A framework for crossover of scaling law using a self-similar solution : dynamical impact of the solid sphere onto the viscoelastic PDMS board.

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https://arxiv.org/abs/2203.00253

Soft matter : Mixed physical property, Scale-locality.

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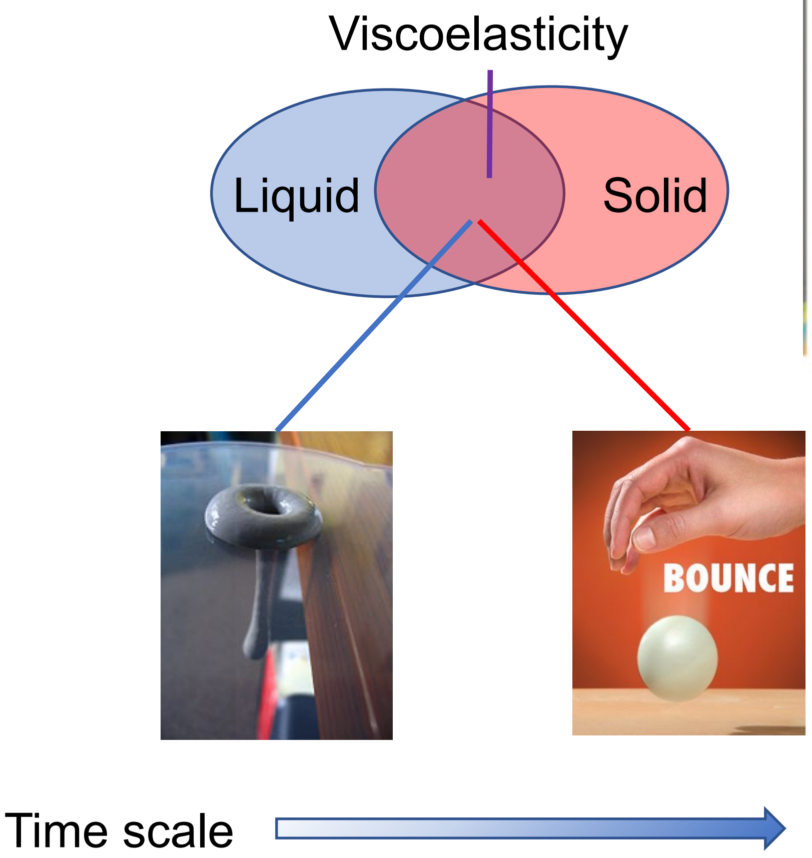
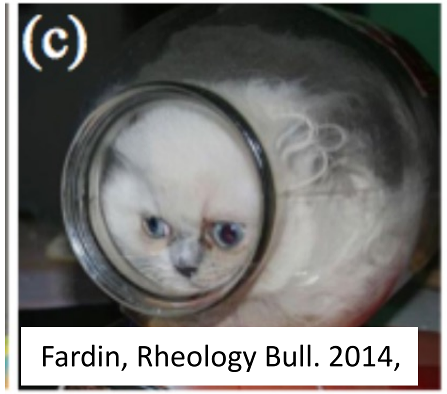
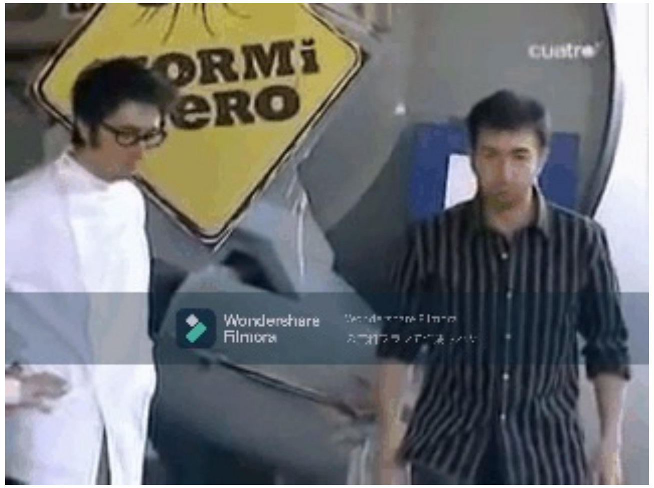
- ***Mixed*** physical property

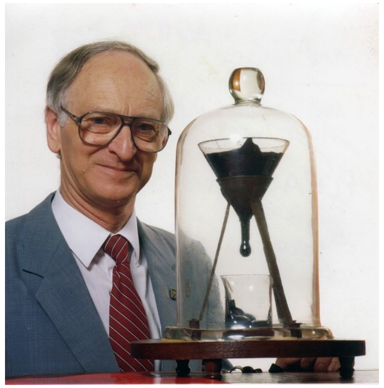




Fardin, Rheology Bull. 2014,

- ***Scale-local*** behavior





$$\frac{dV}{dT} = \begin{cases} 0 & (\mathrm{De} \gg 1) \\ \frac{\pi d^4 \rho g}{128\mu}\left(1+\frac{h}{l}\right) & (\mathrm{De} \sim 1)\,. \end{cases}$$

