2022-9-13
@TITECH

幾何学的位相に駆動されたデーモン

H Hayakawa and R Yoshii, arXiv:2205.15193 (2022).

(H Hayakawa, VMM Paasonen, R Yoshii, arXiv:2112.12370 (2021).)

吉井 涼輔

(山陽小野田市立山口東京理科大学)

共同研究者

早川尚男 (京都大学)

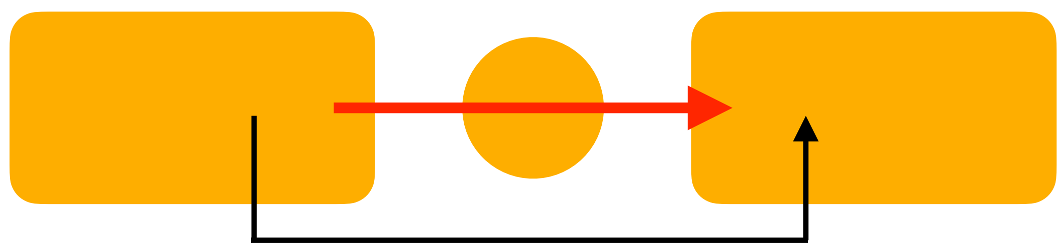
(Ville M. M. Paasonen (Univ. Turku))

導入: Thouless pumping

トポロジカルポンプ

平均電圧がゼロでも電流が生じ得る

Mean current!

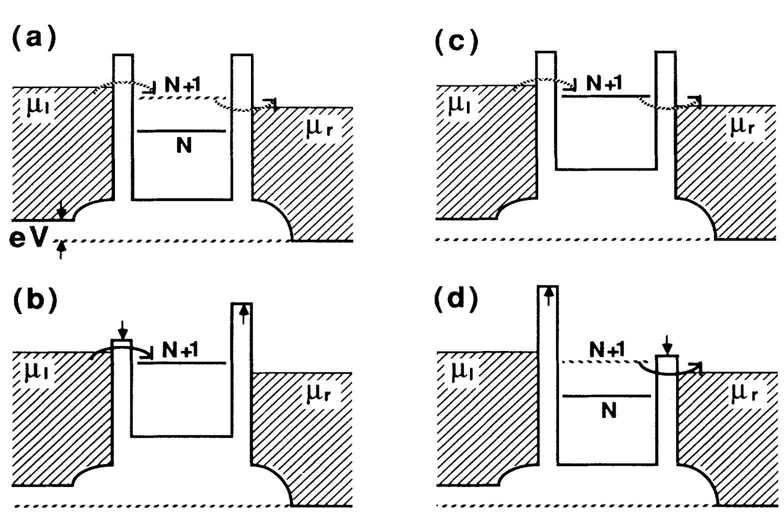


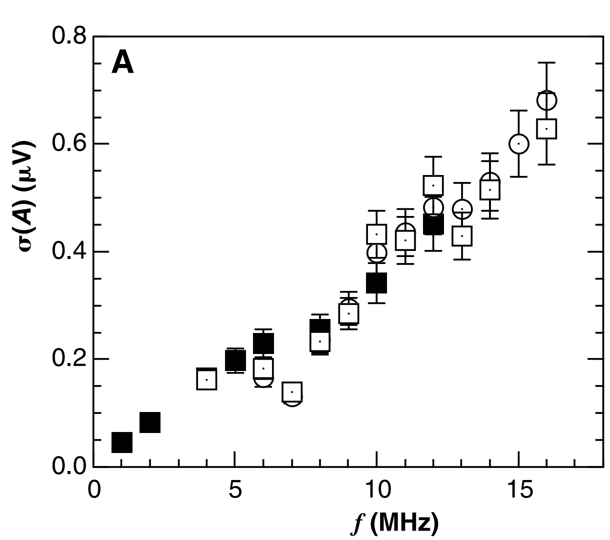
$$\langle V_{\mathrm{bias}}\rangle_{\text{time average}} = 0$$

Thouless, 1983

量子ドット系

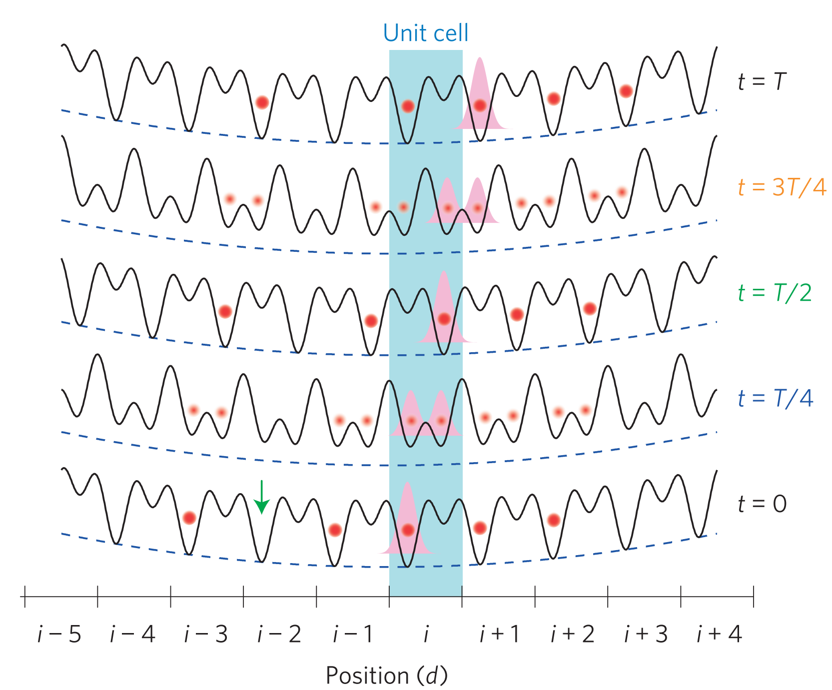
M. Switkes, et.al., Science, **283**, 1905 (1999).



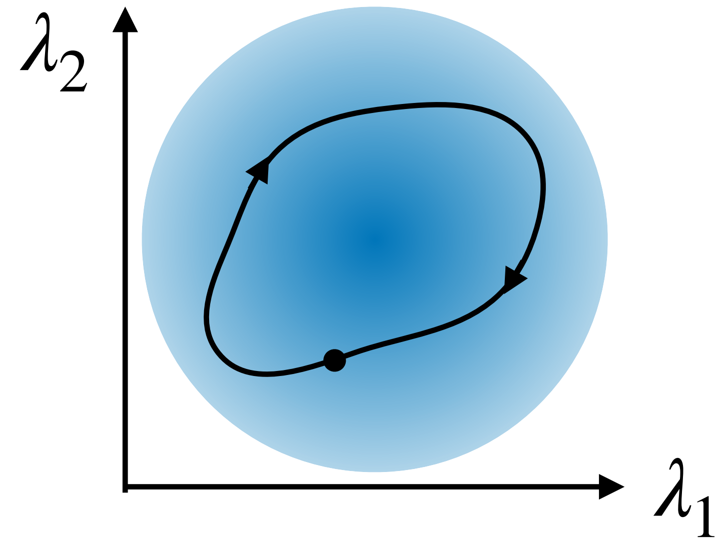


冷却原子系

S. Nakajima, et.al., Nat. Phys., **12**, 296 (2016).



幾何学的な解釈



パラメータ空間内の非自明な曲率

Gauss-Bonnet型の議論

⇨ カレント ~ トラジェクトリで囲まれた部分の曲率の和

Berry-Sinitsyn-Nemenman(BSN)曲率

N. A. Sinitsyn and I. Nemenman PRL **99**, 220408 (2007).

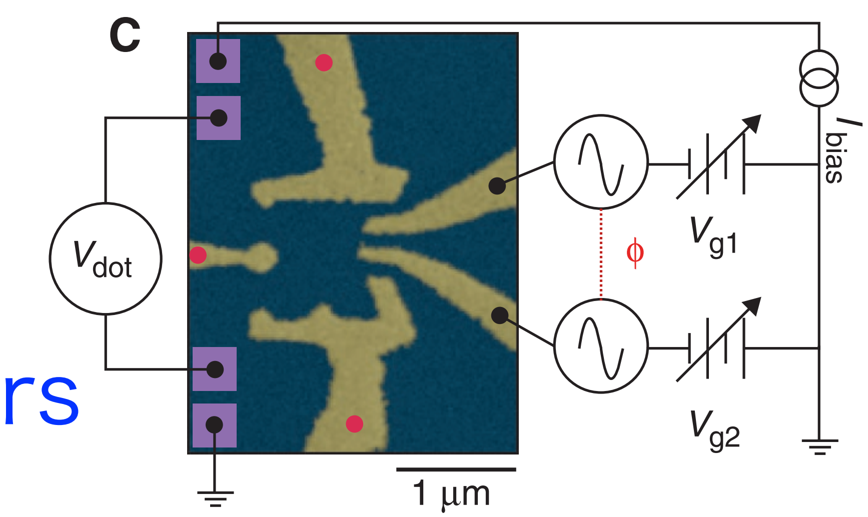
熱力学にベクトルポテンシャルが現れる(スカラーでは不十分)

K. Tomita and H. Tomita PTP **51**, 6 (1974).

熱力学の幾何学的な解釈

- Brandner-Saito (PRL 2020) : 熱浴1つ
- Hino-Hayakawa (PRR2021): 熱浴2つ
- Ito-Dechant (PRX2020): 確率過程

本研究の目的



実験可能な系

量子ドット系 Quantum dot + reservoirs

クーロン相互作用するフェミ流粒子系+幾何学的効果

量子効果?　多体効果?　フェルミ統計?

が熱力学にどう影響を及ぼすか

仕事? 効率? エントロピー生成?

前回の発表(16aB14-3 日本物理学会@岡山大)

エントロピー生成がFisher情報量やHesse行列で書ける

マスター方程式

密度行列ρの時間発展

$$\frac{d\rho}{d\theta} = \hat{\mathscr{K}}\rho$$

$\hat{\mathscr{K}}$: ρに作用する線形作用素
(CPTP性を満たす)

θ: 無次元化した時間

$\hat{\mathscr{K}}$: 周期駆動パラメータを含む

$|r_i(\theta)\rangle\rangle$: $\hat{\mathscr{K}}$の右固有状態

$\langle\langle \ell_i(\theta)|$: $\hat{\mathscr{K}}$の左固有状態

ε_i: $\hat{\mathscr{K}}$の固有値

幾何学的状態

$$|\rho(\theta)\rangle\rangle = |r_0(\theta)\rangle\rangle - \sum_{i\neq 0}\int_0^\theta d\phi e^{\int_\phi^\theta d\chi \epsilon^{-1}\varepsilon_i(\chi)}\mathscr{A}_\mu \frac{d\Lambda^\mu}{d\phi}|r_i(\theta)\rangle\rangle$$

$$\mathscr{A}_\mu = \langle\langle \ell_i(\phi)|\frac{\partial}{d\Lambda^\mu}|r_0(\phi)\rangle\rangle$$

$$\mathscr{F}^i_{\mu\nu} \equiv \frac{\partial \mathscr{A}^i_\nu}{\partial \Lambda_\mu} - \frac{\partial \mathscr{A}^i_\mu}{\partial \Lambda_\nu}$$

幾何学的状態

$$|\rho(\theta)\rangle\rangle = \underline{|r_0(\theta)\rangle\rangle} - \sum_{i\neq 0}\int_0^\theta \underline{d\phi\, e^{\int_\phi^\theta d\chi \epsilon^{-1}\varepsilon_i(\chi)}} \mathscr{A}_\mu \frac{d\Lambda^\mu}{d\phi}|r_i(\theta)\rangle\rangle$$

$$\mathscr{A}_\mu = \langle\langle \ell_i(\phi)| \frac{\partial}{d\Lambda^\mu}|r_0(\phi)\rangle\rangle$$

$$\mathscr{F}^i_{\mu\nu} \equiv \frac{\partial \mathscr{A}^i_\nu}{\partial \Lambda_\mu} - \frac{\partial \mathscr{A}^i_\mu}{\partial \Lambda_\nu}$$

定常状態

指数減衰

幾何学的状態

$$|\rho(\theta)\rangle\rangle = \underline{|r_0(\theta)\rangle\rangle} - \sum_{i\neq 0}\int_0^\theta \underline{d\phi\, e^{\int_\phi^\theta d\chi \epsilon^{-1}\varepsilon_i(\chi)}}\, \mathscr{A}_\mu \frac{d\Lambda^\mu}{d\phi}|r_i(\theta)\rangle\rangle$$

$$\mathscr{A}_\mu = \langle\langle \ell_i(\phi)|\frac{\partial}{d\Lambda^\mu}|r_0(\phi)\rangle\rangle$$

$$\mathscr{F}^i_{\mu\nu} \equiv \frac{\partial \mathscr{A}^i_\nu}{\partial \Lambda_\mu} - \frac{\partial \mathscr{A}^i_\mu}{\partial \Lambda_\nu}$$

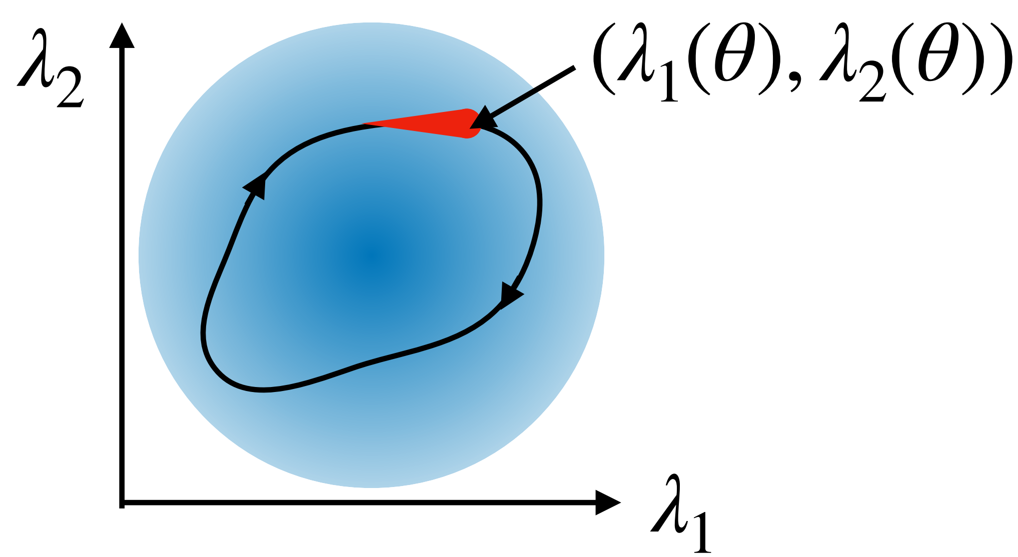
定常状態

指数減衰

$\theta - \epsilon \le \phi \le \theta$だけが寄与

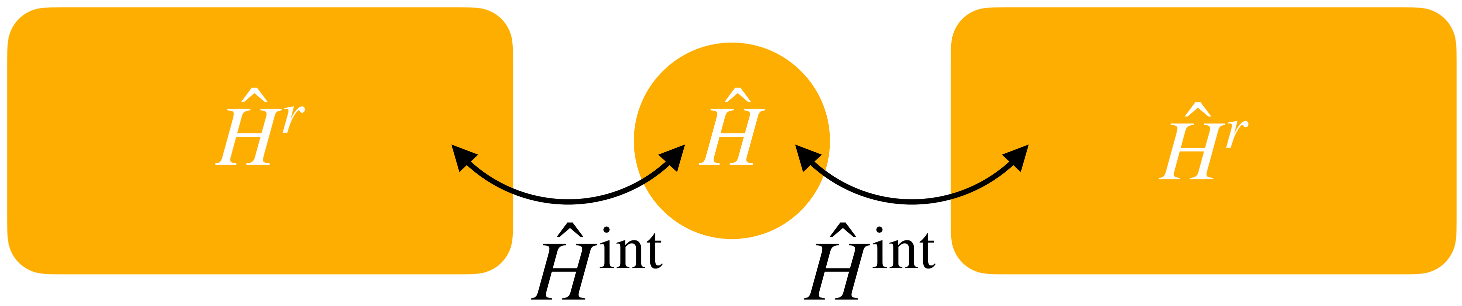
時間的に“ほぼ”局所的になる

“準”定常的



不純物Andersonモデル

$$\hat{H}^{\text{tot}} = \hat{H} + \hat{H}^r + \hat{H}^{\text{int}}$$



$$\hat{H} = \sum_{\sigma} \epsilon_0 \hat{d}^\dagger_\sigma \hat{d}_\sigma + U \hat{n}_\uparrow \hat{n}_\downarrow \quad \left(\hat{n}_{\uparrow/\downarrow} = \hat{d}^\dagger_{\uparrow/\downarrow} \hat{d}_{\uparrow/\downarrow} \right)$$

$\hat{d}^\dagger_\sigma$ $(\hat{d}_\sigma)$: ドット内のスピンσの電子の生成消滅演算子

$$\hat{H}^r = \sum_{\alpha,k,\sigma} \epsilon_k \hat{a}^\dagger_{\alpha,k,\sigma} \hat{a}_{\alpha,k,\sigma}$$

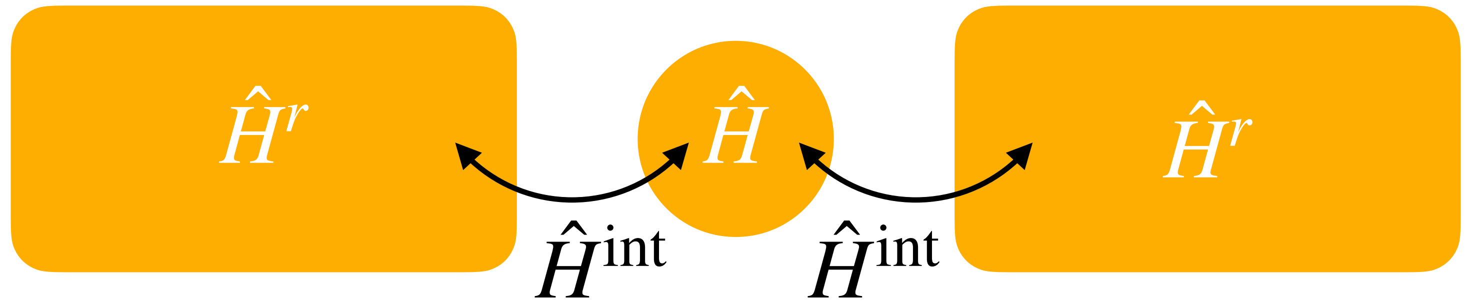
$\hat{a}^\dagger_{\alpha,k,\sigma}$ $(\hat{a}_{\alpha,k,\sigma})$: 熱浴の電子の生成消滅演算子

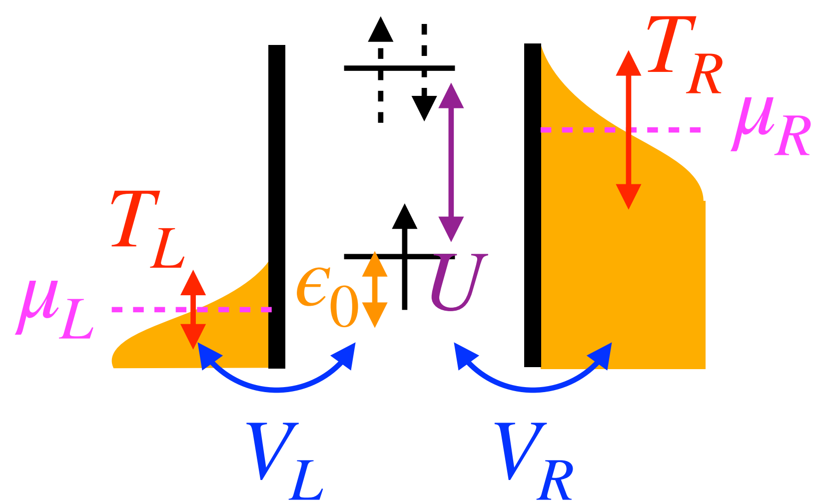
(スピンσ, 波数k, $\alpha = \text{left or right}$)

$$\hat{H}^{\text{int}} = \sum_{\alpha,k,\sigma} V_\alpha \hat{d}^\dagger_\sigma \hat{a}_{\alpha,k,\sigma} + \text{h.c.},$$

