Scaling law for complex shear modulus in frictional granular materials

摩擦を持つ粉体における複素弾性率のスケーリング

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Rheology of granular materials

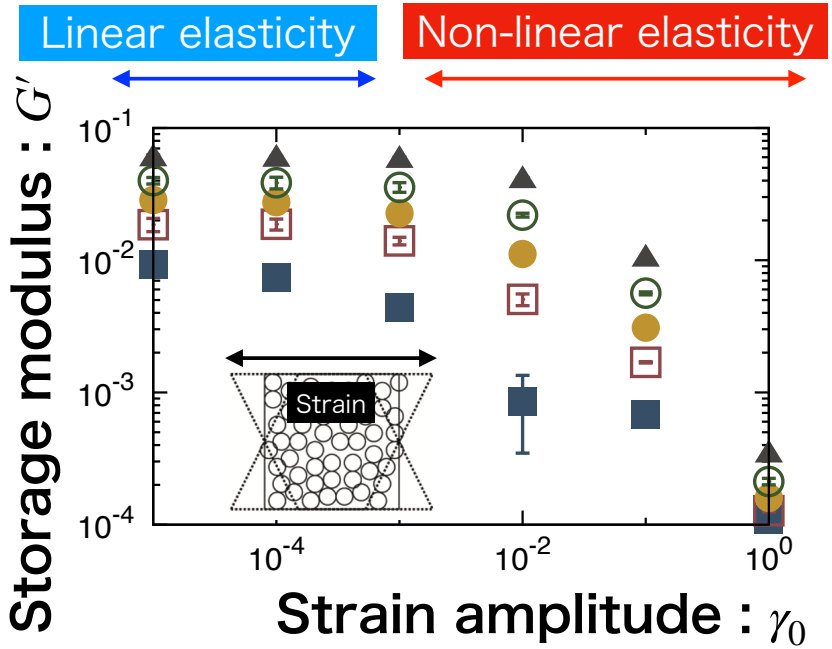
Dense granular materials have rigidity.

Shear modulus: G Packing fraction: ϕ

$G > 0$ for $\phi > \phi_J$

Jamming point: ϕ_J

Frictionless grains



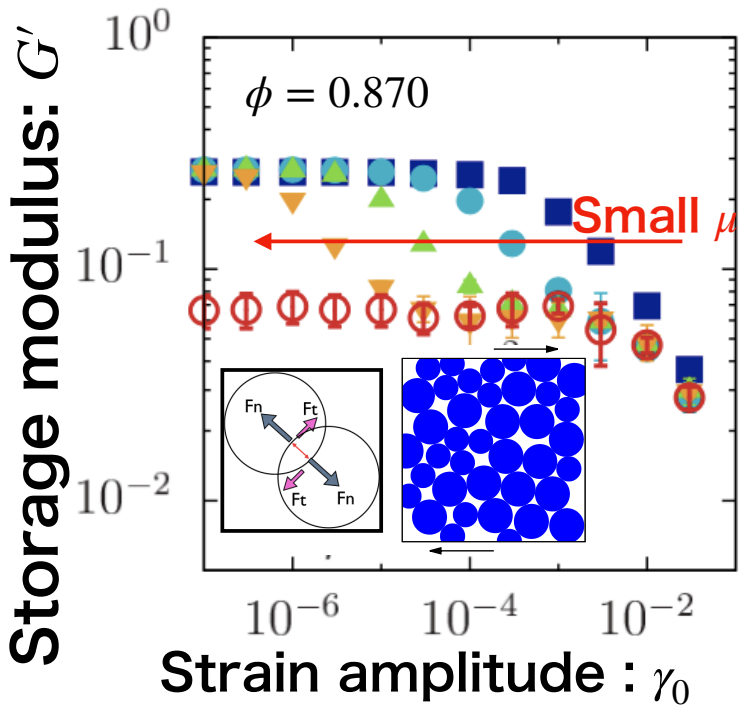
MO and H. Hayakawa, 90, 042202 (2014)

Frictionless grains exhibit non-linear response as γ_0 increases.

J. Boschan, et. al.,(2016), S. Dagois-Bohy, et. al., (2017),
T. Kawasaki and K. Miyazaki, (2020), MO and H. Hayakawa, arXiv:2101.07473