Parnell, T. *et al*. Eur.J.Phys. (1984)

Crossover of scaling law

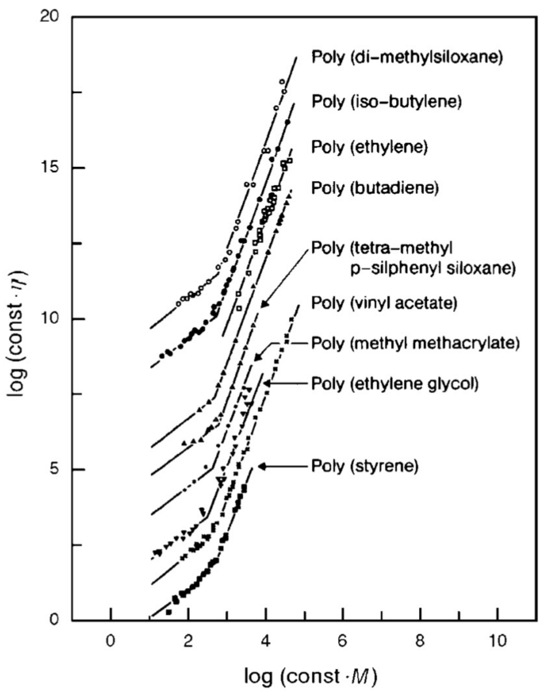
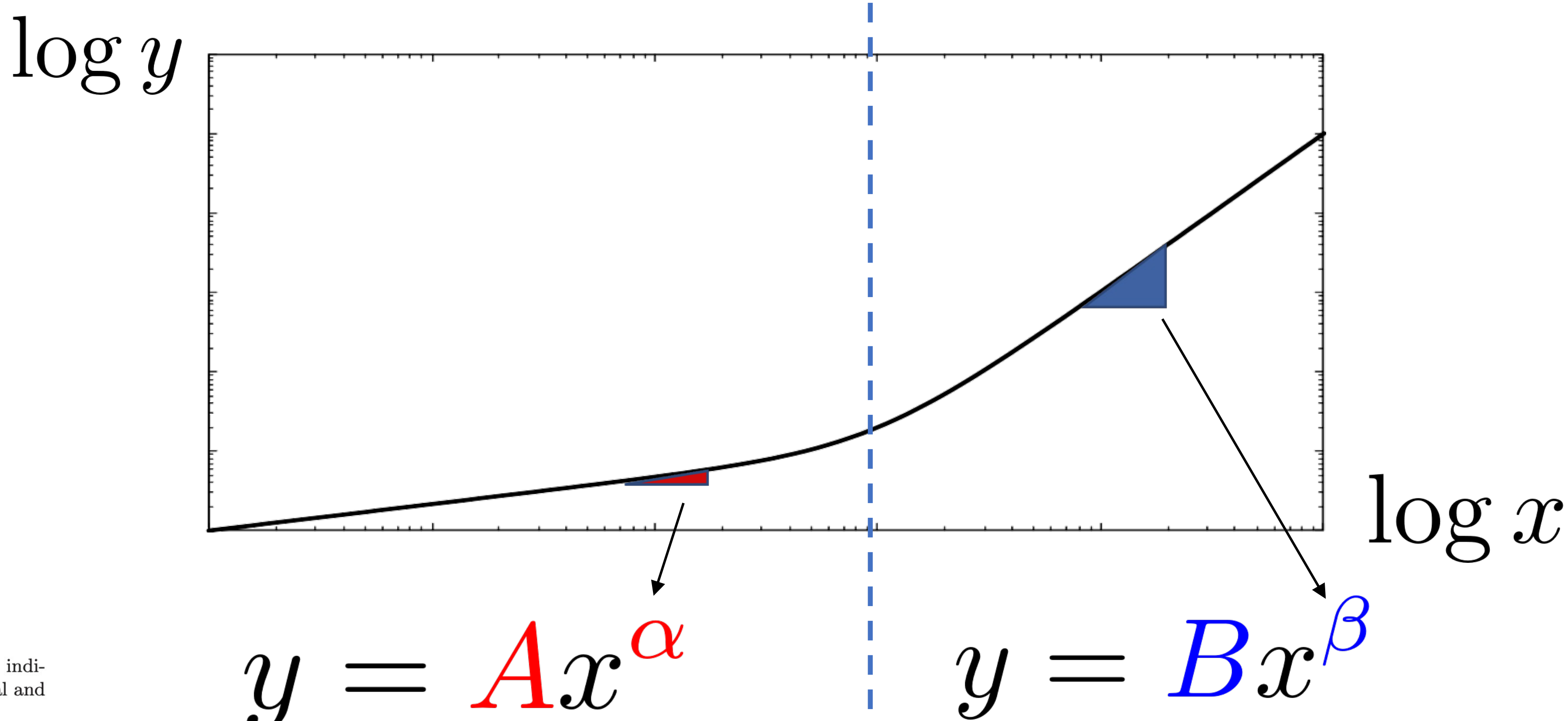