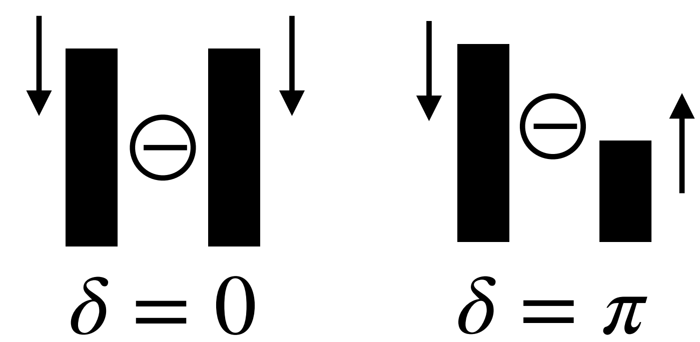
(簡単のため$V_L = V_R$)

変調可能なパラメータ



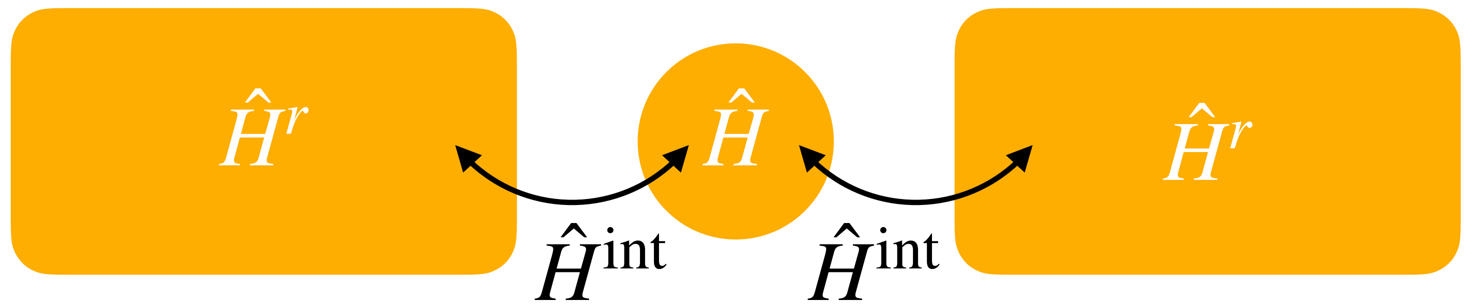


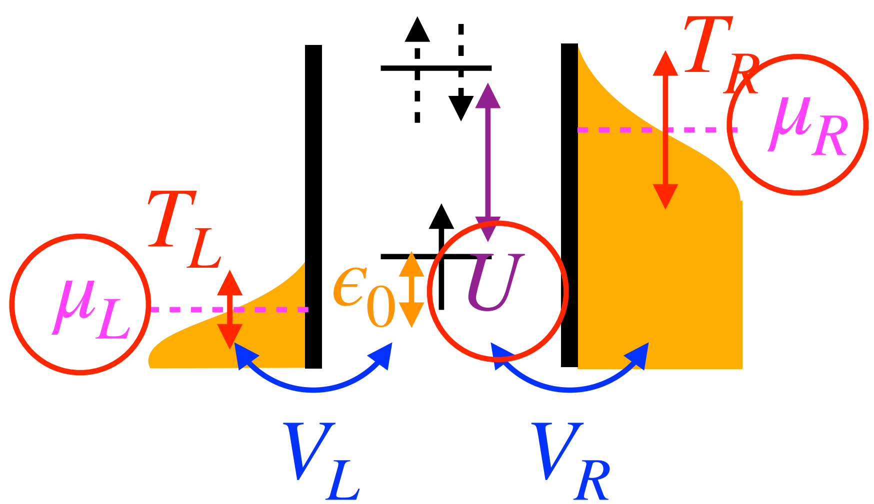
δ: 左右の変調の位相差



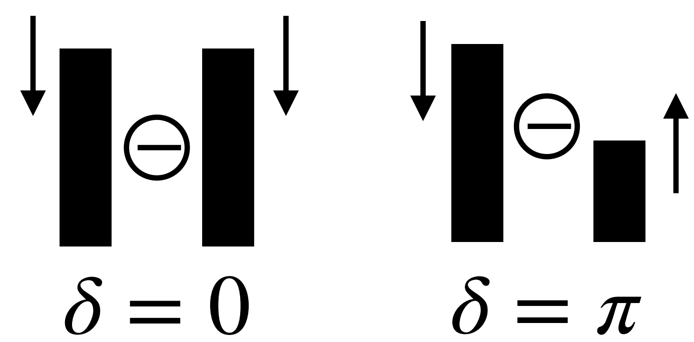
- 実験的には温度制御は難しい
- 仕事を取り出すには量子ドット内のパラメータの変調が必要

変調可能なパラメータ





δ: 左右の変調の位相差

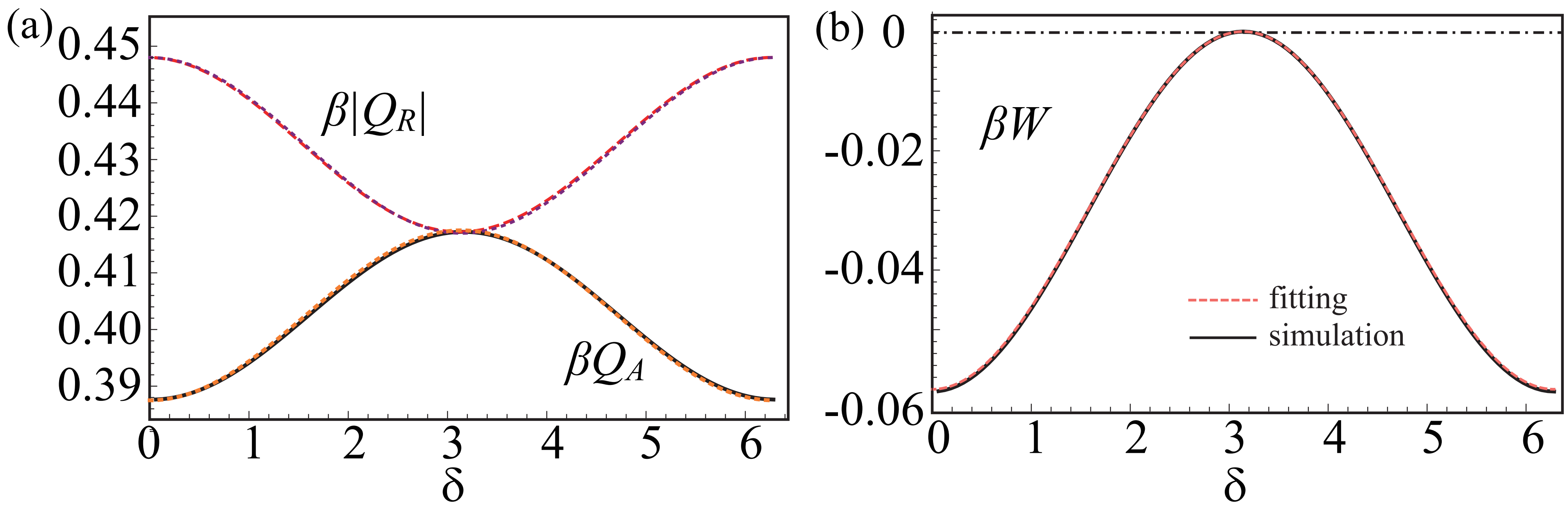


- 実験的には温度制御は難しい
- 仕事を取り出すには量子ドット内のパラメータの変調が必要

U, μ_L, μ_Rを変調させる

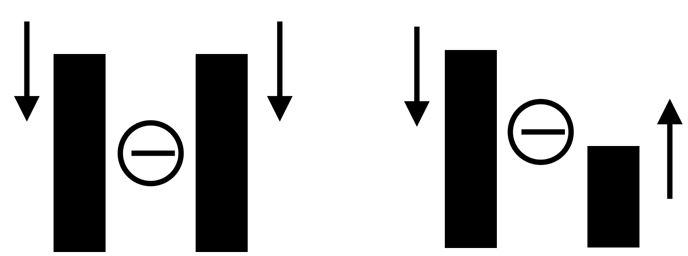
吸熱, 発熱, 仕事

$$Q_A = \oint \frac{\mathscr{P}_\mu + \left|\mathscr{P}_\mu\right|}{2} d\Lambda^\mu, \quad Q_R = \oint \frac{\mathscr{P}_\mu - \left|\mathscr{P}_\mu\right|}{2} d\Lambda^\mu, \quad W = \oint \mathscr{P}_\mu d\Lambda^\mu, \quad \mathscr{P}_\mu = \mathrm{Tr}\left(\rho \frac{\partial H}{\partial \Lambda^\mu}\right)$$

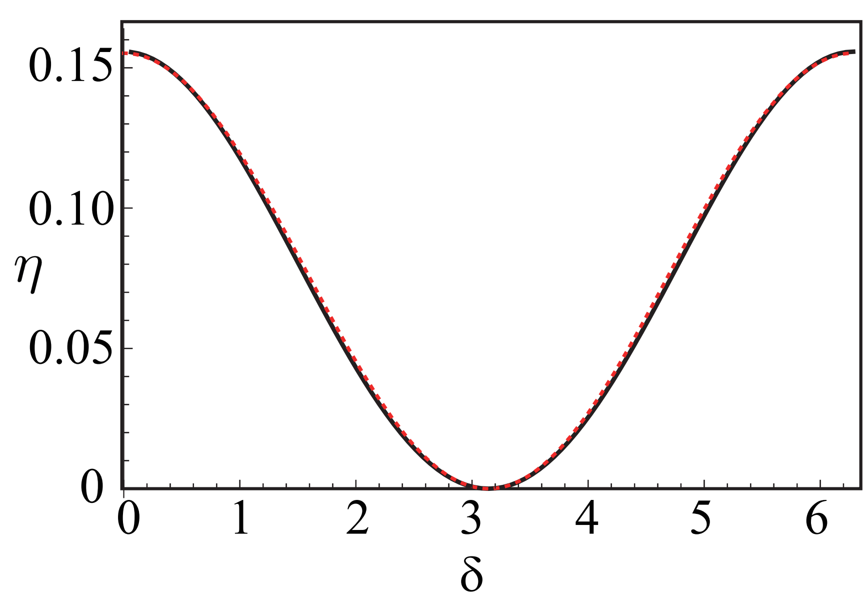


負の仕事: 幾何学的デーモン

効率: $\eta \equiv \dfrac{|W|}{Q_A} = \dfrac{|Q_A + Q_R|}{Q_A}$



$\delta = 0$ $\qquad$ $\delta = \pi$



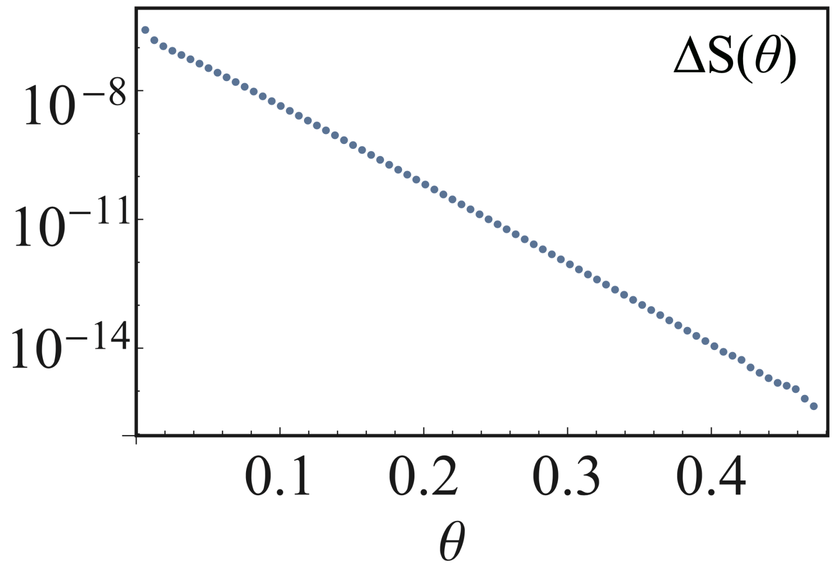
相対エントロピーの時間発展

θ : 無次元化した時間

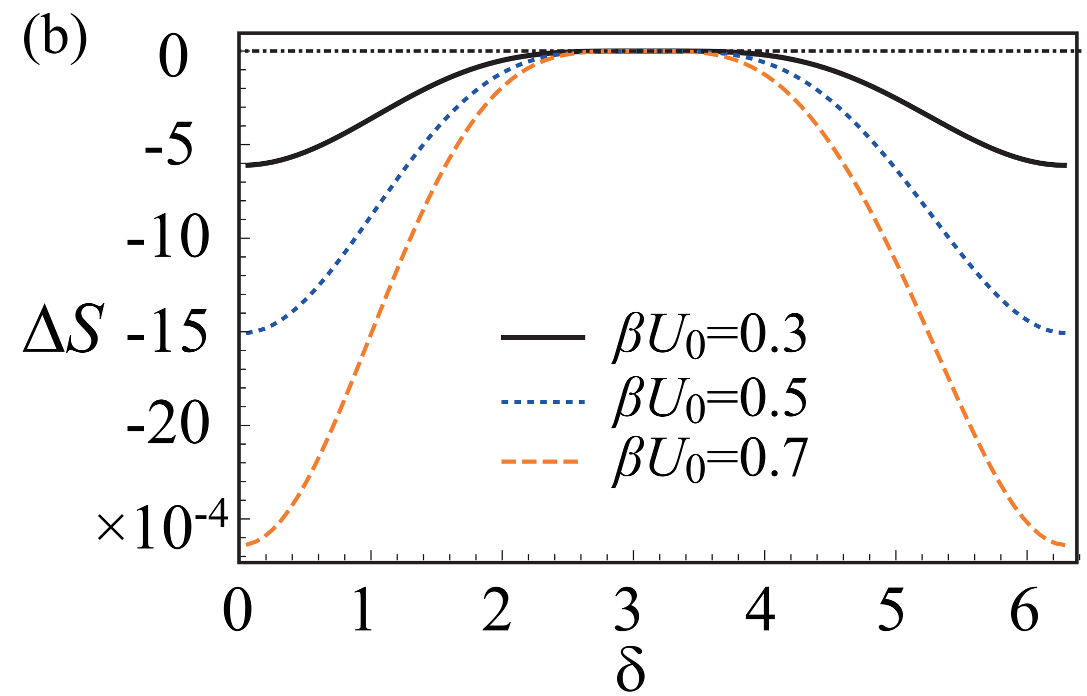
相対エントロピー: $S^{\mathrm{HS}}(\rho(\theta)\,||\,\rho^{\mathrm{SS}}(\theta)) = \mathrm{Tr}\rho(\theta)\left[\ln\rho(\theta) - \ln\rho^{\mathrm{SS}}(\theta)\right]$

“準”定常状態が実現される

準定常状態から変化



指数減衰



$$\Delta S = -S^{\mathrm{HS}}(\rho(2\pi)\,||\,\rho^{\mathrm{SS}}(2\pi)) + S^{\mathrm{HS}}(\rho^{\mathrm{SS}}(0)\,||\,\rho^{\mathrm{SS}}(0))$$

幾何学的効果によるエントロピー減少

→ 負の仕事: 幾何学的デーモン

(※ パラメータ変調のコストは考慮していない)

結論

- パラメータ変調によって定常状態とは違う周期的な状態が実現

$|\rho(\theta)\rangle\rangle$: 時間に対して”準”局所的

- 幾何学的位相由来の幾何学的デーモンの実現

- 相対エントロピーは変調により、負となり得る

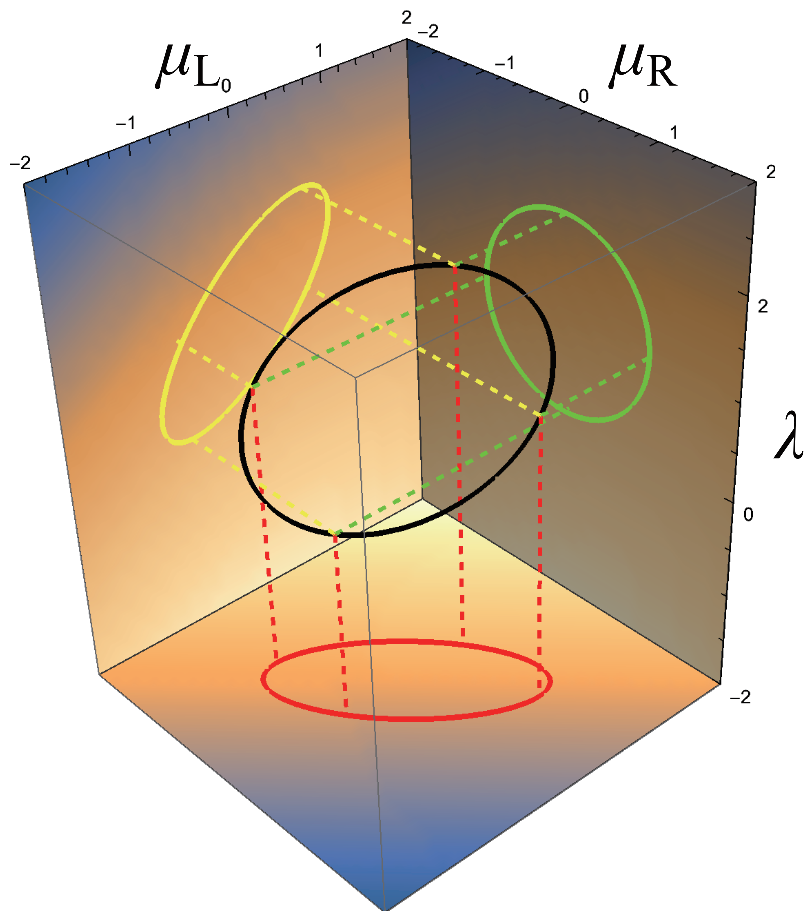
- 取り出せる仕事は時間に対して指数減衰する

変調を止めて定常状態に戻す必要がある

パラメータ変調のさせ方

左右の熱浴の変調のずれ: δ

$$U = U_0(1+\lambda),\ \lambda = \cos\theta, \quad \mu_L = \mu\sin\theta, \quad \mu_R = \mu\sin(\theta+\delta)$$