Nonlinear rheology of **frictional** grains

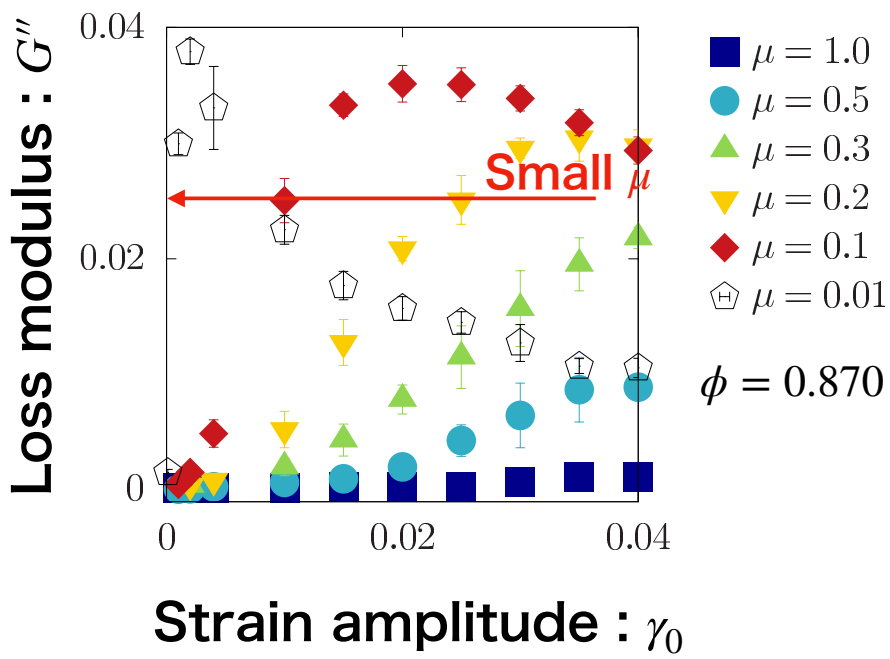
μ: Friction coefficient



MO and H. Hayakawa, Phys. Rev. E 95, 062902 (2017)

- G' decreases as γ_0 increases.
- The critical strain depends on μ.
- G' exhibits two step relaxation.

The number of grains : $N \geq 1000$



MO and H. Hayakawa, Eur. Phys. J. E 44, 106 (2021)

- G'' becomes finite as γ_0 increases.
- The critical strain depends on μ.
- G'' has a peak for small μ.

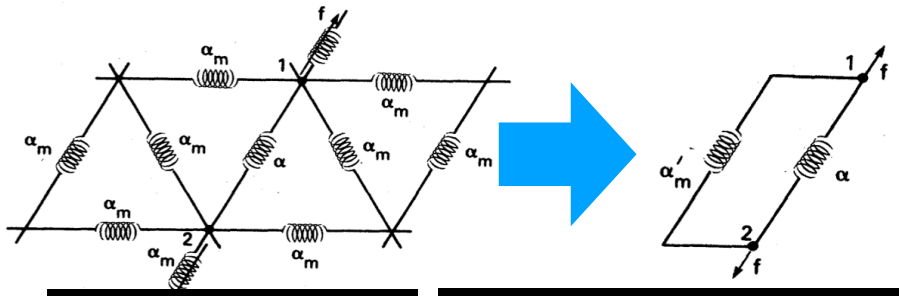
Theoretical analysis is difficult Disorder, non-affine motion, friction, etc.

Approach: simple effective model

Example: Mean field theory for Ising model

Elastic response of amorphous solid

Effective medium theory

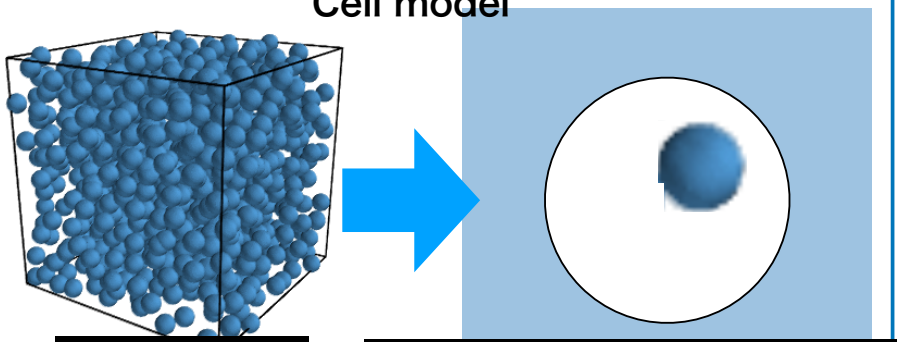


Random spring network | Two nodes with effective spring

Feng, Thorpe, and Garboczi, Phys. Rev. B 31, 276 (1995)

Gas-liquid transition

Cell model

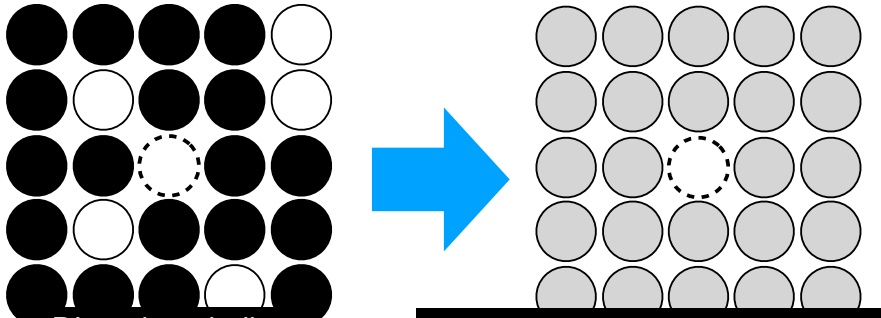


Many molecules | One molecule in an effective media

Lennard-Jones, Devonshire, Proc. R. Soc. Lond A 63, 53 (1937)