Fig. 6.13. Molecular weight dependence of the viscosity as observed for the indicated polymers. For better comparison curves are suitably shifted in horizontal and vertical directions. Data from Berry and Fox [71]

$$\eta \sim \begin{cases} M & (M < M_c) \\ M^{3.2-3.6} & (M > M_c) \end{cases}$$



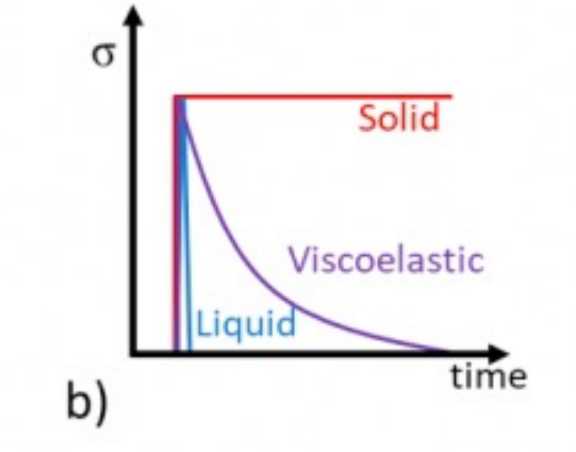
$$y = Ax^{\alpha} \qquad y = Bx^{\beta}$$

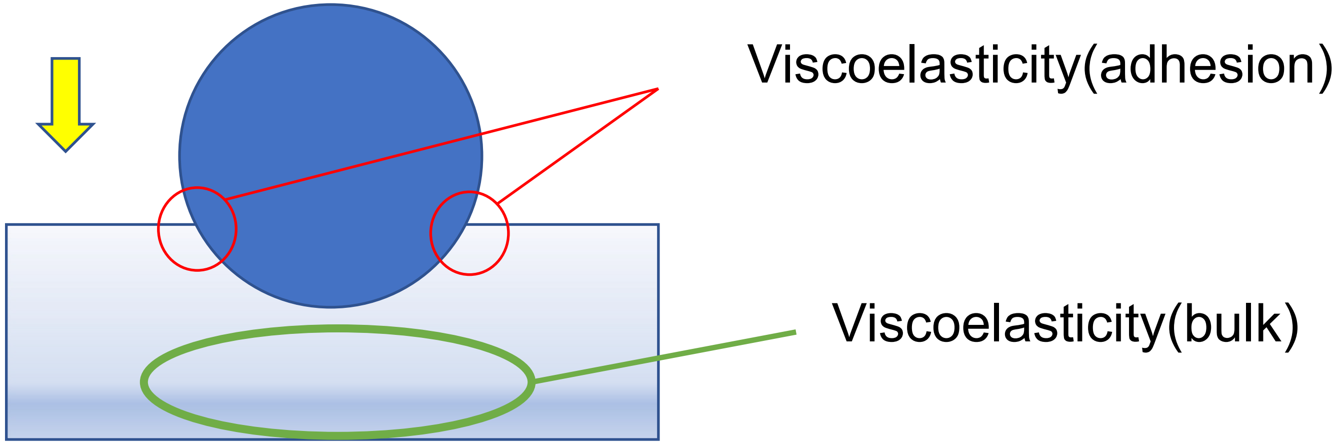
x →

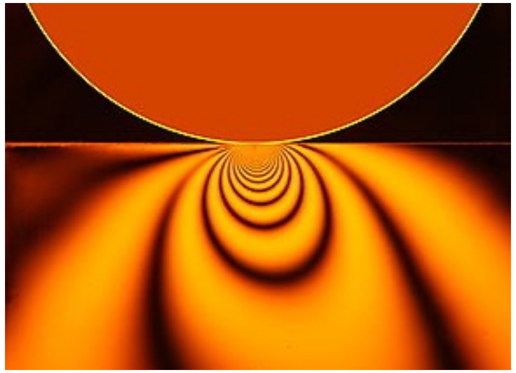
- **Crossover of scaling is the process of transition of scaling law by continuous change of scale parameter**
- **Crossover of scaling law corresponds to a transition of physical law**