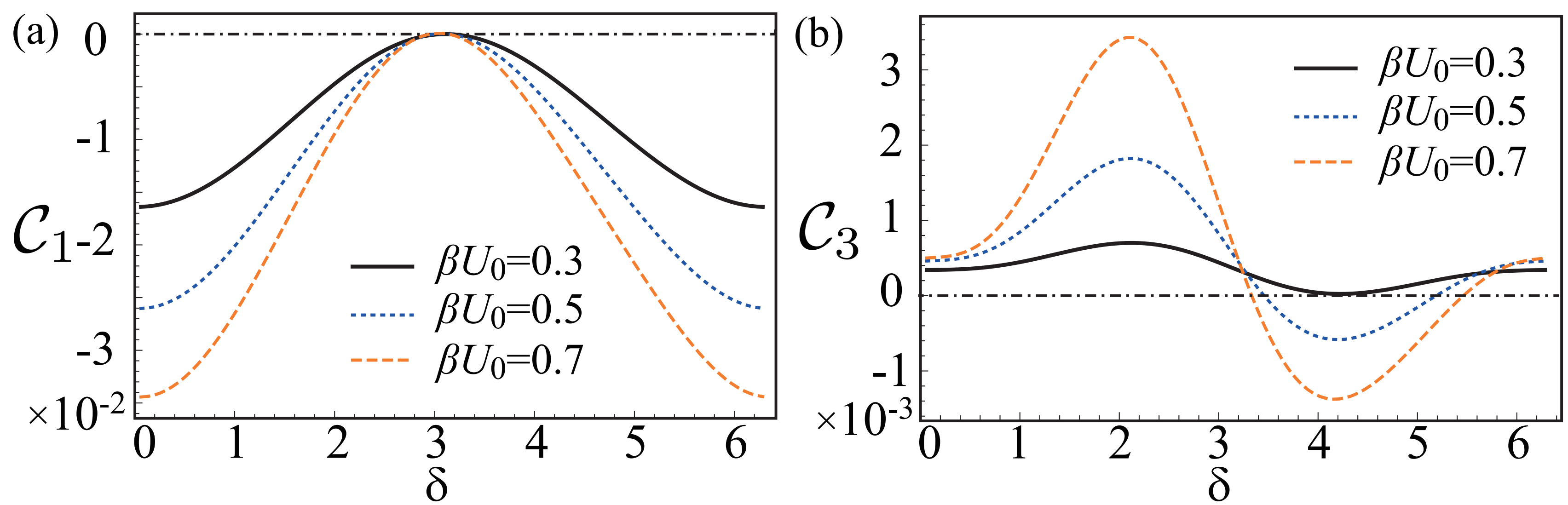


始状態: パラメータ変調のない場合の定常状態

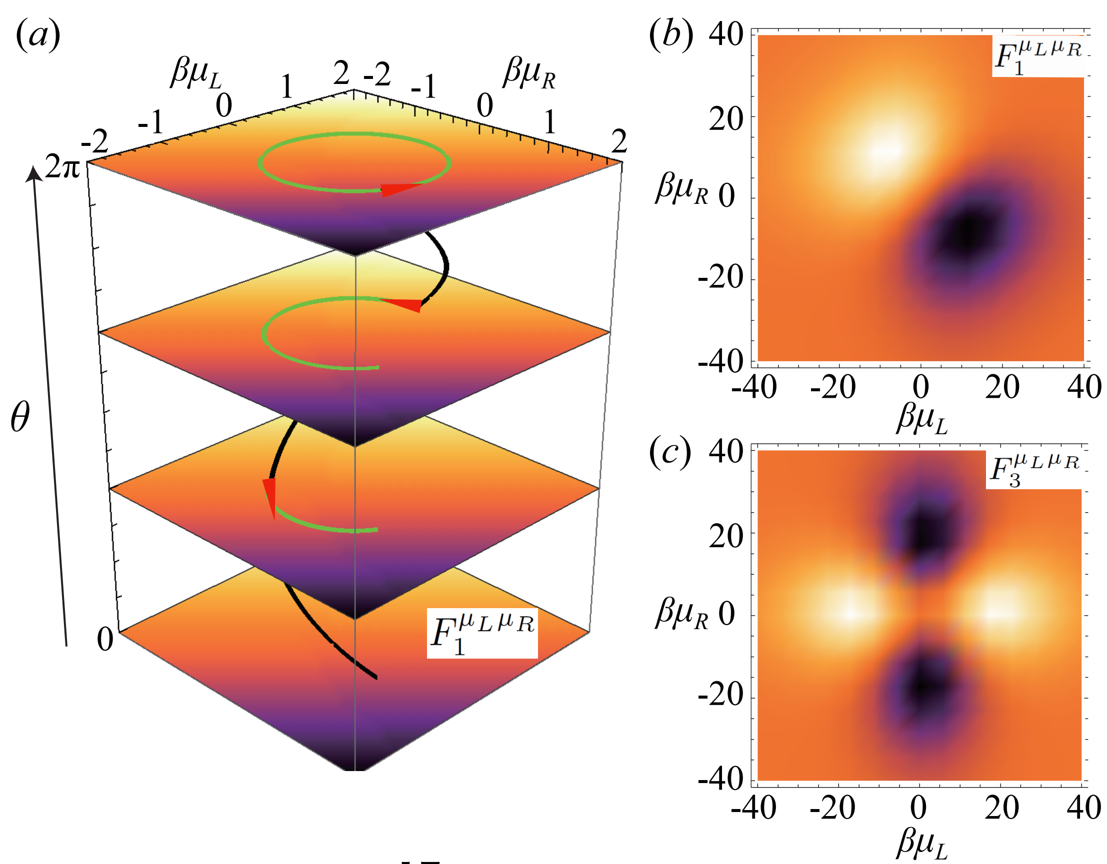
仮定: パラメータ変調は遅いとする

Andersonモデルにおける幾何学的状態と曲率

$$|\rho(\theta)\rangle\rangle = |r_0(\theta)\rangle\rangle + \sum_{i\neq 0} C_i |r_i(\theta)\rangle\rangle$$



$$\mathscr{F}^i_{\mu\nu} \equiv \frac{\partial \mathscr{A}^i_\nu}{\partial \Lambda_\mu} - \frac{\partial \mathscr{A}^i_\mu}{\partial \Lambda_\nu}$$

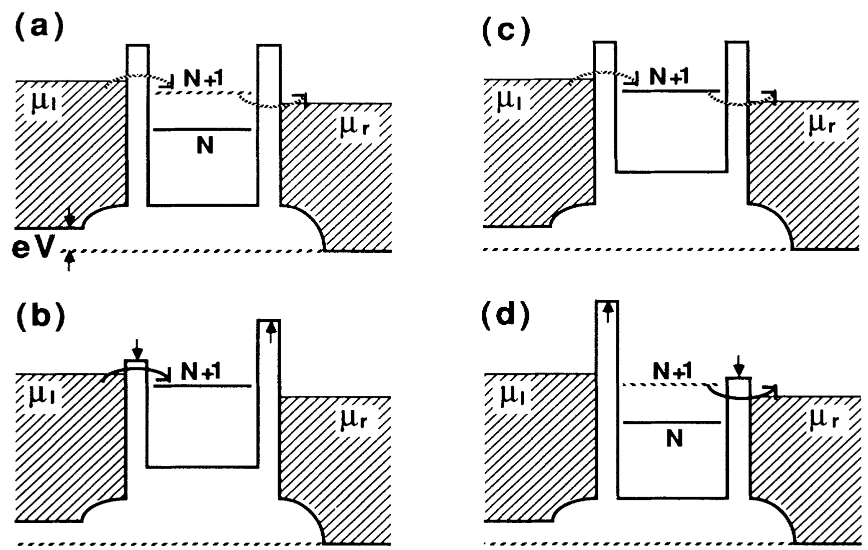


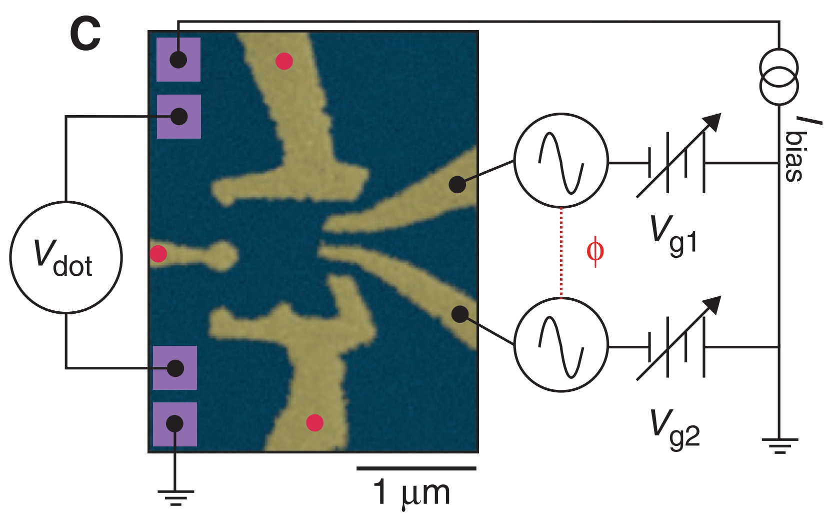
実験での実現

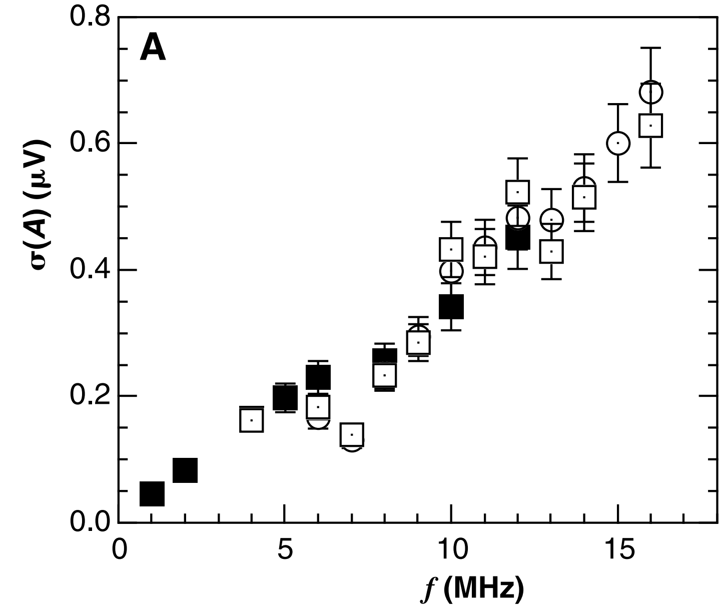
量子ドット系

Quantum dot turnstile

M. Switkes, et.al., Science, **283**, 1905 (1999).

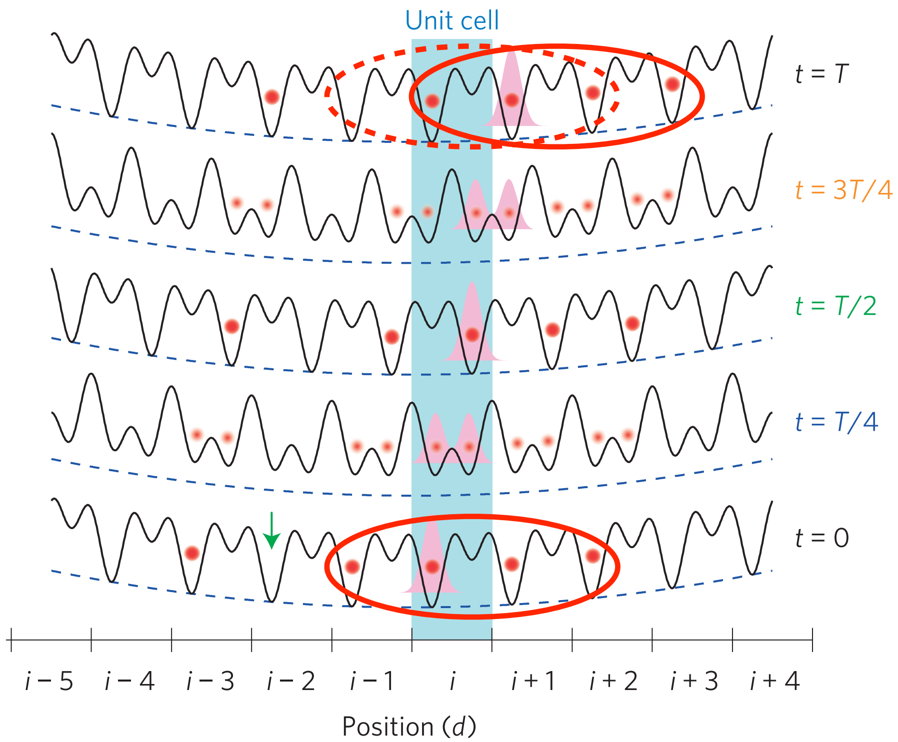


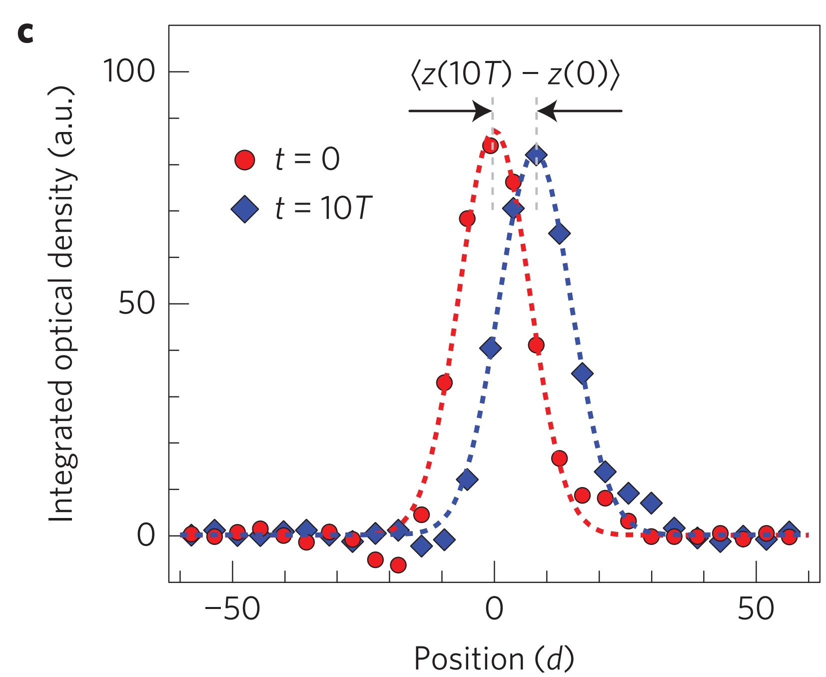




冷却原子系

S. Nakajima, et.al., Nat. Phys., **12**, 296 (2016).





熱浴自由度の逓減

全系の密度行列の時間発展

$$i\frac{d}{dt}\rho^{\mathrm{tot}} = [\hat{H}^{\mathrm{tot}}, \rho^{\mathrm{tot}}]$$

熱浴自由度のトレースアウト

$$\mathrm{Tr}_{\mathrm{reservoirs}}\left[i\frac{d}{dt}\rho^{\mathrm{tot}}\right] = \mathrm{Tr}_{\mathrm{reservoirs}}\left[[\hat{H}^{\mathrm{tot}}, \rho^{\mathrm{tot}}]\right]$$ + Markov approx.

$$\Rightarrow \quad \frac{d}{dt}\begin{pmatrix}\rho_d \\ \rho_\uparrow \\ \rho_\downarrow \\ \rho_e\end{pmatrix} = \Gamma \begin{pmatrix} -2f_-^{(1)} & f_+^{(1)} & f_+^{(1)} & 0 \\ f_-^{(1)} & -f_-^{(0)} - f_+^{(1)} & 0 & f_+^{(0)} \\ f_-^{(1)} & 0 & -f_-^{(0)} - f_+^{(1)} & f_+^{(0)} \\ 0 & f_-^{(0)} & f_-^{(0)} & -2f_+^{(0)} \end{pmatrix} \begin{pmatrix}\rho_d \\ \rho_\uparrow \\ \rho_\downarrow \\ \rho_e\end{pmatrix}$$

縮約密度行列

縮約密度行列は今回の取り扱いでは対角要素のみとなる

$$\rho = \begin{pmatrix} \rho_d & 0 & 0 & 0 \\ 0 & \rho_\uparrow & 0 & 0 \\ 0 & 0 & \rho_\downarrow & 0 \\ 0 & 0 & 0 & \rho_e \end{pmatrix}$$

$$|\rho\rangle\rangle = \begin{pmatrix} \rho_d \\ \rho_\uparrow \\ \rho_\downarrow \\ \rho_e \end{pmatrix}$$

Supervector representation

$\rho_e, \rho_\uparrow, \rho_\downarrow, \rho_d$

Empty

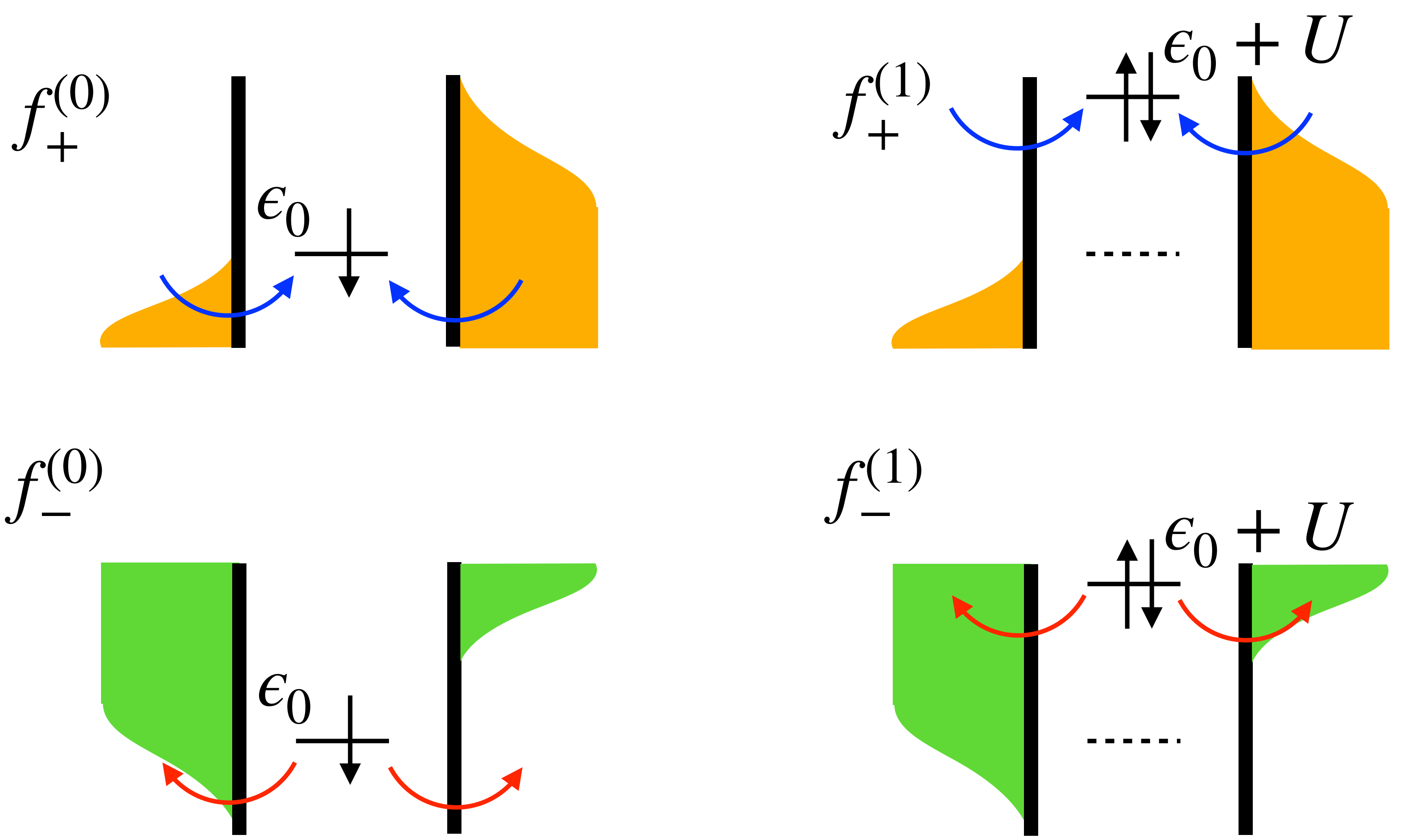
Single occupied

Double occupied

フェルミ分布関数

縮約密度行列はレート方程式に従う

$$f_{\pm}^{(j)} = \frac{1}{1 + \exp[\pm(\epsilon_0 + jU - \mu_L)/T]} + \frac{1}{1 + \exp[\pm(\epsilon_0 + jU - \mu_R)/T]}$$