Electric band structure for disordered solid

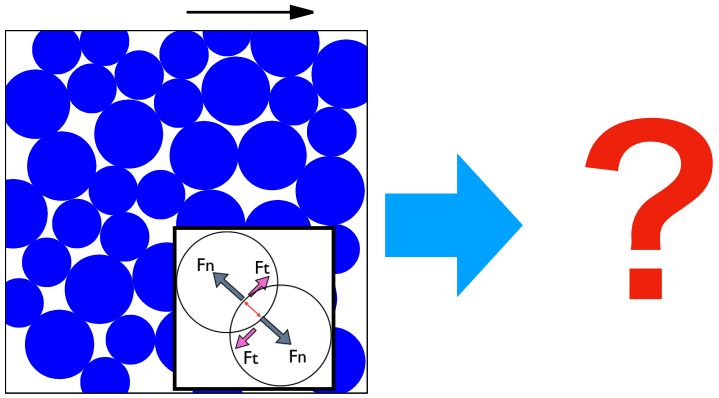
Coherent potential approximation



Disordered alloy | Homogeneous effective medium

Yonezawa, Morigaki, Prog. Theor. Supple. 53, 1 (1973)

Rheology of frictional grains under shear

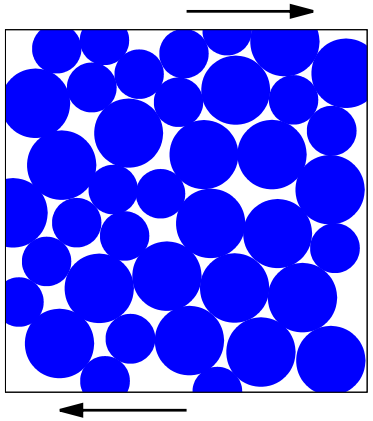


Many grains

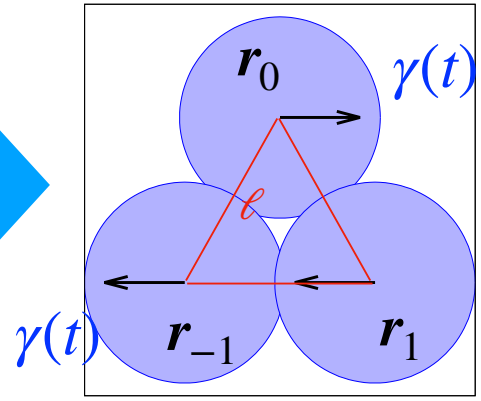
?

3 particle model

Many particle system:
(DEM simulation)



3 particle model:



$l = d(1-\epsilon)$: **Initial distance**

ϵ : **compressive strain** $\propto \phi - \phi_J$

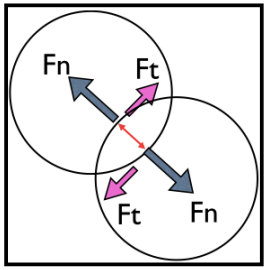
$\boldsymbol{r}_0 = (\sqrt{3}\gamma(t)l/4,\ \sqrt{3}l/4)$

$\boldsymbol{r}_{\pm 1} = (\pm l/2 - \sqrt{3}\gamma(t)l/4,\ -\sqrt{3}l/4)$

Shear strain: $\gamma(t) = \gamma_0 \sin(\omega t)$

γ_0: **Amplitude**, ω: **Frequency**

Particle interaction



The type of the force is the same as in DEM.