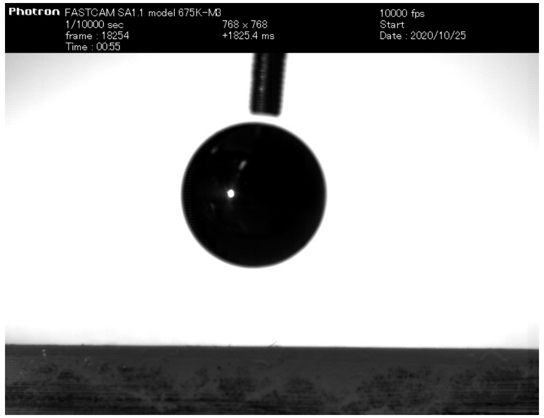
Viscoelascitiy on contact mechanics

Cf. LS Instrument





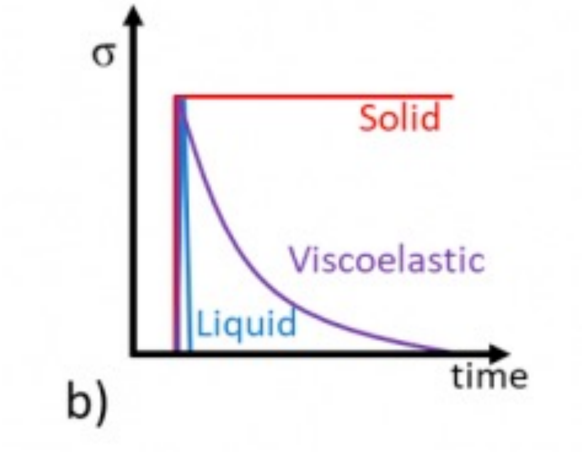


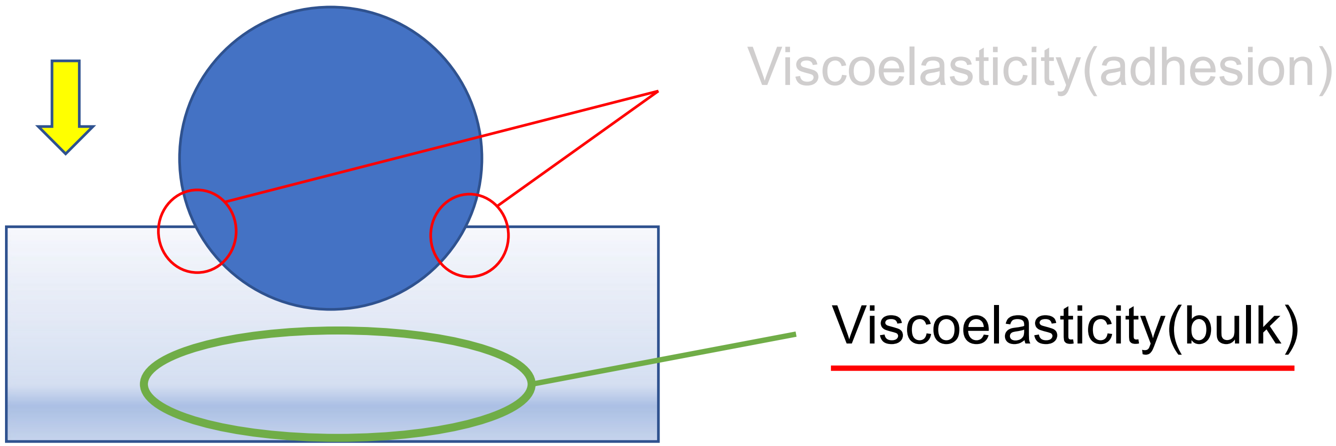


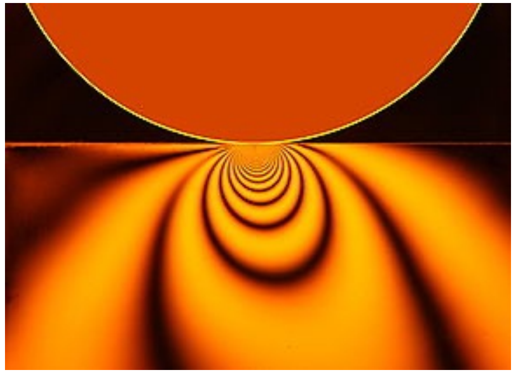
- Velocity-dependent behaviors derived from viscoelasticiy appear.
- The viscoelasticity derived from the adhesion is dominant for contact though here I focus on the viscoelasticity from bulk.

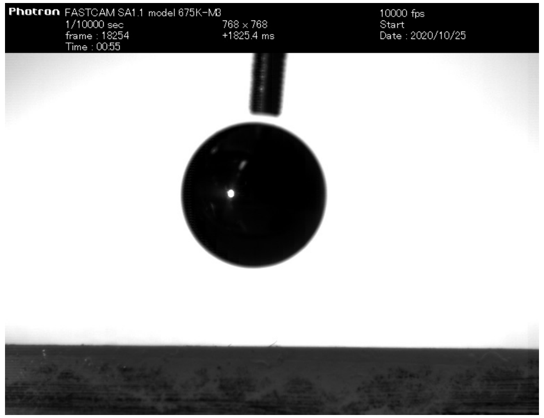
Viscoelascitiy on contact mechanics

Cf. LS Instrument









- Velocity-dependent behaviors derived from viscoelasticiy appear.
- The viscoelasticity derived from the adhesion is dominant for contact though here I focus on the viscoelasticity from bulk.