$f_-^{(j)}$: Hole distributions functions

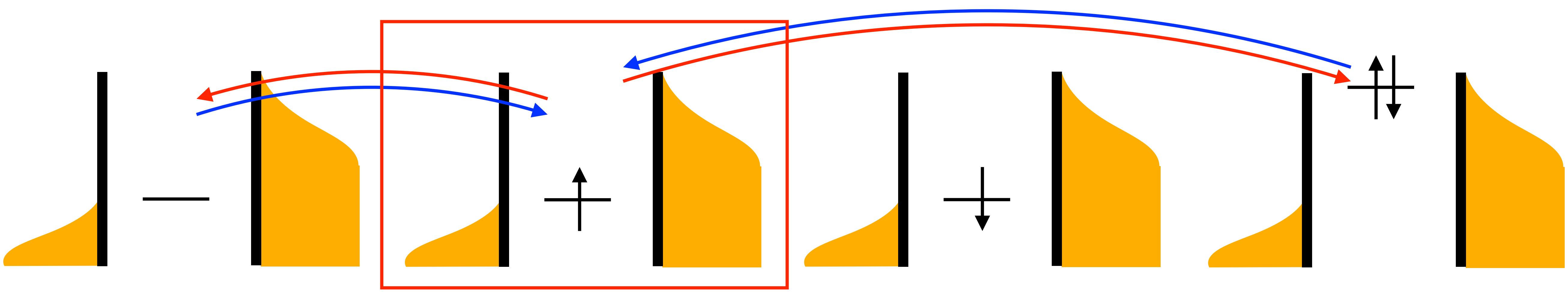
マスター方程式の物理的な解釈

$$\frac{d}{dt}\begin{pmatrix}\rho_d\\ \rho_\uparrow\\ \rho_\downarrow\\ \rho_e\end{pmatrix} = \Gamma\begin{pmatrix}-2f_-^{(1)} & f_+^{(1)} & f_+^{(1)} & 0\\ f_-^{(1)} & -f_-^{(0)}-f_+^{(1)} & 0 & f_+^{(0)}\\ f_-^{(1)} & 0 & -f_-^{(0)}-f_+^{(1)} & f_+^{(0)}\\ 0 & -f_-^{(0)} & -f_-^{(0)} & -2f_+^{(0)}\end{pmatrix}\begin{pmatrix}\rho_d\\ \rho_\uparrow\\ \rho_\downarrow\\ \rho_e\end{pmatrix}$$

$\Gamma = \pi\nu V^2$　ν: 熱浴の状態密度

e.g. 2列目

$$\frac{d}{dt}\rho_\uparrow = \underline{\Gamma f_-^{(1)}\rho_d} \underline{- \Gamma f_-^{(0)}\rho_\uparrow - \Gamma f_+^{(1)}\rho_\uparrow} + \underline{\Gamma f_+^{(0)}\rho_e}$$

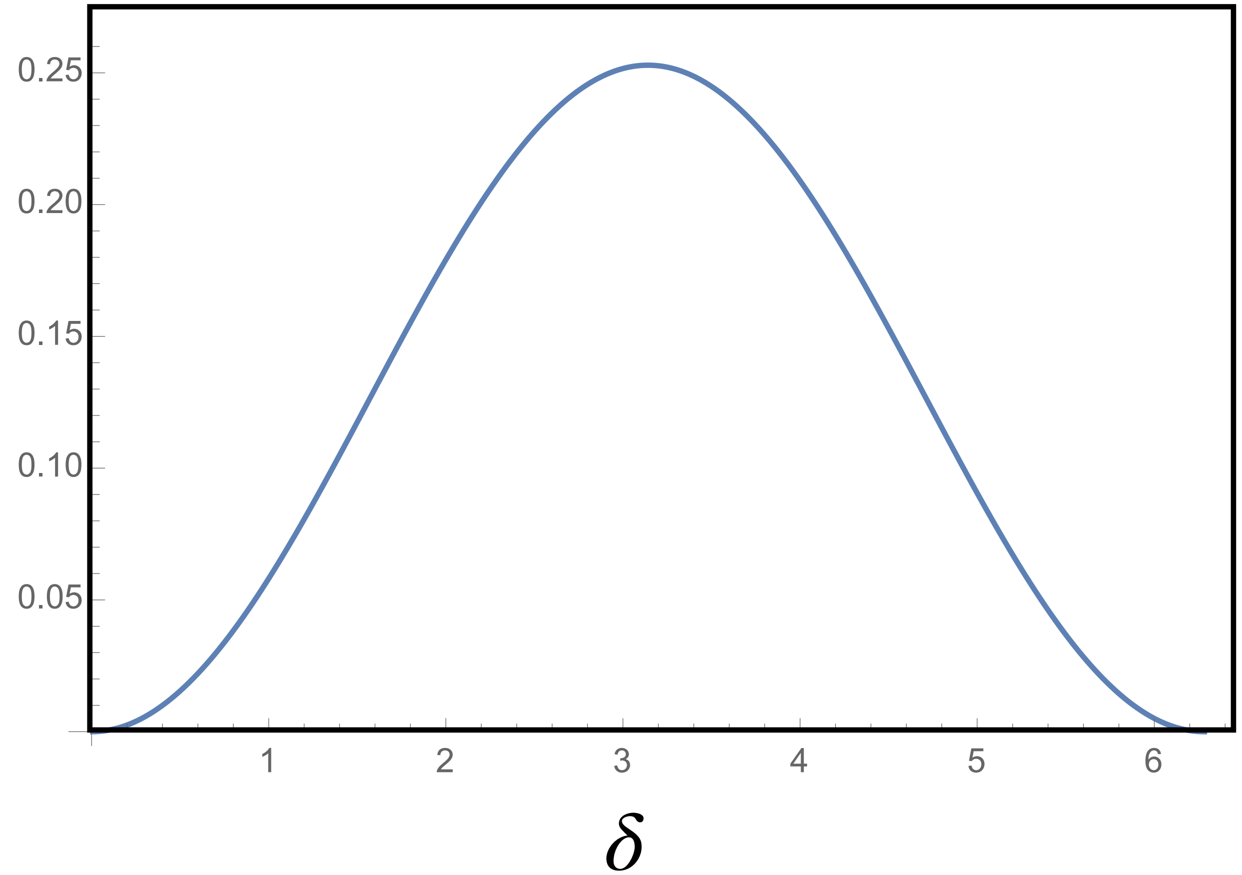


Joule熱

各瞬間のバイアス電圧はノンゼロ + カレント

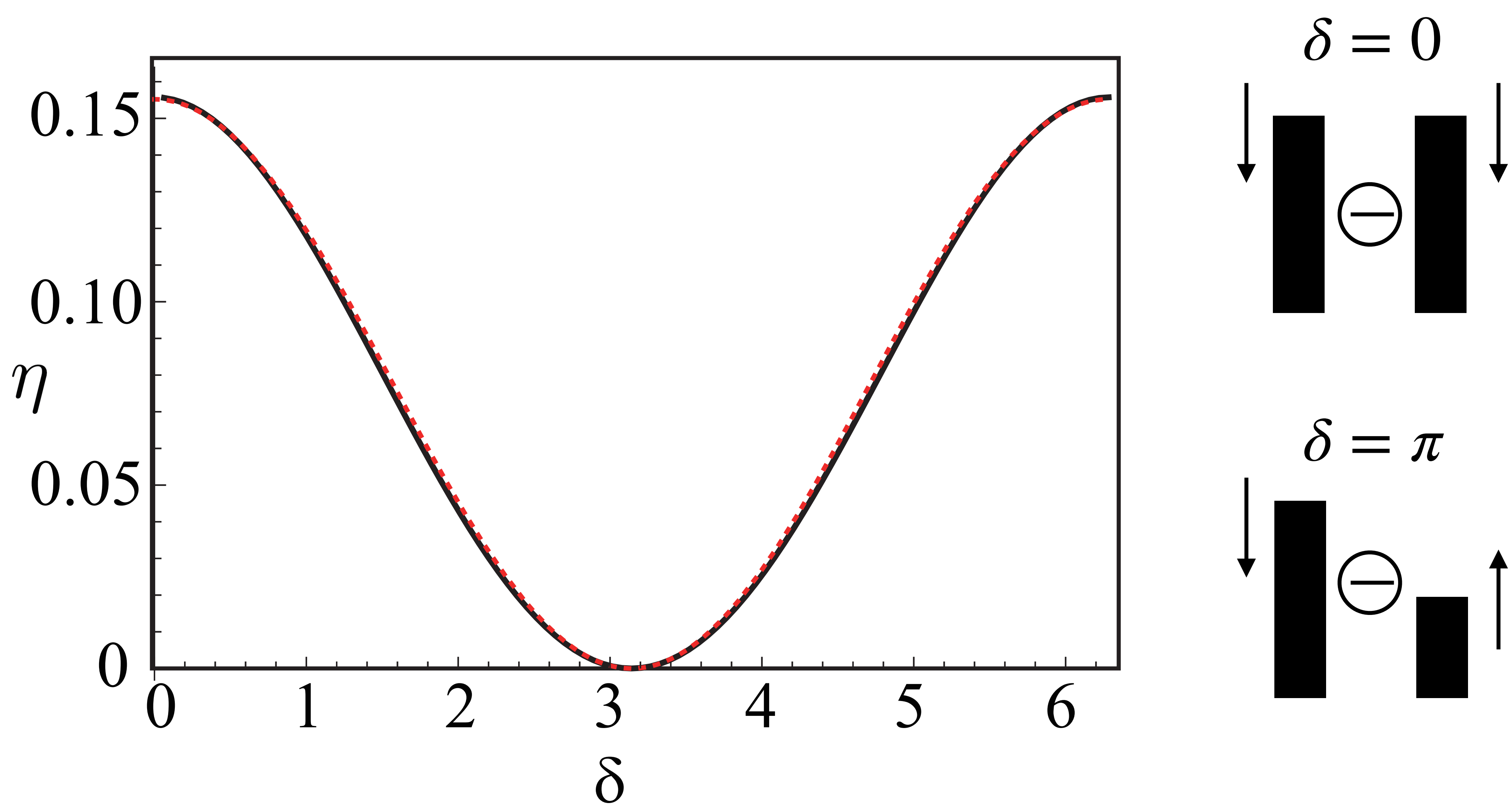
⇒ Joule熱の発生　$Q_{\text{Joule}} \equiv I\Delta V$

Joule熱の1周期積算



効率

効率: $\eta \equiv \dfrac{|W|}{Q_A} = \dfrac{|Q_A + Q_R|}{Q_A}$



$\delta = 2\pi n$で最大, $\delta = (2n+1)\pi$で最小

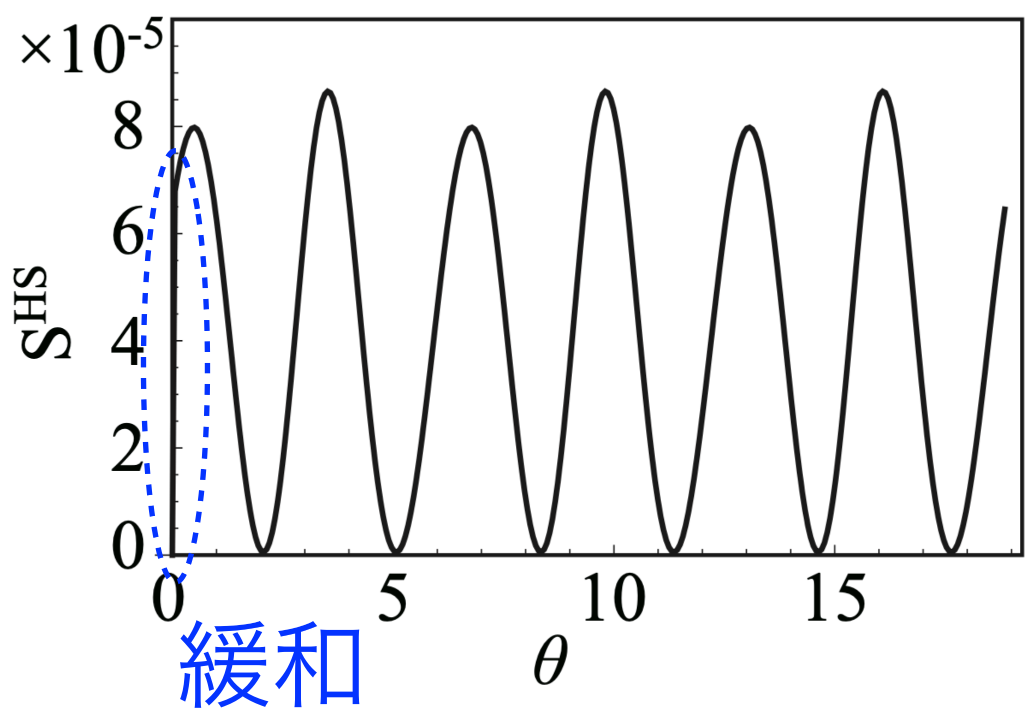
相対エントロピーの時間発展

相対エントロピー: $S^{\mathrm{HS}}(\rho(\theta)\,||\,\rho^{\mathrm{SS}}(\theta)) = \mathrm{Tr}\rho(\theta)\left[\ln\rho(\theta) - \ln\rho^{\mathrm{SS}}(\theta)\right]$

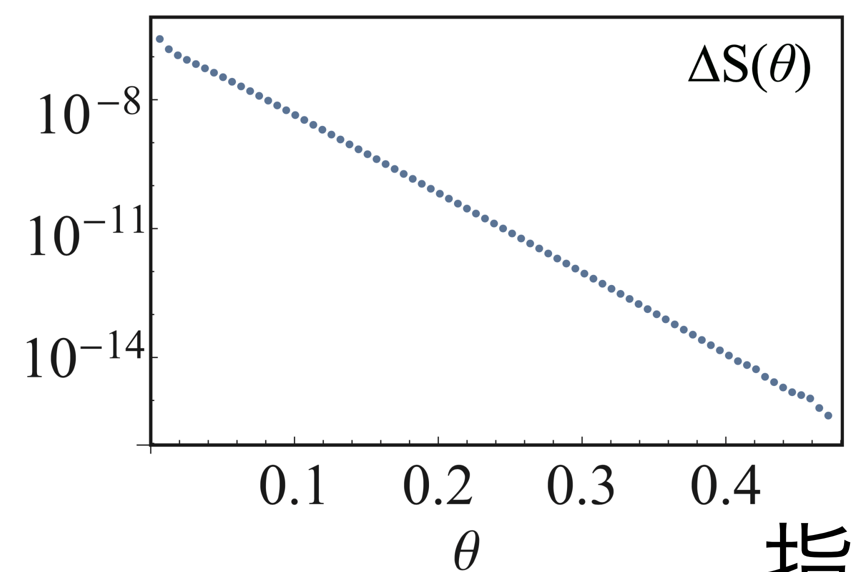
θ : 無次元化した時間

最初の緩和の後、すぐにθに関する周期関数となる

“準”定常状態が実現される



始状態からの変化



指数減衰

(準定常化後: H Hayakawa, VMM Paasonen, R Yoshii, arXiv:2112.12370 (2021).)

MASTER EQUATION FORMALISM

Simplification in semi-classical system

Semiclassical → off-diagonal part is completely decoupled

N^2 components to N components: $|\rho(t)\rangle\rangle = (\rho_{11}, \rho_{22}, \cdots, \rho_{NN})^T$

diagonal part only

$$\frac{d}{dt}|\rho\rangle\rangle = \hat{\mathscr{K}}|\rho\rangle\rangle$$

In this formalism, trace is given by the inner product with

$$\langle\langle \ell_0 | = (1,1,\cdots,1)$$

$$\langle\langle \ell_0 | \rho \rangle\rangle = \rho_{11} + \rho_{22} + \cdots + \rho_{NN} = \mathrm{Tr}\rho$$

(generalization with off-diagonal part is straightforward)

Important properties 1

Trace of the density matrix must always be unity

$$\frac{d}{dt}\langle\langle \ell_0 | \rho \rangle\rangle = \frac{d\langle\langle \ell_0 |}{dt} | \rho \rangle\rangle + \langle\langle \ell_0 | \frac{d | \rho \rangle\rangle}{dt} = \langle\langle \ell_0 | \hat{\mathscr{K}} | \rho \rangle\rangle$$

$$\frac{d}{dt}\langle\langle \ell_0 | \rho \rangle\rangle = 0$$

$$\langle\langle \ell_0 | \hat{\mathscr{K}} = 0$$

$\langle\langle \ell_0 |$: zero mode $\qquad \langle\langle \ell_0 | = (1,1,\cdots,1)$

Constraint on $\hat{\mathscr{K}}$: $\displaystyle\sum_i \hat{\mathscr{K}}_{ij} = 0$

Important properties 2

Suppose the system has unique steady state

By using the right eigenvectors $|r_n\rangle\rangle$ (with eigenvalues λ_n)

$$|\rho(0)\rangle\rangle = \sum_n a_n |r_n\rangle\rangle$$

Time evolution and expected behavior for late time

$$|\rho(t)\rangle\rangle = \sum_n a_n e^{\lambda_n t} |r_n\rangle\rangle \underset{t \to \infty}{\Rightarrow} |r_0\rangle\rangle$$

(a_0 must be chosen to satisfy $\mathrm{Tr}\rho = 1$)

It is achieved only if all $\lambda_{n\neq 0}$ are negative

IN THE PRESENCE OF MODULATION

NON-ADIABATIC GEOMETRICAL PHASE

Adiabatic limit

Non-dimentionalization

$$\frac{d}{dt}|\rho\rangle\rangle = \hat{\mathscr{K}}|\rho\rangle\rangle \;\Rightarrow\; \epsilon\frac{d}{d\theta}|\rho\rangle\rangle = \hat{K}|\rho\rangle\rangle$$

$$\epsilon = \tau_m/\tau_0$$

τ_m, τ_0: characteristic times for modulation and relaxation

Adiabatic limit

$$\frac{d}{d\theta}|\rho\rangle\rangle = \frac{1}{\epsilon}\hat{K}|\rho\rangle\rangle$$

$\underset{\epsilon \to 0}{\rightarrow}$ All eigenstates except for $|r_0\rangle\rangle$ are suddenly dumped

$$|\rho(t)\rangle\rangle_{\epsilon=0} = |r_0(\mathbf{\Lambda(t)})\rangle\rangle$$