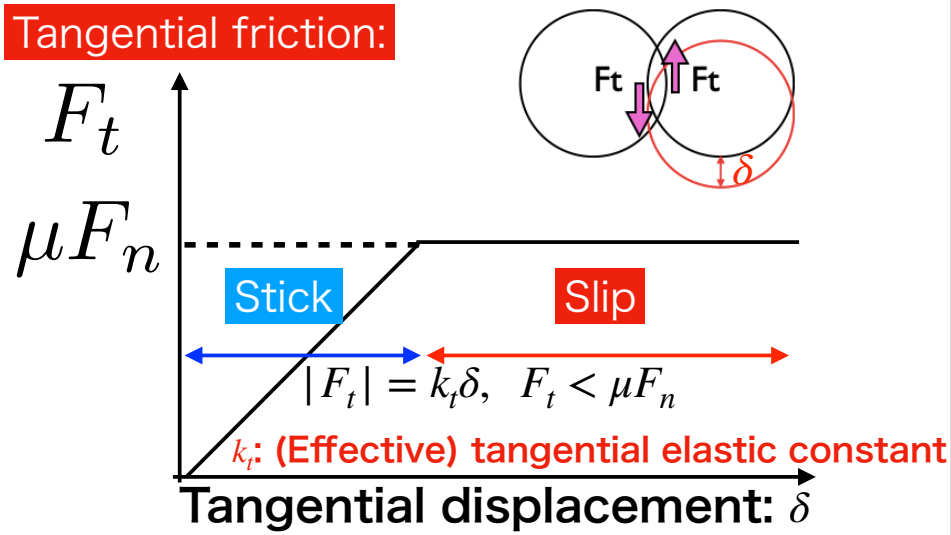
F_n : **Normal repulsive force**

F_t : **Tangential friction**

Coulomb's law $F_t \leq \mu F_n$

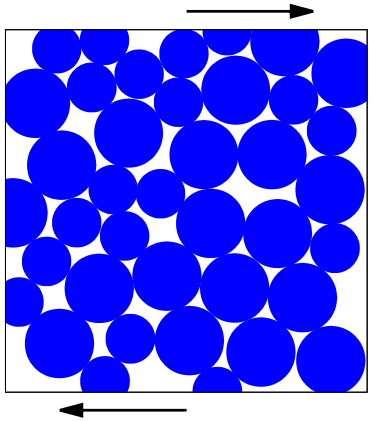
μ : **Friction coefficient**

Tangential friction:

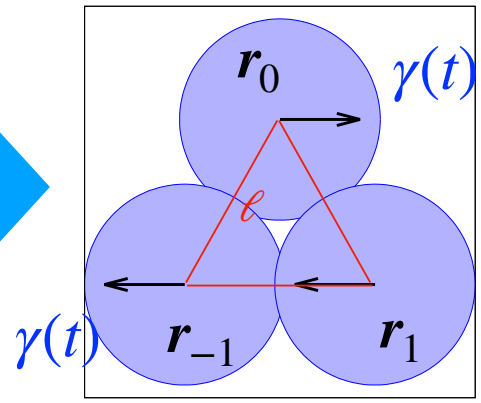


Shear modulus in 3 particle model

Many particle system:
(DEM simulation)



3 particle model:



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Shear strain: $\gamma(t) = \gamma_0 \sin(\omega t)$

γ_0: **Amplitude**, ω: **Frequency**

Interaction force: $\boldsymbol{F}_{ij}$ ⟶ **Shear stress:** $\sigma(t)$, **Pressure:** P

Storage modulus

$$G' = \frac{\omega}{\pi}\int_0^{2\pi/\omega} d\theta\ \frac{\sigma(t)\sin(\omega t)}{\gamma_0}$$

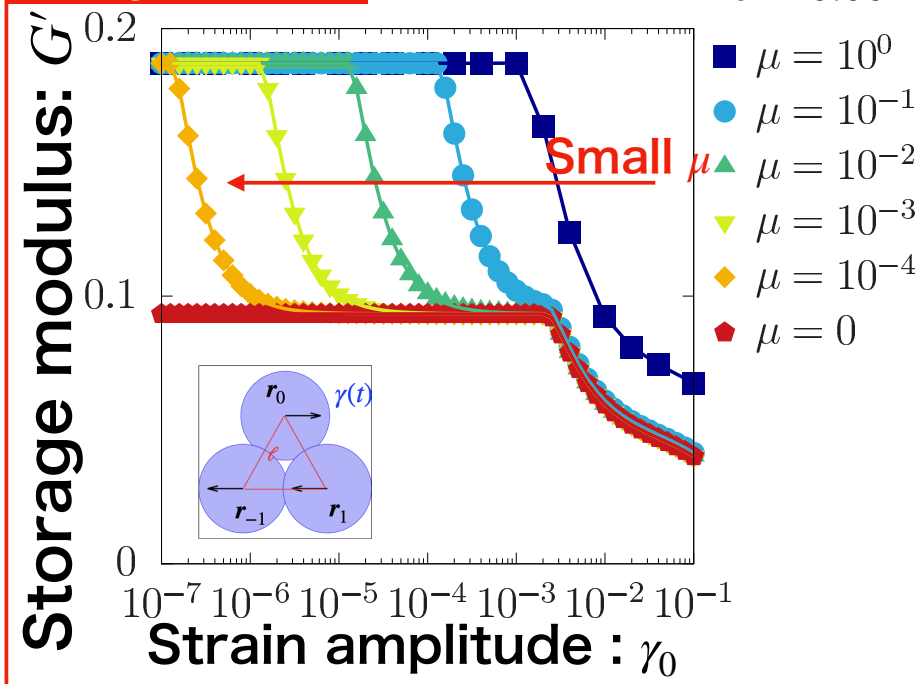
Loss modulus

$$G'' = \frac{\omega}{\pi}\int_0^{2\pi/\omega} d\theta\ \frac{\sigma(t)\cos(\omega t)}{\gamma_0}$$

G' and G'' in 3 particle model

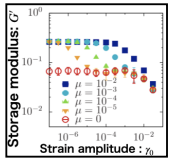
3 particle model qualitatively reproduces results in many particle system.