Research Question

- How do physical phenomena change?
- We have failed to study crossover as continuous process.
- The actual viscoelastic contribution derived from bulk to dynamical impact had been unclear.

⇩

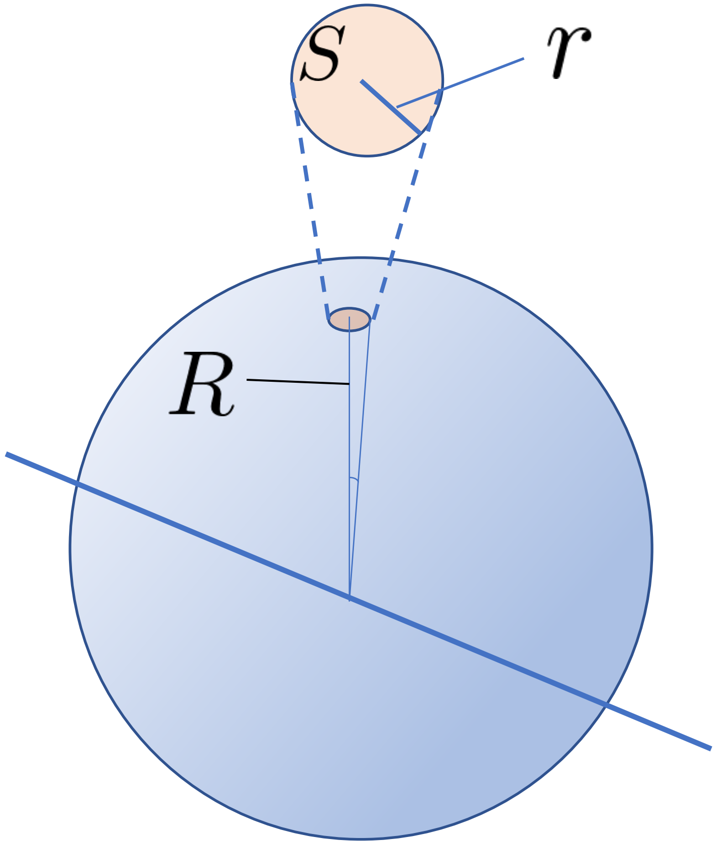
Question to method

- What is a framework of crossover of scaling law?

Question to a phenomenon

- How is the viscoelasticity derived from bulk involved in the dynamical contact?

How the scaling law is formalized: intermediate asymptotics

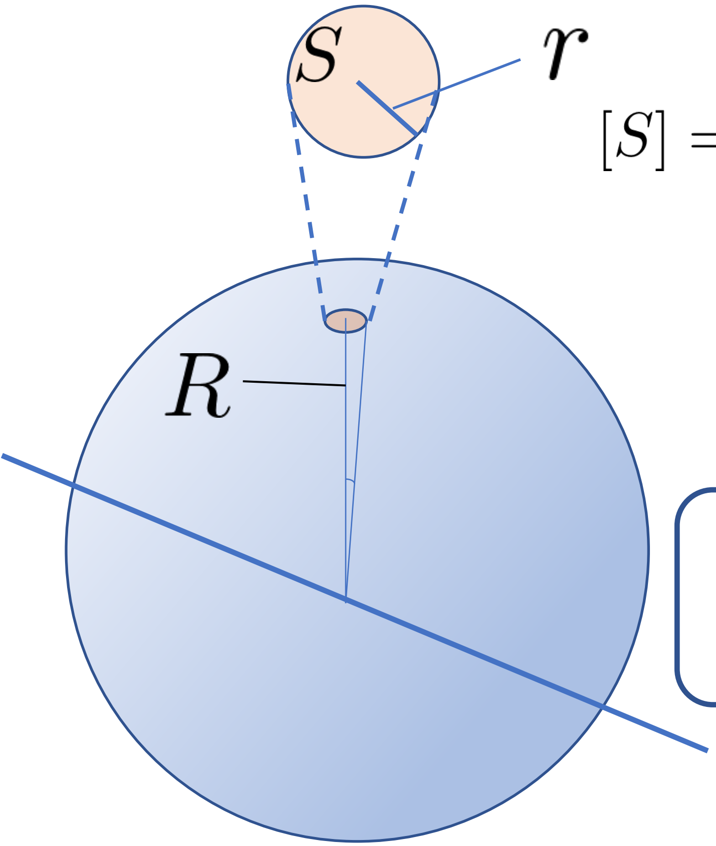


$$S \sim r^{\alpha} \quad ?$$

Barenblatt, *Scaling* (CUP 2003)

