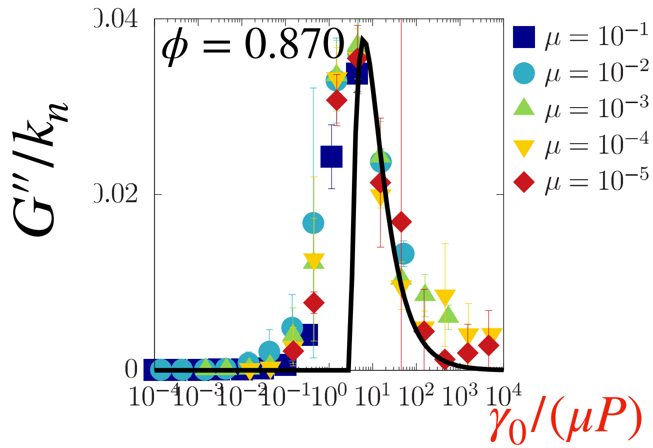
$$\alpha = \frac{8(1+\sqrt{3}/2)^2}{9}$$

$$\lim_{\xi\to 0} C(\xi) = 1, \quad \lim_{\xi\to\infty} C(\xi) = 0,$$

Prediction :

$$G'(\gamma_0, \mu, \phi)/G'_0 = F_1(\gamma_0/\mu P(\phi)) \qquad G''(\gamma_0, \mu, \phi)/k_n = F_2(\gamma_0/\mu P(\phi))$$





Points: DEM
Line: Analysis

The non-linear response results from the tangential friction.

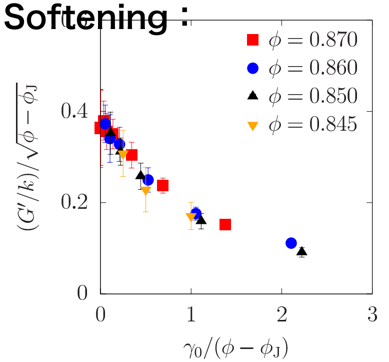
Outline

1. Introduction : Elasticity of jammed grains
2. Frictionless grains arXiv:2101.07473
3. Frictional grains Eur. Phys. J. E 44, 106 (2021)
4. **Summary**

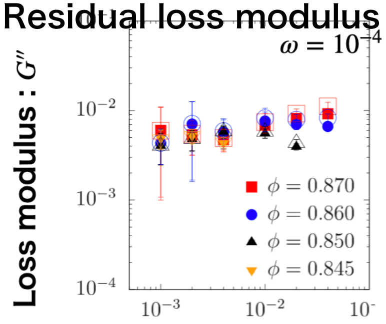
Summary

- Topic： Non-linear response of amorphous particles.
- The storage modulus decreases in the absorbing state. (Softening)
- The loss modulus remains in the absorbing state. (Residual loss modulus)
- Frictionless particles: The softening results from the configurational change during the transient to the absorbing state. The residual loss modulus originates from the loop trajectory.
- Frictional particles: The non-linear response results from the tangential friction between particles.

Softening：



Residual loss modulus：



Analysis (frictionless)：

$$\tilde{r}_i(\theta) = \sum_n \left\{ \boldsymbol{a}_i^{(n)} \sin n\theta + \boldsymbol{b}_i^{(n)} \cos n\theta \right\}$$

$$G_1' = \langle A' \rangle - \sum_i \left\langle \boldsymbol{B}_i' \cdot \frac{\boldsymbol{a}_i^{(1)}}{\gamma_0} \right\rangle$$

$$G_1'' = \sum_i \left\langle \boldsymbol{B}_i'' \cdot \frac{\boldsymbol{b}_i^{(1)}}{\gamma_0} \right\rangle$$

Analysis (frictional)：

$$G'/G_0' = F_1(\gamma_0/\mu P(\phi))$$

$$G''/k_n = F_2(\gamma_0/\mu P(\phi))$$