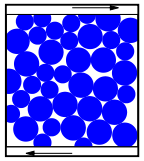
Storage modulus



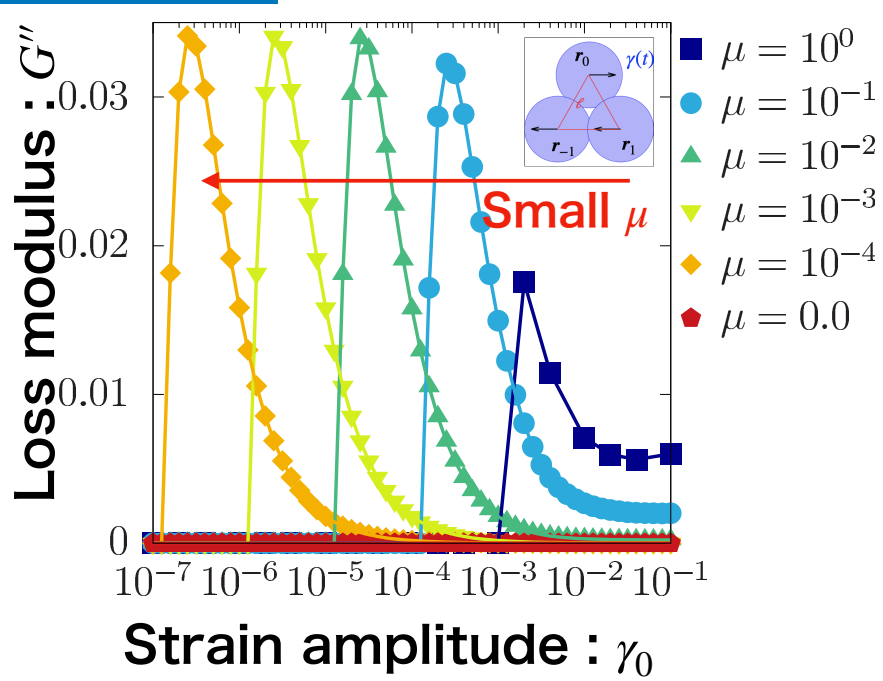
- G' decreases as γ_0 increases.
- The critical strain depends on μ.
- G' exhibits two step relaxation.

Many particle system: (DEM simulation)



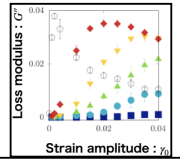


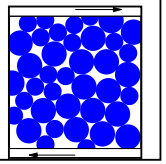
Loss modulus



- G'' becomes finite as γ_0 increases.
- The critical strain depends on μ.
- G'' has a peak for small μ.

Many particle system: (DEM simulation)

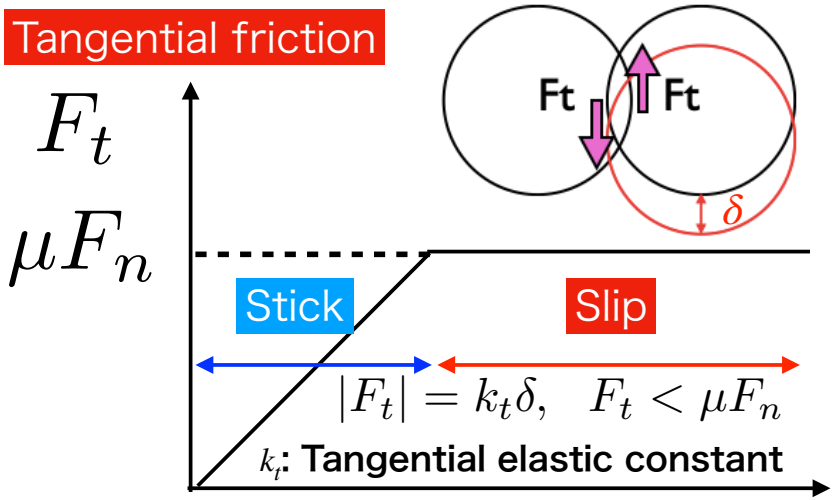




Origin of nonlinear response

Decrease of G' and finite G'' result from the transition in tangential friction.