G.I. Barenblatt

How the scaling law is formalized: intermediate asymptotics



$$[S] = \mathrm{cm}^2,\ [r] = \mathrm{cm},\ [R] = \mathrm{cm}$$

$$\Pi = \frac{S}{r^2} \quad \theta = \frac{r}{R}$$

$$\Pi = \Phi(\theta)$$

Self-similar solution

Barenblatt, *Scaling* (CUP 2003)

How the scaling law is formalized: intermediate asymptotics

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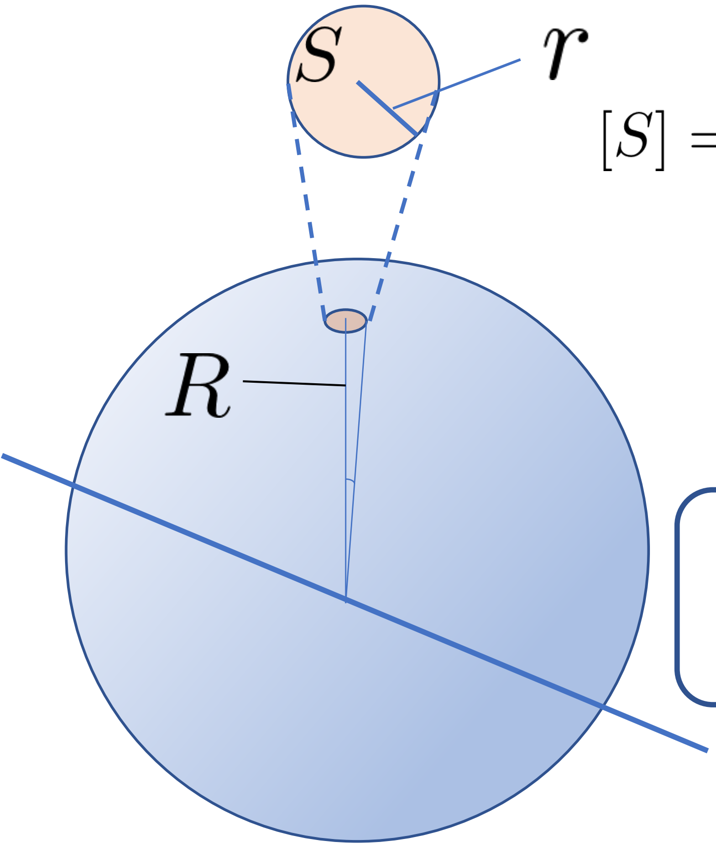
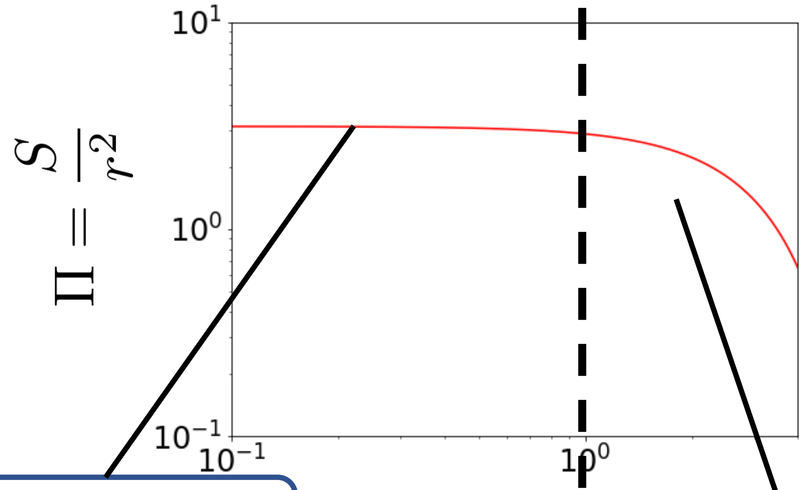


$$[S] = \mathrm{cm}^2,\ [r] = \mathrm{cm},\ [R] = \mathrm{cm}$$

$$\Pi = \frac{S}{r^2} \quad \theta = \frac{r}{R}$$

$$\Pi = \Phi\left(\theta\right)$$



$$\theta = \frac{r}{R}$$

$$\Pi \simeq 3.14$$

$$\Pi \neq \mathrm{const}$$

$$S = 3.14\, r^2$$

$$(0 < r \ll R)$$

Intermediate asymptotics

Barenblatt, *Scaling* (CUP 2003)

- **Intermediate asymptotics** is an asymptotic representation valid in a certain range of physical parameters. → *Convergence of* $\boldsymbol{\Phi}$ *is a condition for a stable scaling law.*