Time evolution of the density matrix

Infinitesimal time evolution

$$\frac{d}{d\theta}|\rho\rangle\rangle = \epsilon^{-1}\hat{K}(\theta)|\rho\rangle\rangle \;\Rightarrow\; |\rho(\theta+\Delta\theta)\rangle\rangle - |\rho(\theta)\rangle\rangle = \epsilon^{-1}\Delta\theta\hat{K}(\theta)|\rho(\theta)\rangle\rangle$$

$$|\rho(\theta+\Delta\theta)\rangle\rangle = \left(1+\epsilon^{-1}\Delta\theta\hat{K}(\theta)\right)|\rho(\theta)\rangle\rangle \simeq \exp(\epsilon^{-1}\Delta\theta\hat{K}(\theta))|\rho(\theta)\rangle\rangle$$

$$= \sum_i |r_i\rangle\rangle\langle\langle\ell_i|\exp(\epsilon^{-1}\Delta\theta\hat{K}(\theta))|\rho(\theta)\rangle\rangle \quad \left(\sum_i |r_i\rangle\rangle\langle\langle\ell_i| = \hat{I}\right)$$

$$= \sum_i \exp(\epsilon^{-1}\varepsilon_i(\theta)\Delta\theta)|r_i(\theta)\rangle\rangle\langle\langle\ell_i(\theta)|\rho(\theta)\rangle\rangle$$

Recursive procedure

$$|\rho(\theta+\Delta\theta)\rangle\rangle = \sum_{i,j} e^{\epsilon^{-1}\varepsilon_i(\theta)\Delta\theta}|r_i(\theta)\rangle\rangle\langle\langle\ell_i(\theta)|$$

$$\times e^{\epsilon^{-1}\varepsilon_j(\theta-\Delta\theta)\Delta\theta}|r_i(\theta-\Delta\theta)\rangle\rangle\langle\langle\ell_i(\theta-\Delta\theta)|\rho(\theta-\Delta\theta)\rangle\rangle$$

Important remark 1

Eigenstates

$$\hat{K}(\theta)\,|\,r_i(\theta)\rangle\rangle = \varepsilon_i(\theta)\,|\,r_i(\theta)\rangle\rangle,\ \langle\langle \ell_i(\theta)\,|\,\hat{K}(\theta) = \varepsilon_i(\theta)\langle\langle \ell_i(\theta)\,|$$

means

$$\hat{K}(\mathbf{\Lambda}_\theta)\,|\,r_i(\mathbf{\Lambda}_\theta)\rangle\rangle = \varepsilon_i(\mathbf{\Lambda}_\theta)\,|\,r_i(\mathbf{\Lambda}_\theta)\rangle\rangle,\ \langle\langle \ell_i(\mathbf{\Lambda}_\theta)\,|\,\hat{K}(\mathbf{\Lambda}_\theta) = \varepsilon_i(\mathbf{\Lambda}_\theta)\langle\langle \ell_i(\mathbf{\Lambda}_\theta\,|$$

$\mathbf{\Lambda}_\theta$: set of parameters at time θ

Above equations: eigenstate for fixed parameters $\mathbf{\Lambda}_\theta$

(eigenstate for snapshot parameters)

Time evolution from the initial state

Path integral-like formula

$$
\begin{aligned}
&|\rho(\theta+\Delta\theta)\rangle\rangle \\
&= \sum_{i,j,\cdots,k} e^{\epsilon^{-1}\varepsilon_i(\theta)\Delta\theta} e^{\epsilon^{-1}\varepsilon_j(\theta)\Delta\theta} \cdots e^{\epsilon^{-1}\varepsilon_k(\theta)\Delta\theta} \\
&\quad \times |r_i(\theta)\rangle\rangle\langle\langle \ell_i(\theta)|r_j(\theta-\Delta\theta)\rangle\rangle\langle\langle \ell_j(\theta-\Delta\theta)|\cdots|r_k(\Delta\theta)\rangle\rangle\langle\langle \ell_k(\Delta\theta)|\rho(0)\rangle\rangle
\end{aligned}
$$

Initial condition

$$
|\rho(0)\rangle\rangle = \sum_i a_i |r_m(0)\rangle\rangle\langle\langle \ell_m(0)|\rho(0)\rangle\rangle = \sum_m a_m |r_m(0)\rangle\rangle
$$

$$
\begin{aligned}
&|\rho(\theta+\Delta\theta)\rangle\rangle \\
&= \sum_{i,j,\cdots,k,m} e^{\epsilon^{-1}\varepsilon_i(\theta)\Delta\theta} e^{\epsilon^{-1}\varepsilon_j(\theta)\Delta\theta} \cdots e^{\epsilon^{-1}\varepsilon_k(\theta)\Delta\theta} a_m \\
&\times |r_i(\theta)\rangle\rangle\langle\langle \ell_i(\theta)|r_j(\theta-\Delta\theta)\rangle\rangle\langle\langle \ell_j(\theta-\Delta\theta)|\cdots|r_k(\Delta\theta)\rangle\rangle\langle\langle \ell_k(\Delta\theta)|r_m(0)\rangle\rangle
\end{aligned}
$$

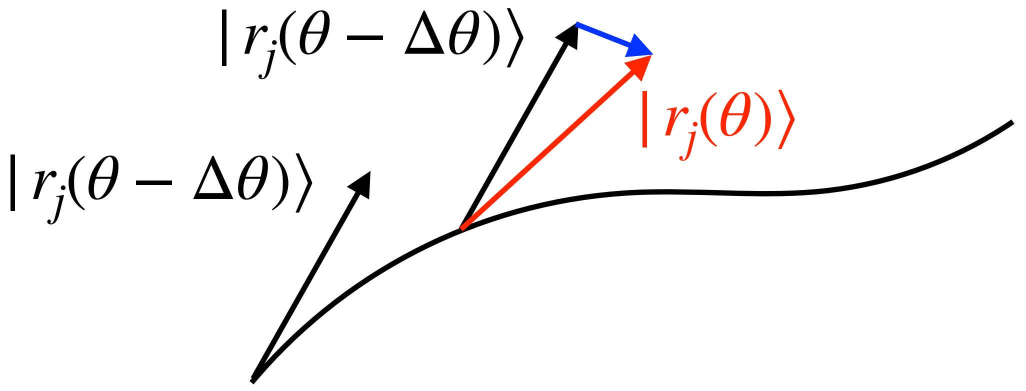
Connection

Connection between different time

$$\langle\langle \ell_i(\theta) | r_j(\theta - \Delta\theta)\rangle\rangle = \langle\langle \ell_i(\theta) | r_j(\theta)\rangle\rangle - \Delta\theta \langle\langle \ell_i(\theta) | \frac{d}{d\theta} | r_j(\theta)\rangle\rangle$$

$$= \delta_{i,j} - \Delta\theta \langle\langle \ell_i(\theta) | \frac{d}{d\theta} | r_j(\theta)\rangle\rangle$$

Similar to translation in curved space



Quasi adiabatic limit

$$|\rho(\theta)\rangle\rangle \simeq \sum_m e^{\int_0^\theta d\phi \epsilon^{-1} \varepsilon_m(\phi)} a_m | r_m(0)\rangle\rangle$$

$$- \sum_{m,i} \int_0^\theta d\phi e^{\int_\phi^\theta d\chi \epsilon^{-1} \varepsilon_i(\chi)} a_m e^{\int_0^\phi d\chi \epsilon^{-1} \varepsilon_m(\chi)} | r_i(\theta)\rangle\rangle\langle\langle \ell_i(\phi) | \frac{d}{d\phi} | r_m(\phi)\rangle\rangle$$

Important remark 2

There is no transition from $j \neq 0 \ \rightarrow \ i = 0$

$$\langle\langle \ell_0(\theta) | r_{j\neq 0}(\theta - \Delta\theta)\rangle\rangle = -\Delta\theta \langle\langle \ell_0(\theta) | \frac{d}{d\theta} | r_j(\theta)\rangle\rangle$$

$$= -\Delta\theta \frac{d}{d\theta} \underbrace{\langle\langle \ell_0(\theta) | r_j(\theta)\rangle\rangle}_{=0} + \Delta\theta \left(\underbrace{\frac{d}{d\theta}\langle\langle \ell_0(\theta) |}_{=0} \right) | r_j(\theta)\rangle\rangle = 0$$

Orthogonal

$$\langle\langle \ell_0(\theta) | = (1, \cdots, 1)$$

Coefficient for $| r_0 \rangle\rangle$ is unchanged

$$| \rho(\theta)\rangle\rangle = a_0(0) | r_0(\theta)\rangle\rangle + \sum_{i\neq 0} a_i(\theta) | r_m(\theta)\rangle\rangle$$

Consistent with $\mathrm{Tr}\rho = 1 \rightarrow \langle\langle \ell_0(\theta) | \rho(\theta)\rangle\rangle = \langle\langle \ell_0(0) | \rho(0)\rangle\rangle = 1$

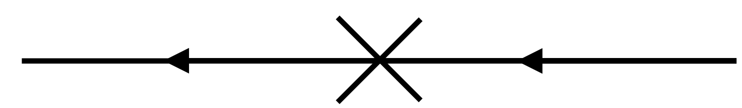
(If $a(0) = 1$)

Special case: initially steady state

K Takahashi, K Fujii, Y Hino, H Hayakawa, PRL (2020).

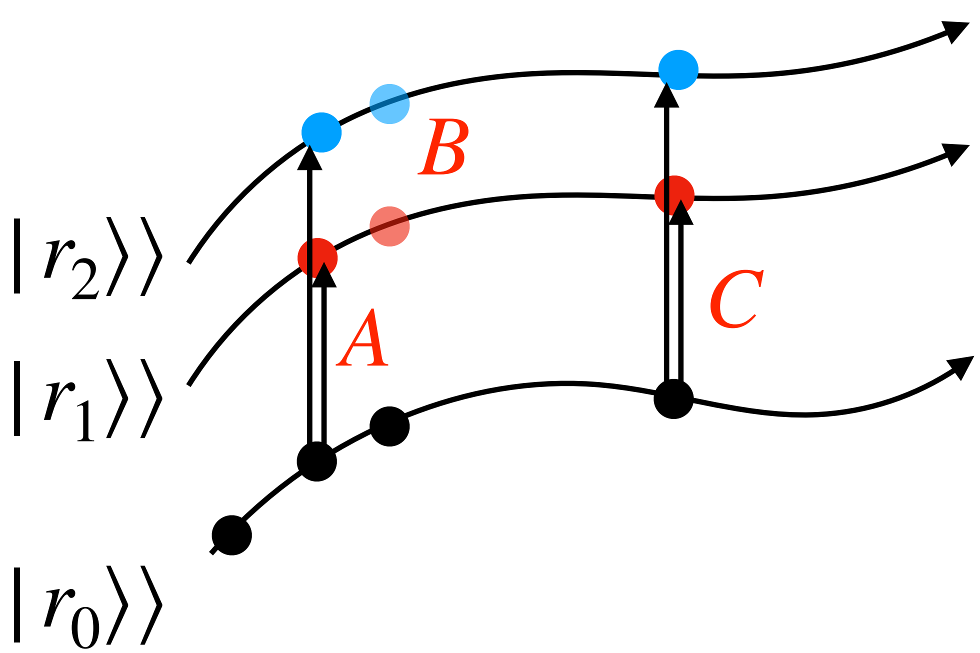
In the case of $|\rho(0)\rangle\rangle = |r_0(0)\rangle\rangle = |\rho^{SS}(0)\rangle\rangle$

$$|\rho(\theta)\rangle\rangle = |r_0(\theta)\rangle\rangle - \underbrace{\sum_{i\neq 0}\int_0^\theta d\phi}_{C}\, \underbrace{e^{\int_\phi^\theta d\chi \epsilon^{-1}\varepsilon_i(\chi)}}_{B} |r_i(\theta)\rangle\rangle \underbrace{\langle\langle \ell_i(\phi)|\frac{d}{d\phi}|r_0(\phi)\rangle\rangle}_{A}$$



"Feynman" diagram

Schematic understanding



A: transition at ϕ

B: exponential dumping

C: A takes place at any time

"Pumping and relaxation"

Slow modulating case

If $\epsilon \ll 1$

$$|\rho(\theta)\rangle\rangle \sim |r_0(\theta)\rangle\rangle - \sum_{i\neq 0}\int_0^\theta d\phi \underline{e^{\int_\phi^\theta d\chi \epsilon^{-1}\varepsilon_i(\chi)}} |r_i(\theta)\rangle\rangle\langle\langle \ell_i(\phi)|\frac{d}{d\phi}|r_0(\phi)\rangle\rangle$$

Only $\theta - \epsilon \leq \phi \leq \theta$ contributes

$$\sim |r_0(\theta)\rangle\rangle - \sum_{i\neq 0}\int_{\theta-\epsilon}^\theta d\phi e^{\int_\phi^\theta d\chi \epsilon^{-1}\varepsilon_i(\chi)} |r_i(\theta)\rangle\rangle\langle\langle \ell_i(\phi)|\frac{d}{d\phi}|r_0(\phi)\rangle\rangle$$

$$\sim |r_0(\theta)\rangle\rangle - \epsilon\sum_{i\neq 0} e^{\varepsilon_i(\theta)} |r_i(\theta)\rangle\rangle\langle\langle \ell_i(\theta)|\frac{d}{d\theta}|r_0(\theta)\rangle\rangle$$

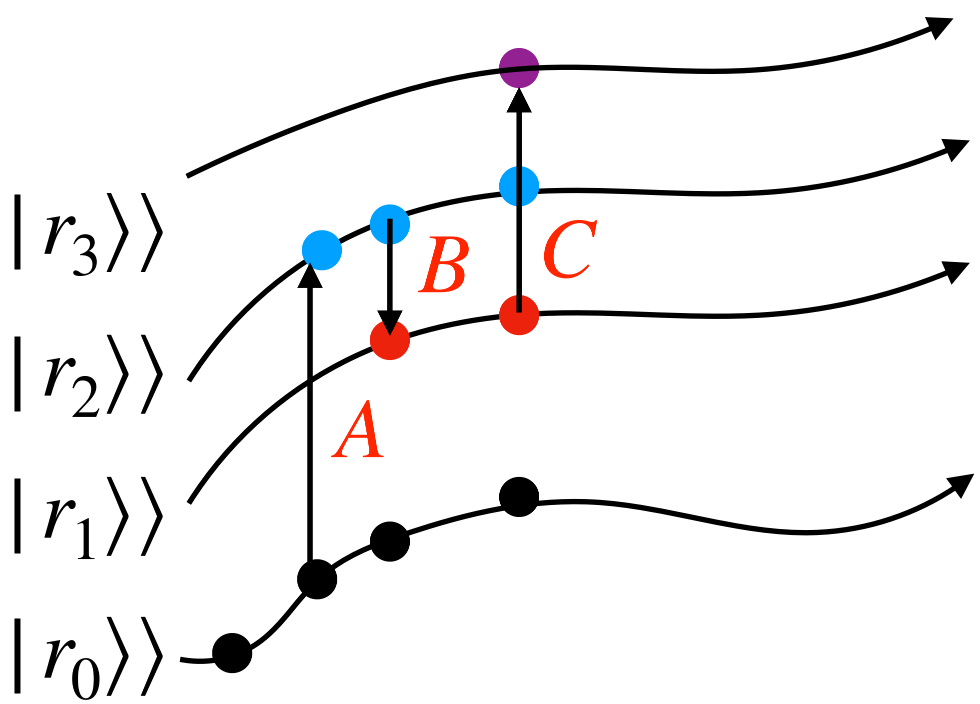
$$\sim |r_0(\theta)\rangle\rangle - \epsilon\sum_{i\neq 0} c_i(\theta)|r_i(\theta)\rangle\rangle$$

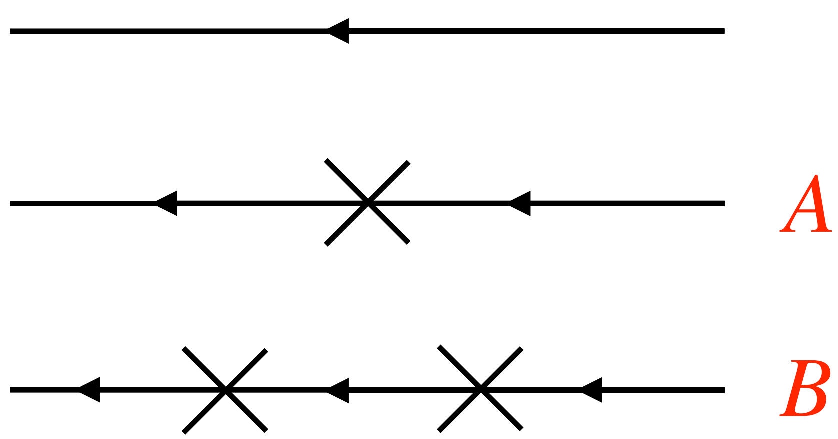
Deviation from $|r_0(\theta)\rangle\rangle$ is $O(\epsilon)$

Important remark 3

For $\epsilon \ll 1$ $\quad |\rho(\theta)\rangle\rangle \sim |r_0(\theta)\rangle\rangle - \epsilon \sum_{i\neq 0} c_i(\theta)|r_i(\theta)\rangle\rangle$

We neglect the higher orders ~ Born approximation





"Feynman" diagram

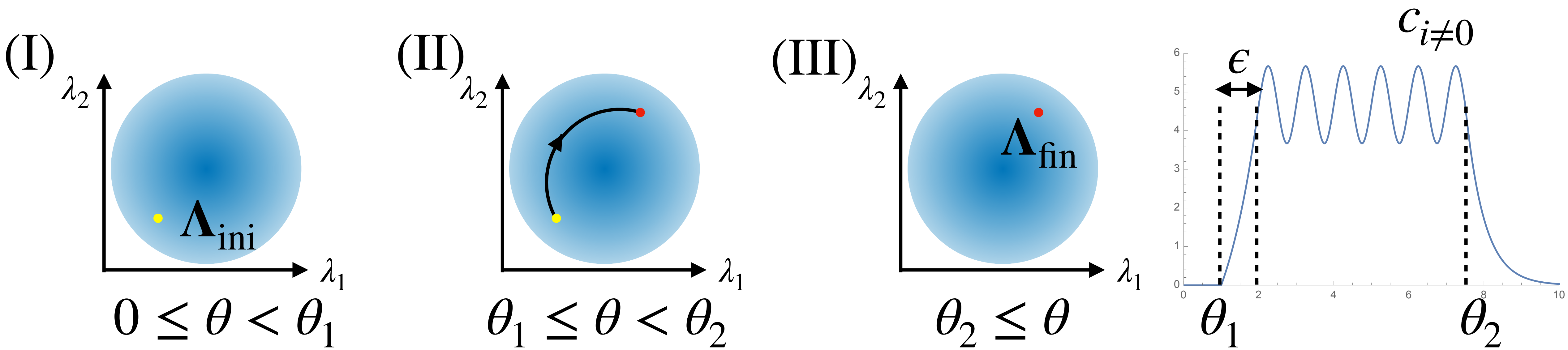
$$|\rho(\theta)\rangle\rangle \sim |r_0(\theta)\rangle\rangle - \sum_{i\neq 0}\int_0^\theta d\phi\, e^{\int_\phi^\theta d\chi \epsilon^{-1}\varepsilon_i(\chi)}|r_i(\theta)\rangle\rangle\langle\langle \ell_i(\phi)|\frac{d}{d\phi}|r_0(\phi)\rangle\rangle$$

$$+\sum_{i\neq 0}\sum_{j\neq 0}\int_\xi^\theta d\phi \int_0^\xi d\zeta\, e^{\int_\phi^\theta d\chi_1 \epsilon^{-1}\varepsilon_i(\chi_1)} e^{\int_\zeta^\xi d\chi_2 \epsilon^{-1}\varepsilon_j(\chi_2)}|r_i(\theta)\rangle\rangle\langle\langle \ell_i(\phi)|\frac{d}{d\phi}|r_j(\phi)\rangle\rangle\langle\langle \ell_j(\zeta)|\frac{d}{d\zeta}|r_0(\zeta)\rangle\rangle$$

$$+\cdots$$

Suggestive example

If we start the modulation at θ_1 and stop at θ_2



$$\text{(I)} \quad |\rho(\theta)\rangle\rangle = |r_0(\mathbf{\Lambda}_{\mathrm{ini}})\rangle\rangle$$

$$\text{(II)} \quad |\rho(\theta)\rangle\rangle = |r_0(\mathbf{\Lambda})\rangle\rangle - \sum_{i \neq 0} \int_{\theta_1}^{\theta} d\phi\, e^{\int_\phi^\theta d\chi \epsilon^{-1} \varepsilon_i(\chi)} |r_i(\theta)\rangle\rangle\langle\langle \ell_i(\phi)| \frac{d}{d\phi} |r_0(\phi)\rangle\rangle$$

$$\text{(III)} \quad |\rho(\theta)\rangle\rangle = |r_0(\mathbf{\Lambda}_{\mathrm{fin}})\rangle\rangle - \sum_{i \neq 0} \int_{\theta_1}^{\theta_2} d\phi\, \underbrace{e^{\int_\phi^\theta d\chi \epsilon^{-1} \varepsilon_i(\chi)}}_{\Rightarrow\, 0\ (\theta \gg \theta_2)} |r_i(\theta)\rangle\rangle\langle\langle \ell_i(\phi)| \frac{d}{d\phi} |r_0(\phi)\rangle\rangle$$