Tangential friction



Tangential displacement: δ

Critical strain for slip

Displacement for slip : $\delta_c = \mu F_n / k_t$

Pressure : $P \sim F_n / d$

Normal force: F_n, Diameter of grain: d

Strain for slip : $\gamma_c \sim \delta_c / d \sim \mu P / k_t$

Analytical solution:

Assumption: $\mu \ll 1$

$$\gamma_c = \frac{9}{32}\frac{\mu P}{k_t}$$

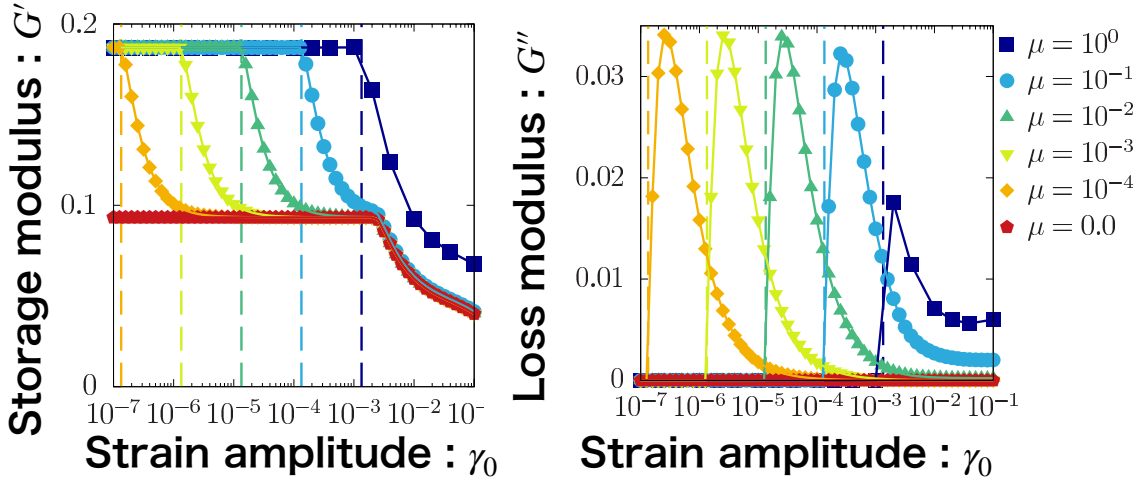
Dashed lines: γ_c for each μ

G' decreases above γ_c.

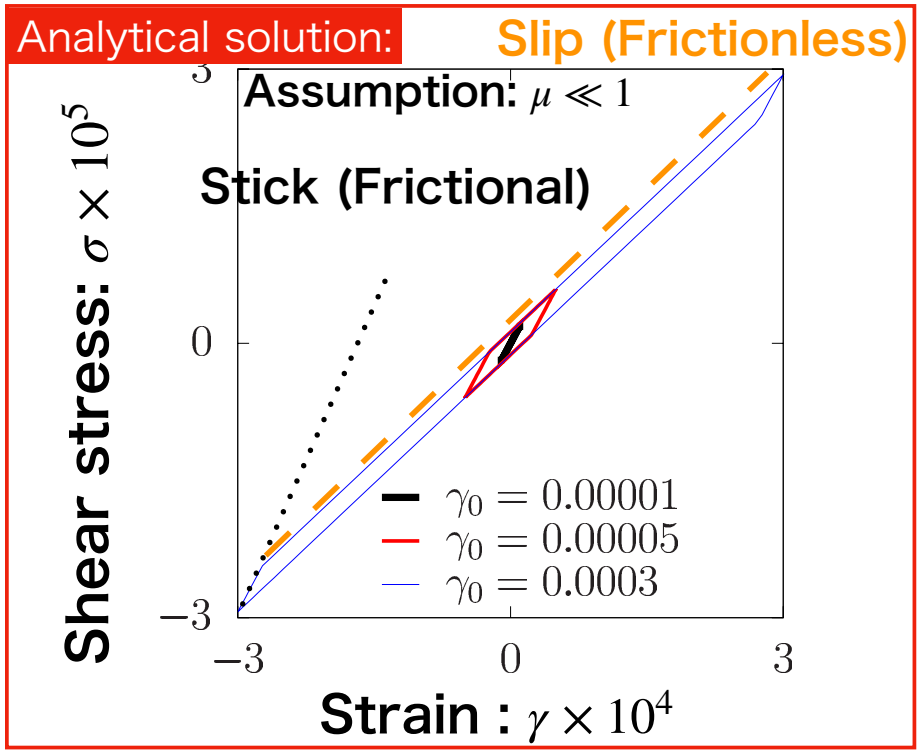
G'' becomes finite above γ_c.