Intermediate asymptotics

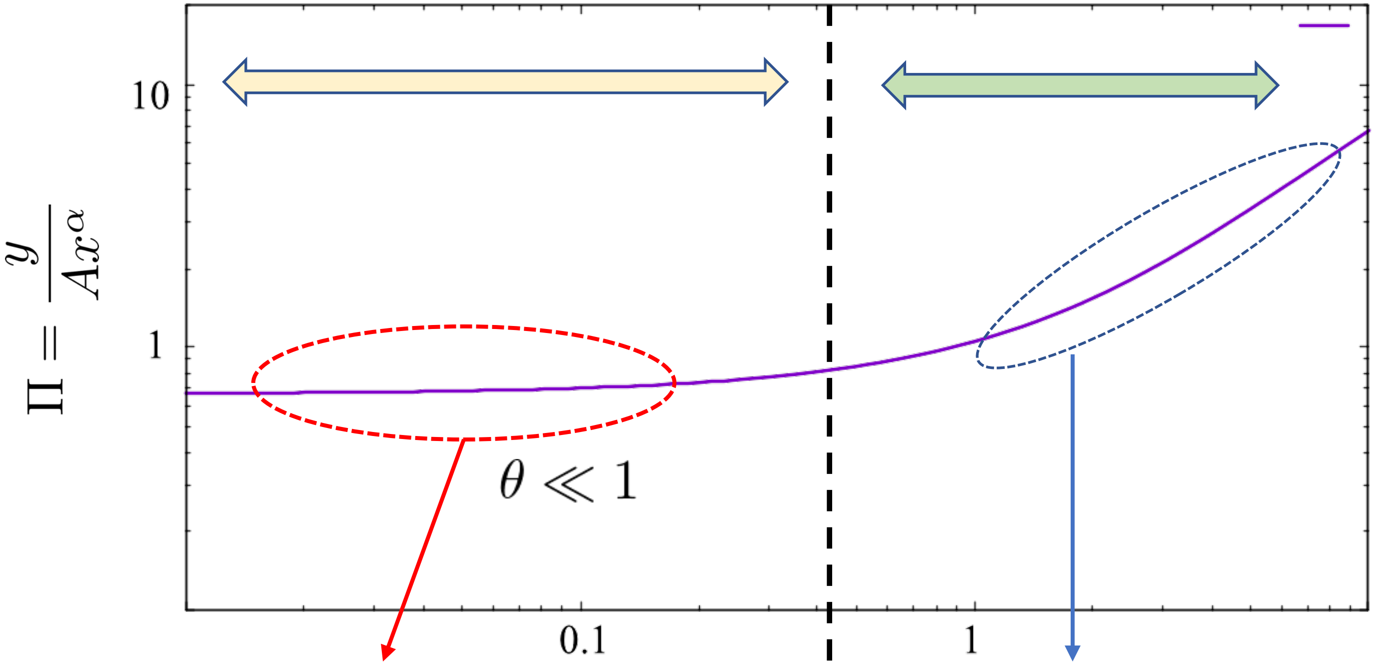
- Intermediate asymptotic is an asymptotics representation of a function valid in a certain range of independent variables.

- Ideal gas equation is an intermediate asymptotic in which the volume of molecules b and the molecular interaction a are negligible on a van der Waals equation.

$$p = \frac{nRT}{V - nb} - \frac{an^2}{V^2} \to \frac{nRT}{V} \quad \left(\frac{an^2}{V} \ll p \ll \frac{RT}{b} \right)$$

A framework for crossover of scaling law

Φ idealized region | nonidealized region



$$\theta = \frac{z}{x^{\beta}}$$

$$\Pi = \Phi(\theta)$$

Self-similar solution

$$\Phi = \text{const}$$

$$\Phi \neq \text{const}$$

$$y = Ax^{\alpha} \longrightarrow y = Bx^{\beta}$$

Intermediate asymptotics

- **Φ** ≠ const, corresponds to a crossover of scaling law?

Research Strategy

Strategy

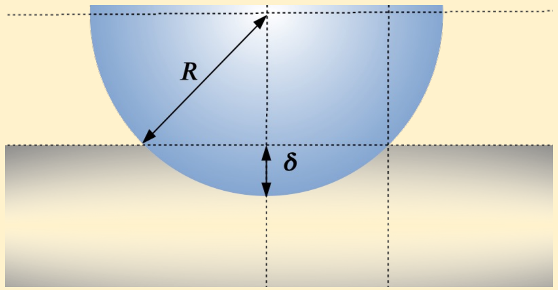
- A stability of scaling laws can be understood by intermediate asymptotics. The convergence of self-similar solution to a finite limit gives the asymptotic expression of scaling law.

$$\lim_{\theta \to 0} \Phi(\theta) = \mathrm{const} \quad \Longleftrightarrow \quad y = Ax^{\alpha} \quad (\theta \ll 1)$$