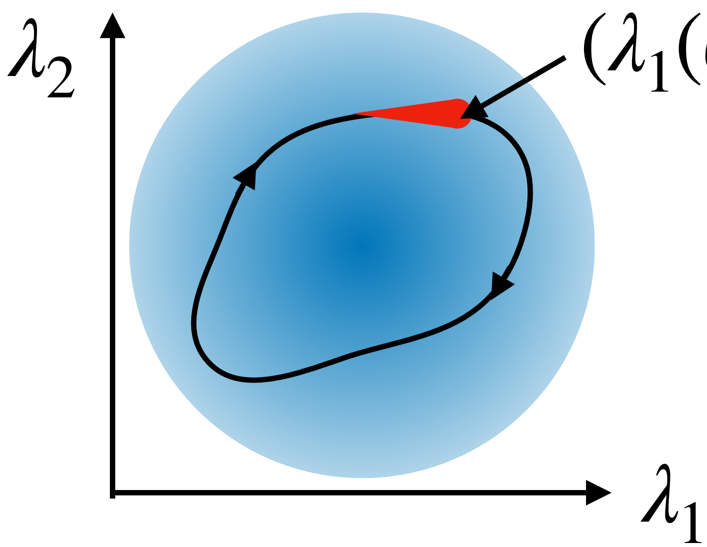
$$\Rightarrow |r_0(\mathbf{\Lambda}_{\mathrm{fin}})\rangle\rangle$$

Expected behavior

If we continuously modulate parameters

$$|\rho(\theta)\rangle\rangle = |r_0(\theta)\rangle\rangle - \sum_{i\neq 0}\int_0^\theta d\phi \underline{e^{\int_\phi^\theta d\chi \epsilon^{-1}\varepsilon_i(\chi)}} |r_i(\theta)\rangle\rangle\langle\langle \ell_i(\phi)|\frac{d}{d\phi}|r_0(\phi)\rangle\rangle$$

Exponential factor cut off the contribution from far past



$|\rho(\theta)\rangle\rangle$: quasi local in time

$|\rho(\theta)\rangle\rangle$ becomes quasi steady state

which wears the effect from the past

Similar to Fermi liquid

Geometrical interpretation

In the case of $|\rho(0)\rangle\rangle = |r_0(0)\rangle\rangle = |\rho^{\mathrm{SS}}(0)\rangle\rangle$

$$|\rho(\theta)\rangle\rangle = |r_0(\theta)\rangle\rangle + \sum_{i\neq 0} C^i |r_i(\theta)\rangle\rangle$$

$$C^i = -\int_0^\theta d\phi e^{\int_\phi^\theta d\chi \epsilon^{-1}\varepsilon_i(\chi)} \underline{\langle\langle \ell_i(\phi)|\frac{d}{d\phi}|r_0(\phi)\rangle\rangle}$$

"Vector potential"

$$\langle\langle \ell_i(\phi)|\frac{d}{d\phi}|r_0(\phi)\rangle\rangle = \frac{d\Lambda^\mu}{d\phi}\underline{\langle\langle \ell_i(\phi)|\frac{\partial}{d\Lambda^\mu}|r_0(\phi)\rangle\rangle} \equiv \mathscr{A}^i_\mu$$

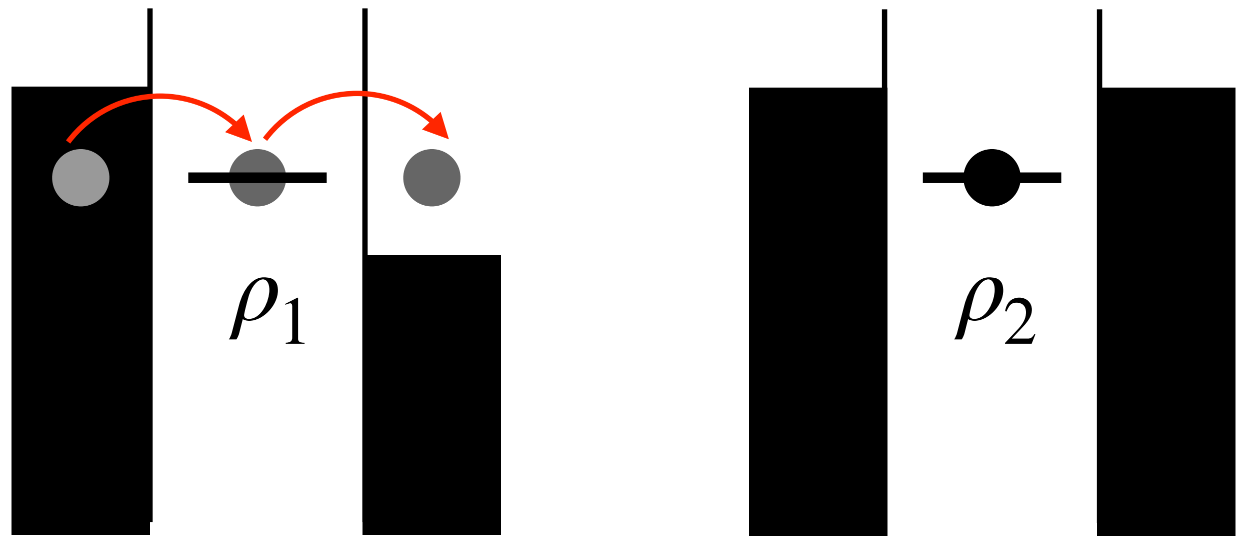
"Curvature"

$$\mathscr{F}^i_{\mu\nu} \equiv \frac{\partial \mathscr{A}^i_\nu}{\partial \Lambda_\mu} - \frac{\partial \mathscr{A}^i_\mu}{\partial \Lambda_\nu}$$

RELATIVE ENTROPY

Distance between two distribution

How to measure the distance between two distribution?



Naive idea: compare von-Neumann entropies

$$S_1^{\mathrm{vN}} = \mathrm{Tr}(-\rho_1 \ln \rho_1) \qquad S_2^{\mathrm{vN}} = \mathrm{Tr}(-\rho_2 \ln \rho_2)$$

Shortcoming: it is not positive-semidefinite

$$\Delta S^{\mathrm{vN}}(\rho || \sigma) = -\rho \ln \rho + \sigma \ln \sigma$$

If $\Delta S^{\mathrm{vN}}(\rho || \sigma) > 0$, then $\Delta S^{\mathrm{vN}}(\sigma || \rho) < 0$

Useful tool: Relative entropy

Quantum Relative entropy

(Kullback-Leibler-Umegaki relative entropy)

$$\Delta S^{\mathrm{KL}}(\rho||\sigma) = \mathrm{Tr}[\rho(\ln\rho - \ln\sigma)]$$

Positive-semidefiniteness

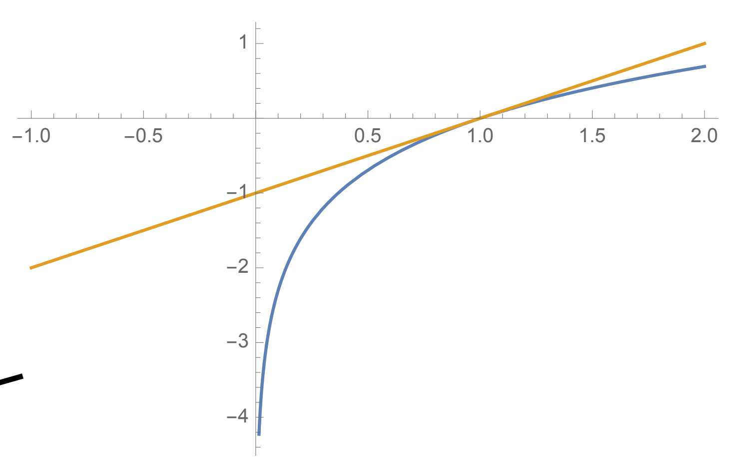
$$\Delta S^{\mathrm{KL}}(\rho||\sigma) \geq 0 \qquad \Delta S^{\mathrm{KL}}(\sigma||\rho) \geq 0$$

(Note that $\Delta S^{\mathrm{KL}}(\rho||\sigma) \neq \Delta S^{\mathrm{KL}}(\sigma||\rho)$ in general)

Diagonal case (in our study, it is the case)

$$-\Delta S^{\mathrm{KL}}(\rho||\sigma) = -\sum_i [\rho_{ii}(\ln\rho_{ii} - \ln\sigma_{ii})]$$

$$= \sum_i \rho_{ii}\ln\frac{\sigma_{ii}}{\rho_{ii}} \leq \sum_i \rho_{ii}\left(\frac{\sigma_{ii}}{\rho_{ii}} - 1\right) = \mathrm{Tr}\sigma - \mathrm{Tr}\rho = 0$$



$$\ln x \leq x - 1$$

Meaning of Relative entropy

Uncertainty of information $\sim -\ln\rho$

Entropy $= \langle -\rho\ln\rho\rangle = -\mathrm{Tr}\rho\ln\rho$

---

ex.) $\rho_1 = (0,1),\ \rho_2 = (1/4,3/4),\ \rho_3 = (1/2,1/2)$

$$-\mathrm{Tr}\rho_1\ln\rho_1 = 0, \qquad -\mathrm{Tr}\rho_2\ln\rho_2 = 2\ln 2 - \frac{3}{4}\ln 3, \qquad -\mathrm{Tr}\rho_3\ln\rho_3 = 2\ln 2$$

Certain $\longrightarrow$ Uncertain

If we obtain information and ρ becomes $\rho_3 \ \to \ \rho_1$

Reduction of the uncertainty $\sim -[-\ln\rho_1 - (-\ln\rho_3)]$

Average information gain

$$\langle \ln\rho_1 - \ln\rho_3\rangle = \mathrm{Tr}\rho_1(\ln\rho_1 - \ln\rho_3) = \Delta S^{\mathrm{KL}}(\rho_1||\rho_3)$$

Distance from the steady state

Relative entropy between ρ and ρ^{SS}

$$S^{\mathrm{KL}}(\rho\,||\,\rho^{\mathrm{SS}}) = \mathrm{Tr}\rho\left(\ln\rho - \ln\rho^{\mathrm{SS}}\right)$$

In the relaxation process $\rho = \rho^{\mathrm{SS}} + \sum_i \rho^{(i)} e^{\lambda_i t}\ (\lambda_i < 0)$

$$S^{\mathrm{KL}}(\rho\,||\,\rho^{\mathrm{SS}}) = \sum_n \left(\rho_{nn}^{\mathrm{SS}} + \sum_i \rho_{nn}^{(i)} e^{\lambda_i t}\right)\left[\ln\left(\rho_{nn}^{\mathrm{SS}} + \sum_k \rho_{nn}^{(k)} e^{\lambda_k t}\right) - \ln\rho_{nn}^{\mathrm{SS}}\right]$$

$$\xrightarrow[t\to\infty]{} \sum_n \rho_{nn}^{\mathrm{SS}}\left(\ln\rho_{nn}^{\mathrm{SS}} - \ln\rho_{nn}^{\mathrm{SS}}\right) = 0$$

$$S^{\mathrm{KL}}(\rho\,||\,\rho^{\mathrm{SS}}) \geq 0 \xrightarrow[\text{relaxation}]{} S^{\mathrm{KL}}(\rho^{\mathrm{SS}}\,||\,\rho^{\mathrm{SS}}) = 0$$

Monotonicity of relative entropy

Complete Positive Trace Preserving (CPTP) map

$$\mathscr{E}:\ \rho \to \mathscr{E}(\rho)\ \ \left(\mathrm{Tr}\mathscr{E}(\rho)=1,\ \mathscr{E}(\rho)\text{ is positive semi-definite}\right)$$

Monotonic property of relative entropy

$$S^{\mathrm{KL}}(\rho||\sigma) \geq S^{\mathrm{KL}}(\mathscr{E}(\rho)||\mathscr{E}(\sigma))$$

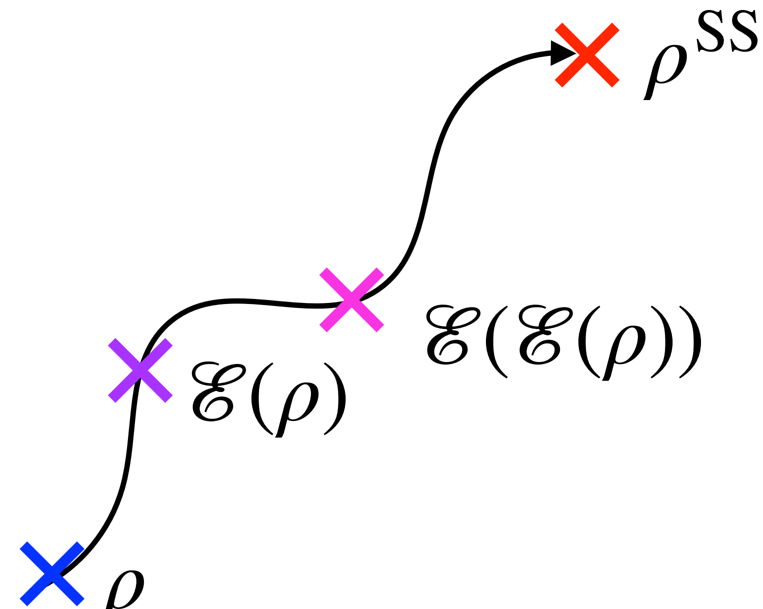
(Proof is found in arXiv: 2112.12370)

In the case $\mathscr{E}$ is time evolution, ρ^{SS} is the fixed point

$$\mathscr{E}(\rho^{\mathrm{SS}}) = \rho^{\mathrm{SS}}$$

$$S^{\mathrm{KL}}(\rho||\rho^{\mathrm{SS}}) \geq S^{\mathrm{KL}}(\mathscr{E}(\rho)||\rho^{\mathrm{SS}})$$

“H-theorem“



“Entropy” of the arbitrary state

Monotonically increasing quantity in time evolution

$$S(\rho||\rho^{\mathrm{SS}}) = -S^{\mathrm{KL}}(\rho||\rho^{\mathrm{SS}}) = -\mathrm{Tr}\rho\left(\ln\rho - \ln\rho^{\mathrm{SS}}\right)$$

Remark 1

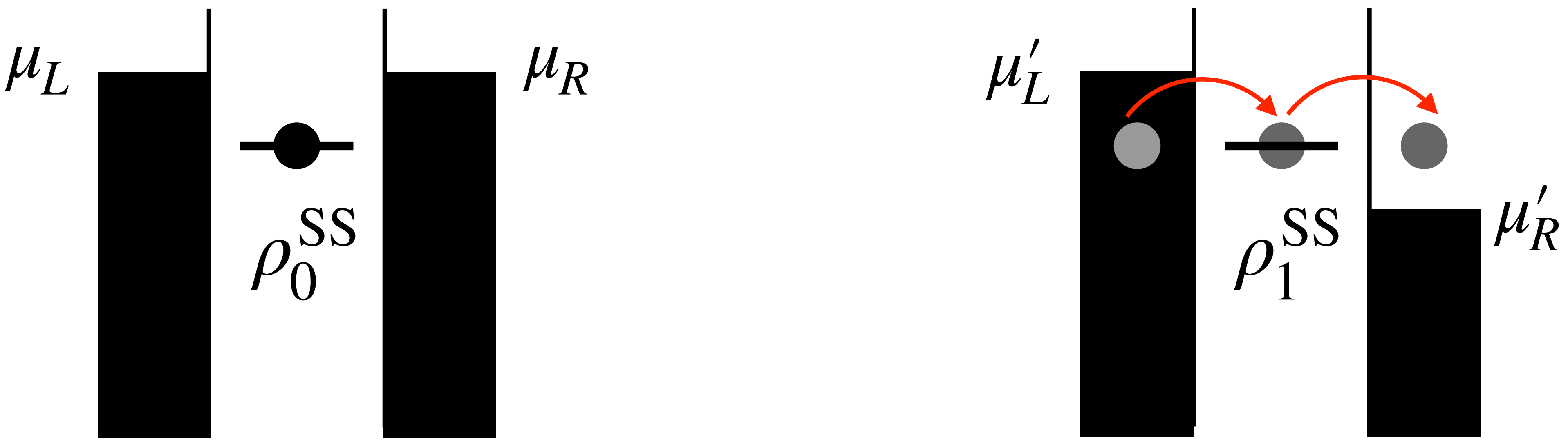
$S(\rho||\rho^{\mathrm{SS}})$ is negative semi-definite

Remark 2

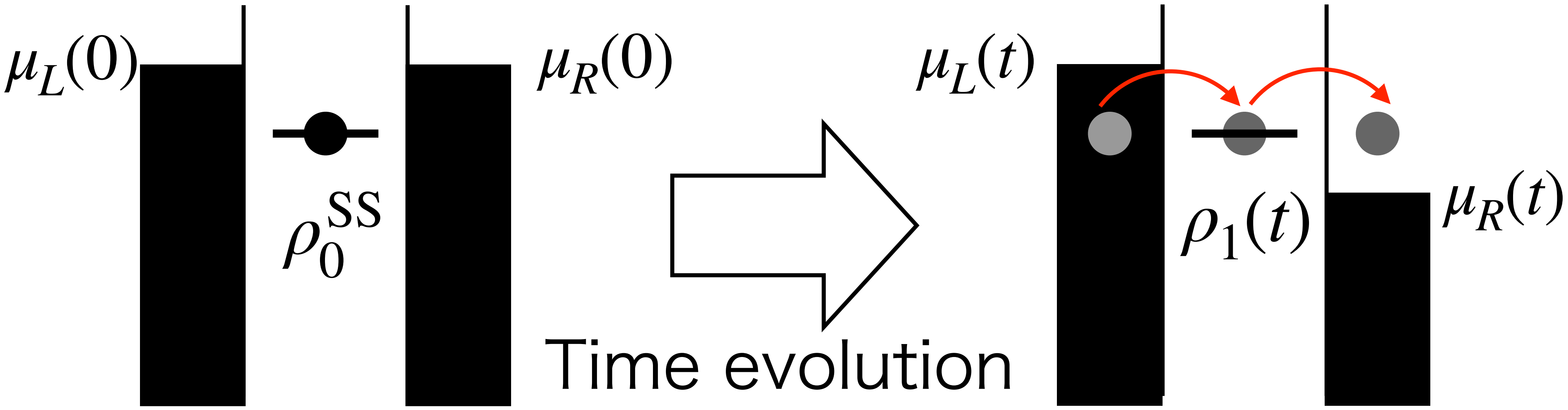
Maximum value of $S(\rho||\rho^{\mathrm{SS}})$ is zero