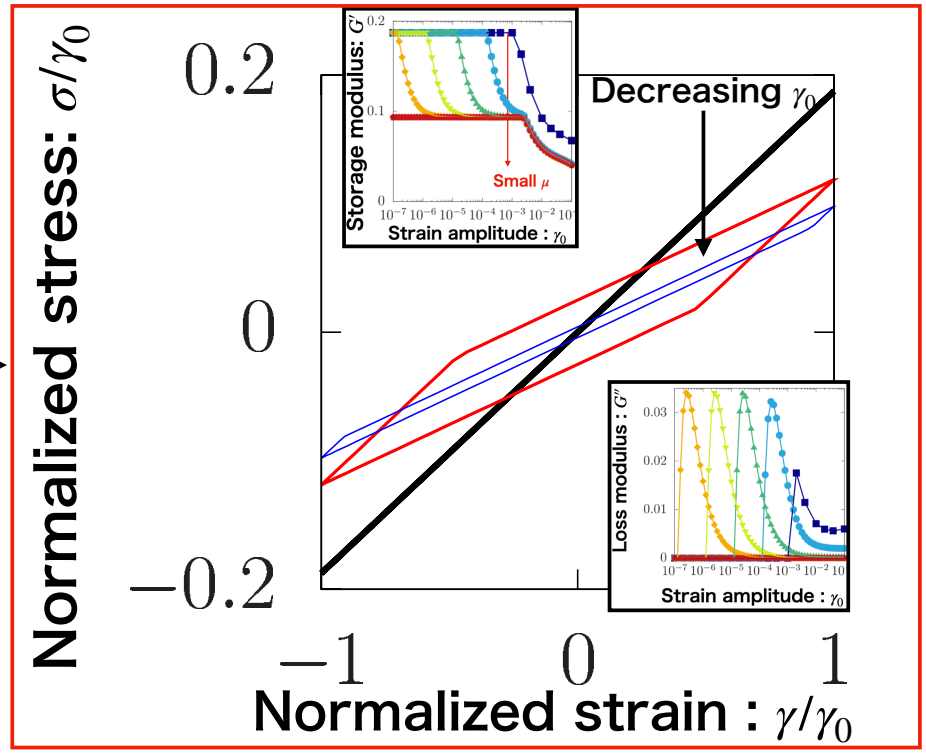
↓

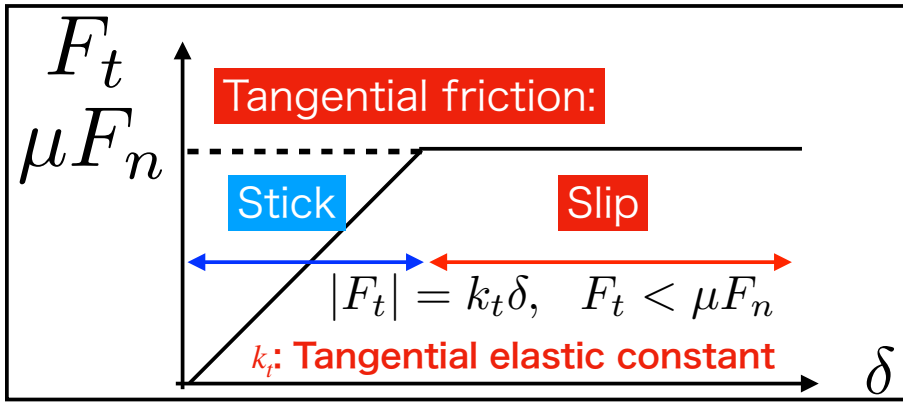
Origin of nonlinear response : Slip in tangential friction.



Theoretical explanation for γ_0 dependence







$$G' = \frac{\omega}{\pi}\int_0^{2\pi/\omega} d\theta \, \frac{\sigma(t)\sin(\omega t)}{\gamma_0} \simeq \text{[Maximum of } \frac{\sigma}{\gamma_0}\text{]}$$

G' decreases as γ_0 increases.

$$G'' = \frac{\omega}{\pi}\int_0^{2\pi/\omega} d\theta \, \frac{\sigma(t)\cos(\omega t)}{\gamma_0} = \text{[Area of } \frac{\sigma}{\gamma_0} vs \frac{\gamma}{\gamma_0}\text{]}/\pi$$

There is a peak in G''.

Prediction by 3 particle model: scaling law

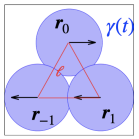
Analytical solution of 3 particle model:

Assumption: $\mu \ll 1$

Prediction: Scaling laws for G' and G''

$$G'(\gamma_0, \mu, \phi) = G'_M(\mu, \phi)\ \mathcal{F}_1\left(\boxed{\frac{k_t \gamma_0}{\mu P(\mu, \phi)}}\right) \sim \gamma_0/\gamma_c$$

$$G''(\gamma_0, \mu, \phi) = G''_M(\mu, \phi)\ \mathcal{F}_2\left(\frac{k_t \gamma_0}{\mu P(\mu, \phi)}\right)$$



Critical strain: $\gamma_c = \frac{9}{32}\frac{\mu P}{k_t}$

μ: **Friction coefficient** P: **Pressure**

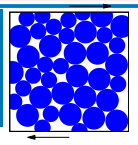
k_t: **Tangential elastic constant**

$G'_M(\mu, \phi)$: **Maximum value of** G'

$G''_M(\mu, \phi)$: **Maximum value of** G''