In the presence of modulation

Steady states in two parameter setting

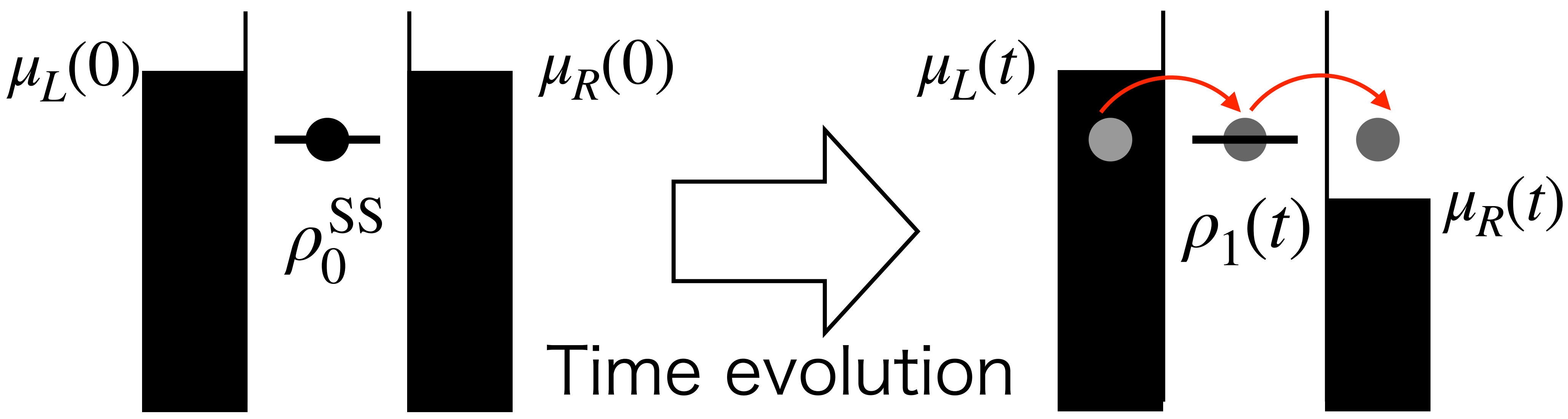


Time evolution of the steady state



Two possibility of the definition of entropy

Definition 1



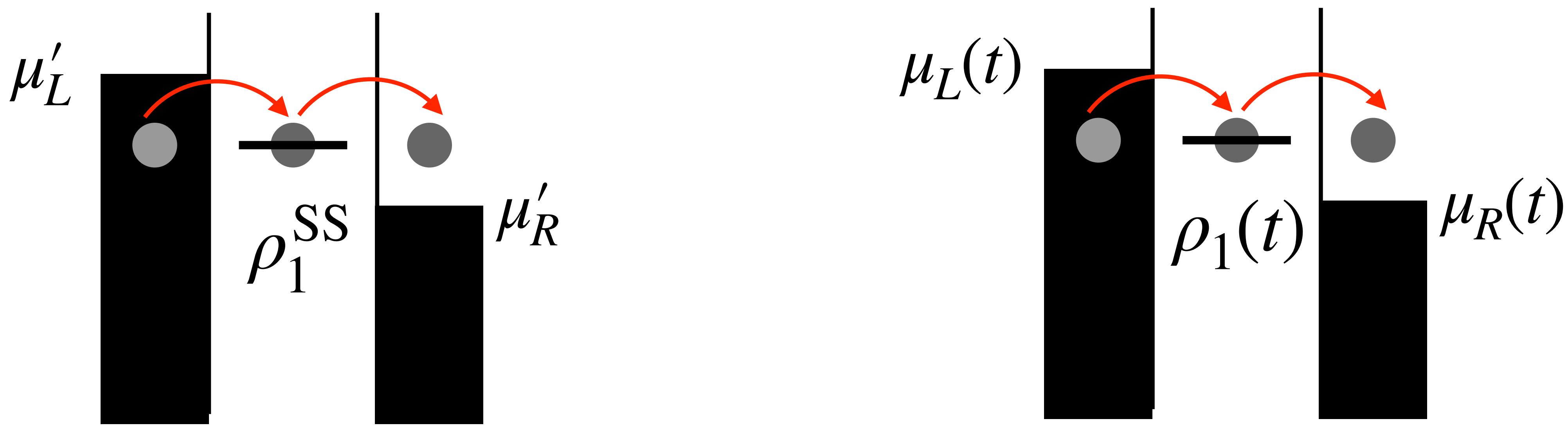
$S(\rho_1(t)||\rho_0^{\mathrm{SS}})$: Deviation from the initial state

😃 $S(\rho_1(t)||\rho_0^{\mathrm{SS}})$ monitors the distance from the initial state

😥 even if $\rho_1(t)$ coincides ρ_1^{SS}, $S(\rho_1(t)||\rho_0^{\mathrm{SS}})$ is nonzero

Two possibility of the definition of entropy

Definition 2



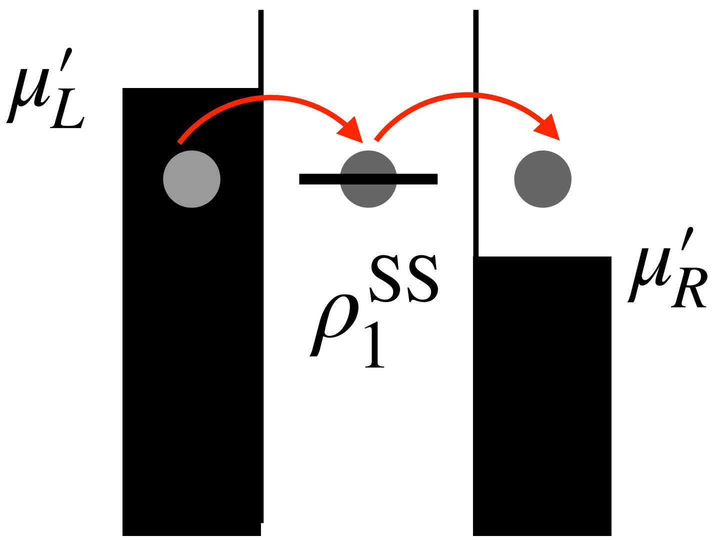
$S(\rho_1(t)||\rho_1^{\mathrm{SS}})$: Deviation from the steady state

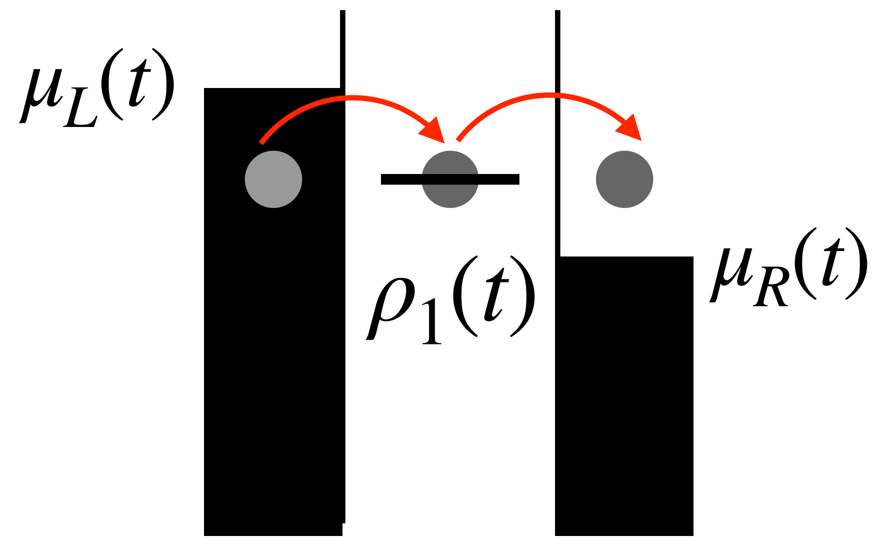
😃 $S(\rho_1(t)||\rho_1^{\mathrm{SS}})$ characterize the distance from steady state

😥 difficult to compare the different time

Our choice

Definition 2



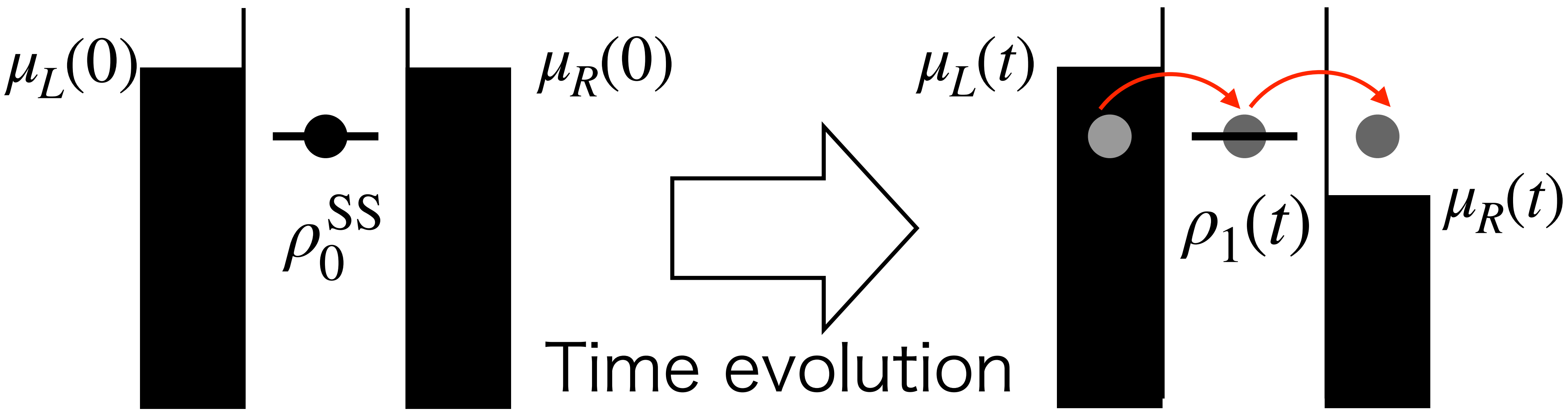


$S(\rho_1(t)||\rho_1^{\mathrm{SS}})$

- Deviation from the steady state $S(\rho_1^{\mathrm{SS}}||\rho_1^{\mathrm{SS}}) = 0$
- House keeping part is subtracted

Important remark

In the presence of the parameter modulation

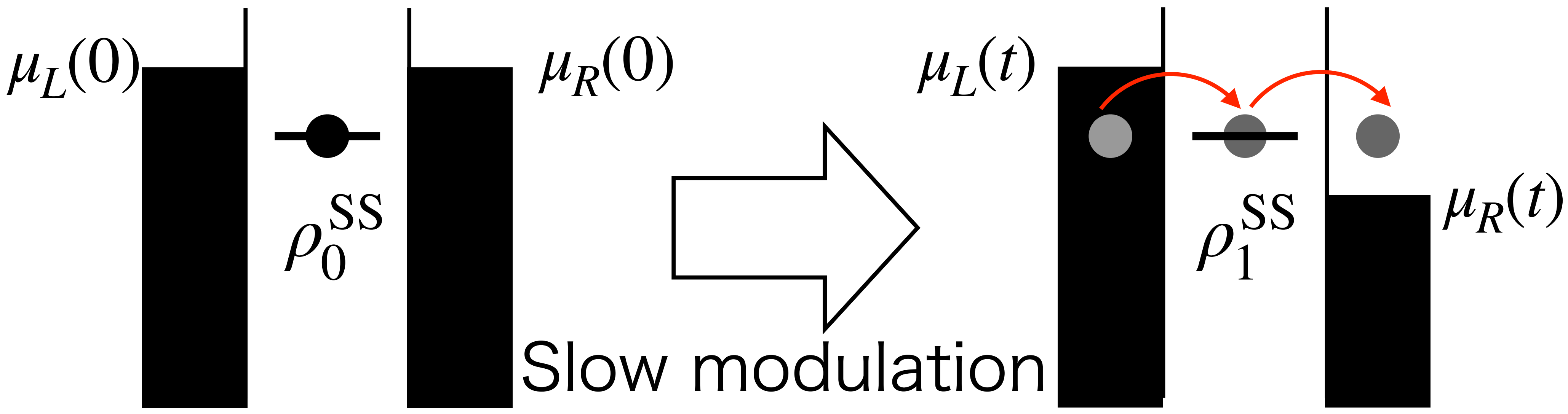


$$S(\rho||\rho_0^{\mathrm{SS}}) \leq S(\mathscr{E}(\rho)||\mathscr{E}(\rho_0^{\mathrm{SS}})) \neq S(\mathscr{E}(\rho)||\rho_0^{\mathrm{SS}})$$

ρ_0^{SS} is no longer the fixed point

Expected behavior

If Modulation is much slower than other time scales



$$S(\rho(0)\,||\,\rho_0^{\mathrm{SS}}) = 0 \qquad S(\rho(t)\,||\,\rho_1^{\mathrm{SS}}) = 0$$

The effect of the modulation can be detected

$$S(\rho(t)\,||\,\rho_1^{\mathrm{SS}}) \neq 0$$

RESULT 1: GENERAL PROPERTY

After quasi steady state is achieved

Case of Slow modulation

Expansion in ϵ

$$\rho = \rho^{\mathrm{SS}} + \epsilon\rho^{(1)} + \cdots$$

ϵ : time scale of the parameter modulation

$\epsilon = 0 \Rightarrow$ adiabatic limit

Relative entropy deviation from steady state

$$\Delta S(\rho\,||\,\rho^{\mathrm{SS}}) = S(\rho^{\mathrm{SS}}\,||\,\rho^{\mathrm{SS}}) - S(\rho\,||\,\rho^{\mathrm{SS}})$$

$$= \mathrm{Tr}\rho\left(\ln\rho - \ln\rho^{\mathrm{SS}}\right)$$

$$= \frac{1}{2}\epsilon^2\mathrm{Tr}[\rho^{(1)}\left(\rho^{\mathrm{SS}}\right)^{-1}\rho^{(1)}] + O(\epsilon^3)$$

Linear response calculation

Supervector formalism

$$\epsilon \frac{d}{d\theta} |\rho\rangle\rangle = \hat{K} |\rho\rangle\rangle, \qquad |\rho\rangle\rangle = |\rho^{\mathrm{SS}}\rangle\rangle + \epsilon |\rho^{(1)}\rangle\rangle + \cdots$$

Spectral decomposition

$$\hat{K} = \sum_m \varepsilon_m |r_m\rangle\rangle\langle\langle \ell_m|$$

ε_m: eigenvalues of $\hat{K}$

$|r_m\rangle\rangle$, $\langle\langle \ell_m|$: right and left eigenvectors

$\hat{K}^+ \equiv \sum_{m\neq 0} \varepsilon_m^{-1} |r_m\rangle\rangle\langle\langle \ell_m|$: pseudo inverse of $\hat{K}$

$$\Rightarrow \quad |\rho^{(1)}\rangle\rangle = \frac{d\Lambda_\mu}{d\theta} \hat{K}^+ \frac{\partial}{\partial \Lambda_\mu} |\rho^{\mathrm{SS}}\rangle\rangle \equiv \frac{d\Lambda_\mu}{d\theta} |\partial^\mu \rho^{\mathrm{SS}}\rangle\rangle$$

Geometrical interpretation

Quantum relative entropy for slow modulation

$$\Delta S(\rho||\rho^{\mathrm{SS}}) = \frac{1}{2}\epsilon^2 \mathrm{Tr}\left[\left(\partial^\mu \rho^{\mathrm{SS}}\right)\left(\rho^{\mathrm{SS}}\right)^{-1}\left(\partial^\nu \rho^{\mathrm{SS}}\right)\right]\frac{d\Lambda_\mu}{d\theta}\frac{d\Lambda_\nu}{d\theta} + O(\epsilon^3)$$

$$= \frac{1}{2}\epsilon^2 \mathrm{Tr}\rho^{\mathrm{SS}}\left[\left(\partial^\mu \ln \rho^{\mathrm{SS}}\right)\left(\partial^\nu \ln \rho^{\mathrm{SS}}\right)\right]\frac{d\Lambda_\mu}{d\theta}\frac{d\Lambda_\nu}{d\theta} + O(\epsilon^3)$$

$$\Delta S(\rho||\rho^{\mathrm{SS}}) = \frac{1}{2}\epsilon^2 g^{\mu\nu}\frac{d\Lambda_\mu}{d\theta}\frac{d\Lambda_\nu}{d\theta} + O(\epsilon^3)$$

$$g^{\mu\nu} = \mathrm{Tr}\rho^{\mathrm{SS}}\left[\left(\partial^\mu \ln \rho^{\mathrm{SS}}\right)\left(\partial^\nu \ln \rho^{\mathrm{SS}}\right)\right]$$:Fisher information matrix

$$= -\mathrm{Tr}\rho^{\mathrm{SS}}\left[\partial^\mu \partial^\nu \ln \rho^{\mathrm{SS}}\right]$$:Hessian matrix of $\ln \rho^{\mathrm{SS}} \times (-1)$

Stability of Steady State

Averaged quantum relative entropy in one-cycle

$$\bar{S}_{\text{cycle}} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{1}{2}\epsilon^2 g^{\mu\nu}\frac{d\Lambda_\mu}{d\theta}\frac{d\Lambda_\nu}{d\theta} + O(\epsilon^3)$$

Positive semi-definiteness

$$g^{\mu\nu}\frac{d\Lambda_\mu}{d\theta}\frac{d\Lambda_\nu}{d\theta} = \mathrm{Tr}\left[\rho^{\mathrm{SS}}\left(\frac{d\Lambda_\mu}{d\theta}\partial^\mu \ln\rho^{\mathrm{SS}}\right)^2\right] \geq 0$$

"Entropy" increases towards any direction

It implies the stability of steady state

Thermodynamic length

Lower bound on "entropy" production

$$\bar{S}_{\rm cycle} = \frac{\epsilon^2}{2}\int_0^{2\pi}\frac{d\theta}{2\pi}g^{\mu\nu}\frac{d\Lambda_\mu}{d\theta}\frac{d\Lambda_\nu}{d\theta}$$

Cauchy-Schwartz inequality

$$\int_0^{2\pi}\frac{d\theta}{2\pi}g^{\mu\nu}\frac{d\Lambda_\mu}{d\theta}\frac{d\Lambda_\nu}{d\theta} \geq \left(\int_0^{2\pi}\frac{d\theta}{2\pi}\sqrt{g^{\mu\nu}\frac{d\Lambda_\mu}{d\theta}\frac{d\Lambda_\nu}{d\theta}}\right)^2$$

$$\bar{S}_{\rm cycle} = \frac{\epsilon^2}{2}\int_0^{2\pi}\frac{d\theta}{2\pi}g^{\mu\nu}\frac{d\Lambda_\mu}{d\theta}\frac{d\Lambda_\nu}{d\theta} \geq \epsilon^2\mathscr{L}^2$$

Thermal length

$$\mathscr{L} = \frac{1}{\sqrt{2}}\int_0^{2\pi}\frac{d\theta}{2\pi}\sqrt{g^{\mu\nu}\frac{d\Lambda_\mu}{d\theta}\frac{d\Lambda_\nu}{d\theta}} = \frac{1}{\sqrt{2}}\oint\sqrt{g^{\mu\nu}d\Lambda_\mu d\Lambda_\nu}$$

WORK AND EFFICIENCY

Work relations

The “work” (see, for instance, Jarzynski 1997)

$$W = \int dt\dot{E}(t) = \int dt\langle \dot{E}(t)\rangle = \int dt\langle \dot{H}(t)\rangle$$

If the time dependence only appears through parameters

$$\rho(t),\, H(t) \to \rho(\Lambda^\mu(t)),\, H(\Lambda^\mu(t))$$

Chain rule $\dot{H}(t) = \dfrac{d\Lambda^\mu}{dt}\dfrac{\partial H}{\partial \Lambda^\mu}$ yields

$$W = \int \mathrm{Tr}\left(\rho \frac{\partial H}{\partial \Lambda^\mu}\right)\frac{d\Lambda^\mu}{dt}dt$$

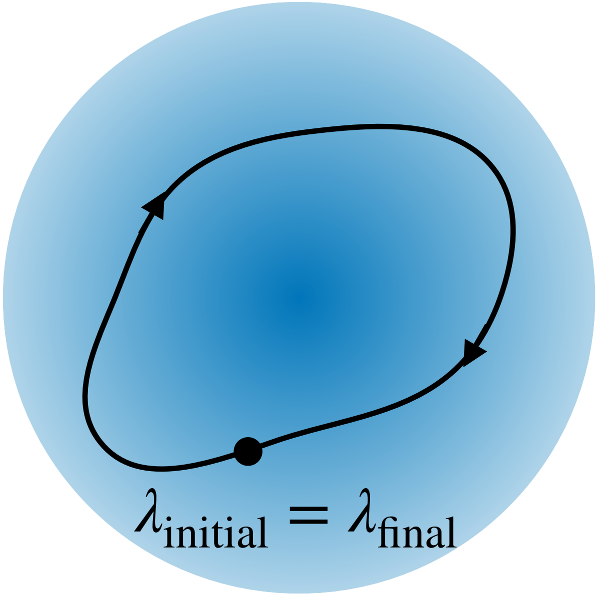
Geometrical interpretation

Work represented by contour integral

$$W = \int \mathrm{Tr}\left(\rho \frac{\partial H}{\partial \Lambda^\mu}\right) \frac{d\Lambda^\mu}{dt} dt = \int \mathrm{Tr}\left(\rho \frac{\partial H}{\partial \lambda^\mu}\right) d\Lambda^\mu$$

In the case of the cyclic modulation $\lambda_{\text{initial}} = \lambda_{\text{final}}$

$$W = \oint d\Lambda^\mu \, \mathrm{Tr}\left(\rho \frac{\partial H}{\partial \Lambda^\mu}\right)$$



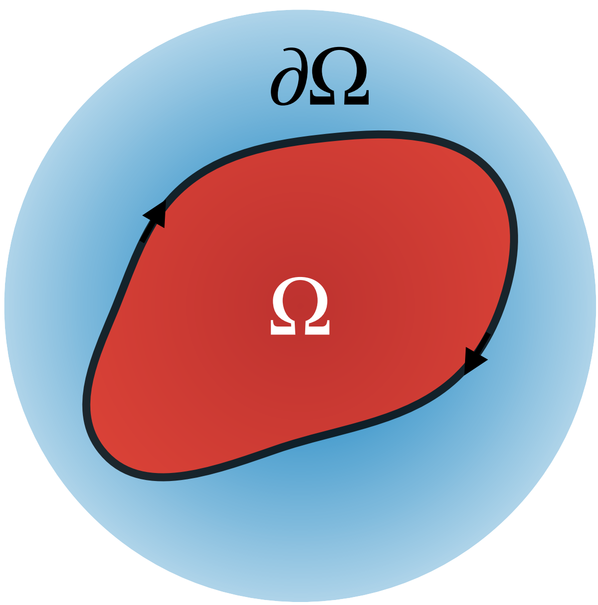
Work is given by the contour integral of the vector field

$$W = \oint \mathscr{P}_\mu d\Lambda^\mu \qquad \mathscr{P}_\mu = \mathrm{Tr}\left(\rho \frac{\partial H}{\partial \Lambda^\mu}\right)$$

Work-density-tensor in parameter space

By using the Stokes' theorem

$$W = \oint_{\partial\Omega} \mathscr{P}_\mu d\Lambda^\mu$$



$$W = \int_{\Omega} \mathscr{R}_{\mu\nu} dS^{\mu\nu}$$

$\mathscr{R}_{\mu\nu} = \dfrac{\partial \mathscr{P}_\nu}{\partial \Lambda_\mu} - \dfrac{\partial \mathscr{P}_\mu}{\partial \Lambda_\nu}$: curvature associated with the work

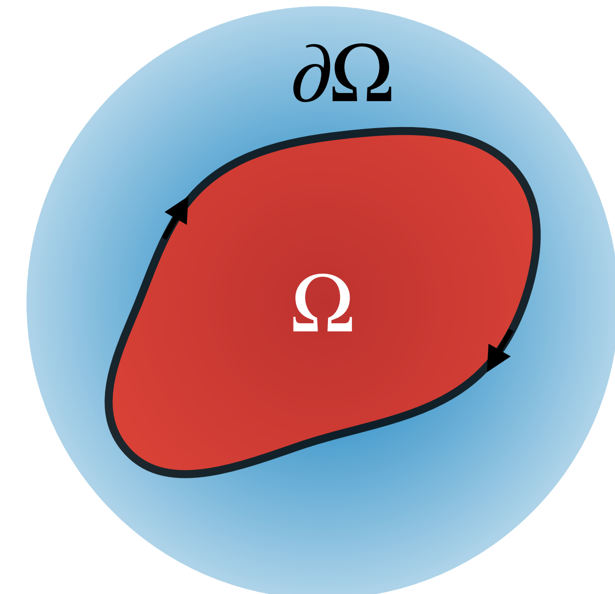
$$dS^{\mu\nu} = \frac{1}{2} d\Lambda_\mu \wedge d\Lambda_\nu$$

Work in one-cycle

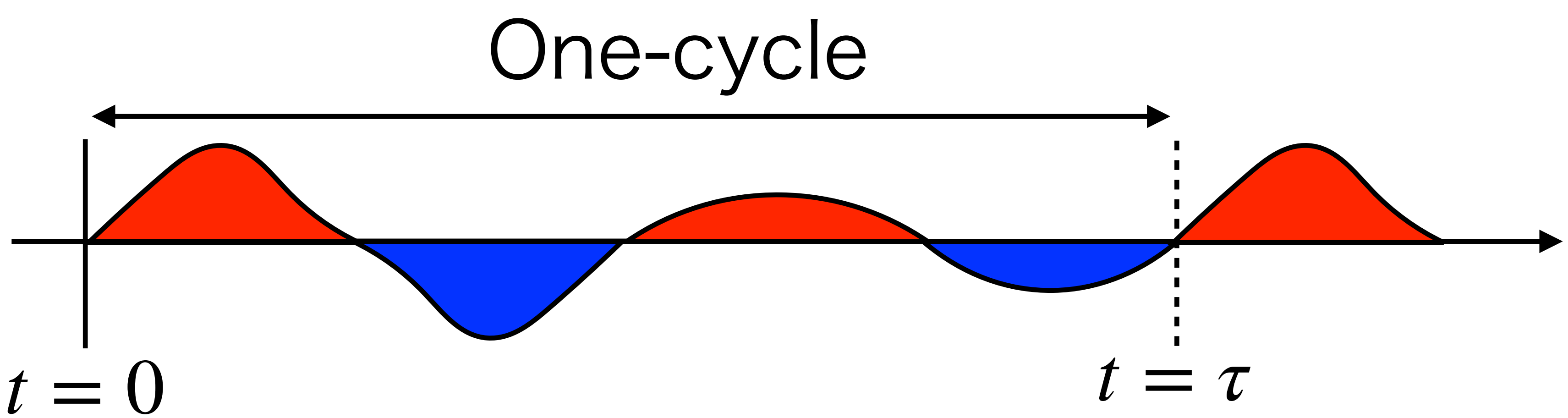
Various representations of work

$$W = \int dt \langle \rho(t) \dot{H}(t) \rangle = \oint \mathscr{P}_\mu d\Lambda^\mu = \int_\Omega \mathscr{R}_{\mu\nu} dS^{\mu\nu}$$

$$\mathscr{P}_\mu = \mathrm{Tr}\left(\rho \frac{\partial H}{\partial \Lambda^\mu} \right) \qquad \mathscr{R}_{\mu\nu} = \frac{\partial \mathscr{A}_\nu}{\partial \Lambda_\mu} - \frac{\partial \mathscr{A}_\mu}{\partial \Lambda_\nu}$$



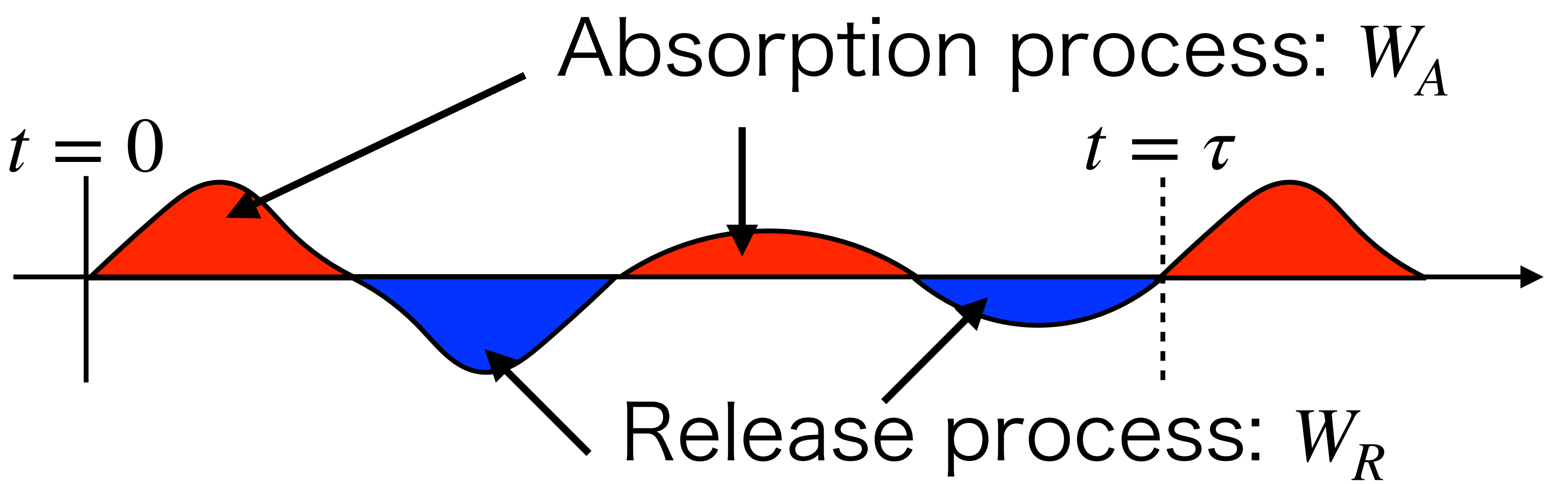
"Work" is either positive or negative (opposite direction)



If $W < 0$, work is extracted from the system

Efficiency

Absorption process and release process



$$Q_A = \oint \frac{\mathscr{P}_\mu + \left|\mathscr{P}_\mu\right|}{2} d\Lambda^\mu, \quad Q_R = \oint \frac{\mathscr{P}_\mu - \left|\mathscr{P}_\mu\right|}{2} d\Lambda^\mu$$

Efficiency: $\eta \equiv \dfrac{|W|}{Q_A} = \dfrac{|Q_A + Q_B|}{Q_A}$

RESULT 2: EFFICIENCY AND POWER

"First low" in the presence of modulation

The "work", "entropy" production, and heat

$$W = \oint d\Lambda^\mu \ \mathrm{Tr}\left(\rho \frac{\partial H}{\partial \Lambda^\mu}\right)$$

$$\bar{S}_{\mathrm{cycle}} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{1}{2}\epsilon^2 g^{\mu\nu}\frac{d\Lambda_\mu}{d\theta}\frac{d\Lambda_\nu}{d\theta} + O(\epsilon^3)$$

$$\Rightarrow \quad Q \equiv W + T\bar{S}_{\mathrm{cycle}}$$

Efficiency is given by

$$\eta^{\mathrm{eff}} \equiv \frac{W}{W + T\bar{S}_{\mathrm{cycle}}} \simeq 1 - \frac{T\bar{S}_{\mathrm{cycle}}}{W} \leq 1 - \epsilon^2 \frac{T\mathscr{L}^2}{W}$$

$$\eta^{\mathrm{eff}} \leq 1 - \epsilon^3 \frac{T\mathscr{L}^2}{P}$$ $P = \epsilon W$: power

Trade-off relation

Upper bound on power

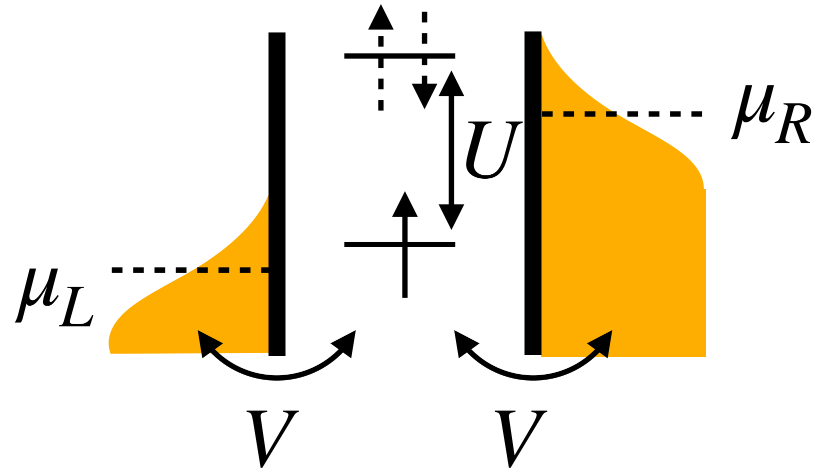
$$\eta^{\mathrm{eff}} \leq 1 - \epsilon^3 \frac{T\mathscr{L}^2}{P}, \qquad \epsilon = \frac{P}{W}$$

$$\Rightarrow \quad \eta^{\mathrm{eff}} \leq 1 - \epsilon^3 \frac{T\mathscr{L}^2}{P} = 1 - P^2 \frac{T\mathscr{L}^2}{W^3}$$

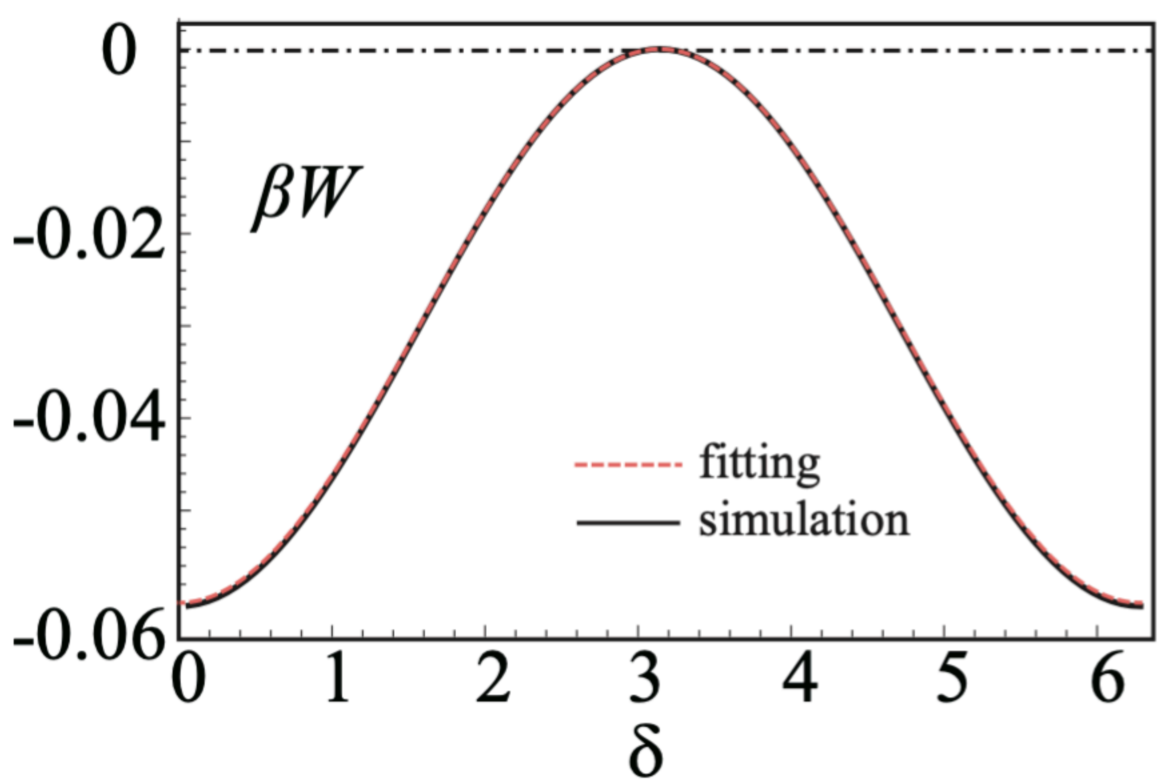
$$\Rightarrow \quad P^2 \leq \frac{W^3}{T\mathscr{L}^2}\left(1 - \eta^{\mathrm{eff}}\right)$$

$$\Rightarrow \quad P \leq \frac{W}{\mathscr{L}}\sqrt{\frac{W}{T}}\sqrt{1 - \eta^{\mathrm{eff}}}$$

Larger efficiency ⇒ lower power



μ_R
U
μ_L
V
V



βW
fitting
simulation
0
-0.02
-0.04
-0.06
0 1 2 3 4 5 6
δ

$$|\rho(\theta)\rangle\rangle = |r_0(\theta)\rangle\rangle - \sum_{i\neq 0}\int_0^\theta d\phi e^{\int_\phi^\theta d\chi \epsilon^{-1}\varepsilon_i(\chi)}|r_i(\theta)\rangle\rangle\langle\langle \ell_i(\phi)|\frac{d}{d\phi}|r_0(\phi)\rangle\rangle$$

$$-\sum_{i\neq 0}\int_0^\theta d\phi e^{\int_\phi^\theta d\chi \epsilon^{-1}\varepsilon_i(\chi)}|r_i(\theta)\rangle\rangle\langle\langle \ell_i(\phi)|\frac{d}{d\phi}|r_0(\phi)\rangle\rangle$$

$$\sim -\sum_{i\neq 0}\int_0^\theta d\phi e^{\int_\phi^\theta d\chi \epsilon^{-1}\varepsilon_i(\theta)}|r_i(\theta)\rangle\rangle\langle\langle \ell_i(\theta)|\frac{d}{d\theta}|r_0(\theta)\rangle\rangle$$

$$\simeq \epsilon\sum_{i\neq 0}\frac{1}{\varepsilon_i}\left(1 - e^{\epsilon^{-1}\varepsilon_i\theta}\right)|r_i(\theta)\rangle\rangle\langle\langle \ell_i(\theta)|\frac{d}{d\theta}|r_0(\theta)\rangle\rangle$$