$\mathcal{F}_1(x)$, $\mathcal{F}_2(x)$: **Scaling functions**

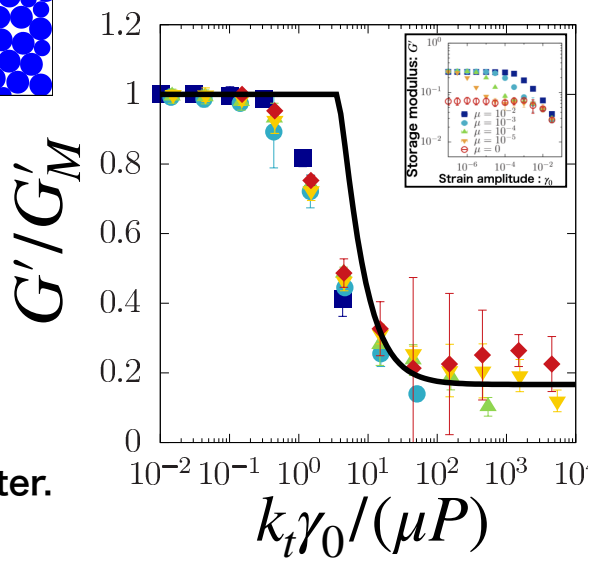
Many particle system: (DEM simulation)

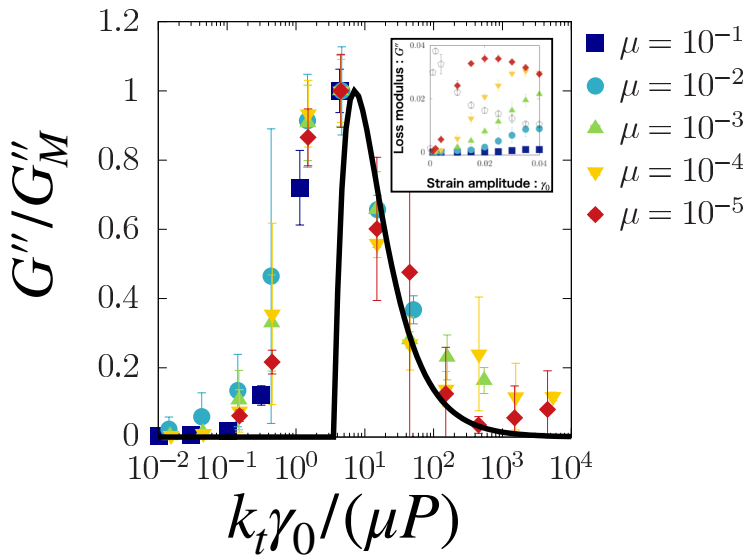


Scaling laws are satisfied in DEM.

Lines: 3 particle model

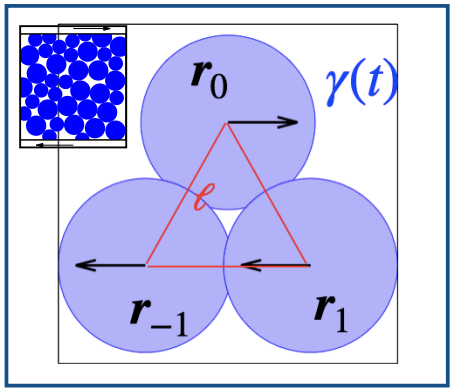
k_t **is used as a fitting parameter.**

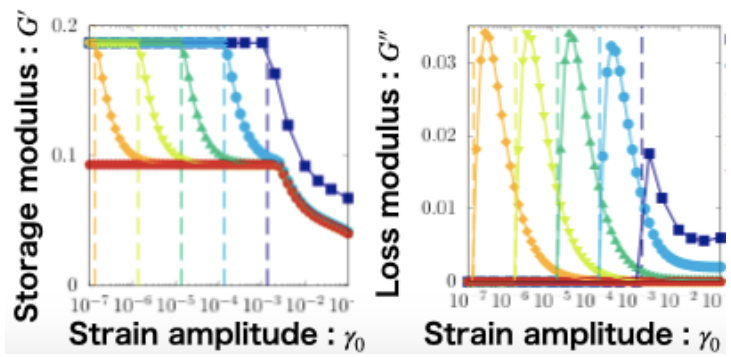




Summary

- Topic : Non-linear rheology of frictional granular materials.
- We have proposed 3 particle model.
- 3 particle model qualitatively reproduces G' and G'' in DEM simulations.
- Non-linear response results from the transition in the tangential friction.
- We derive scaling laws, which are satisfied in DEM simulations.
- Problem: Disorder and non-affine motions are neglected.
- Future work: Self-consistent determination of a fitting parameter k_t.





Scaling laws:

$$G' = G'_M \, F_1\left(\frac{k_t \gamma_0}{\mu P(\mu, \phi)}\right)$$

$$G'' = G''_M \, \mathscr{F}_2\left(\frac{k_t \gamma_0}{\mu P(\mu, \phi)}\right)$$