- Incomplete convergence of scaling law generates the transition of scaling law?

$$\Phi(\theta) \neq \mathrm{const} \quad \Longleftrightarrow \quad y = Ax^{\alpha} \longrightarrow Bx^{\gamma} \quad ?$$

- What is a self-similar solution that describes the transition from elastic impact to viscoelastic impact?



$$\Longrightarrow \quad \Psi = \Phi(Z) \quad ?$$

Experimental set up

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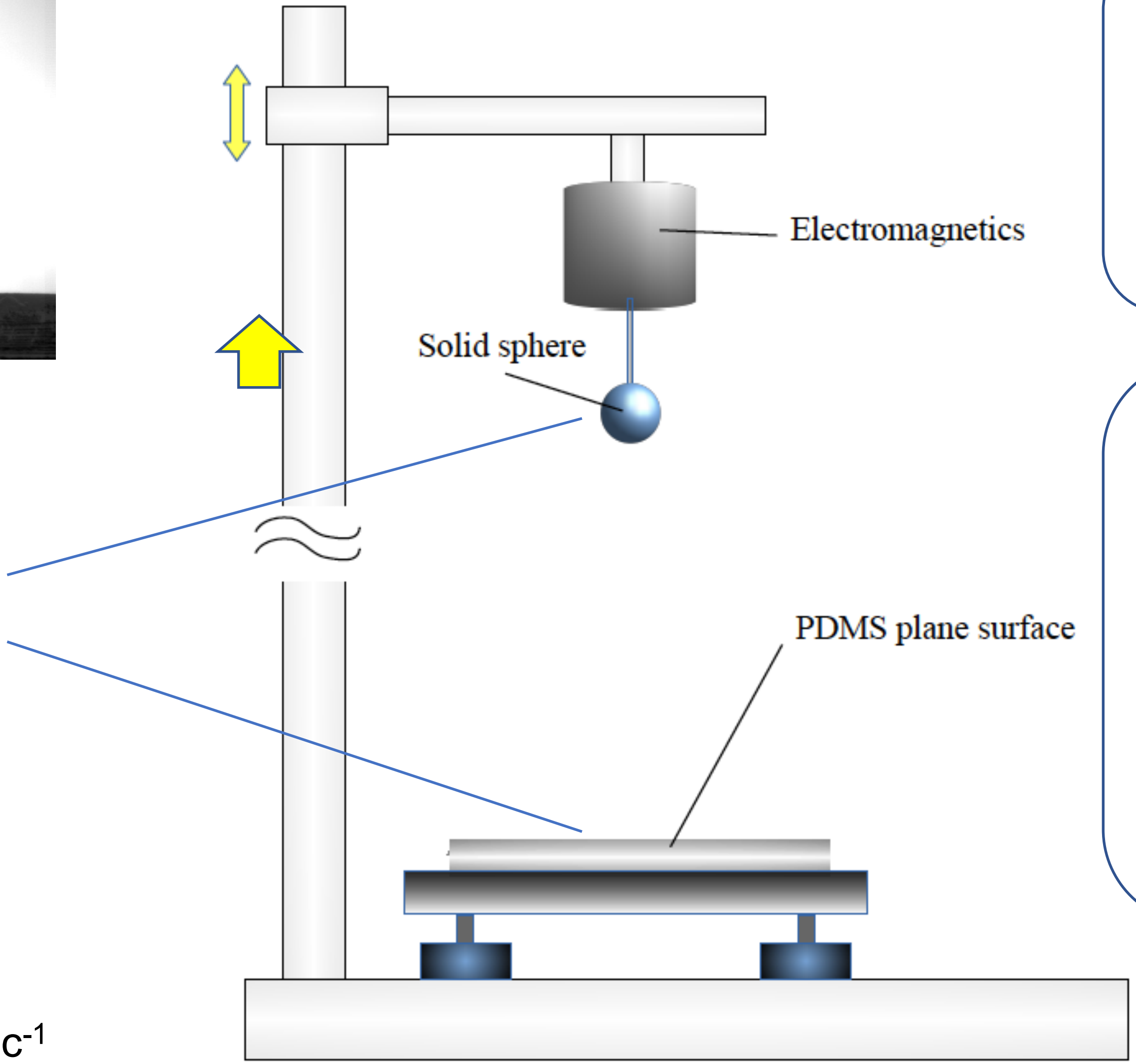
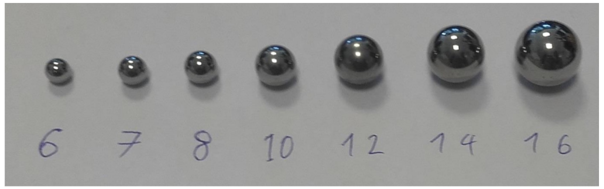
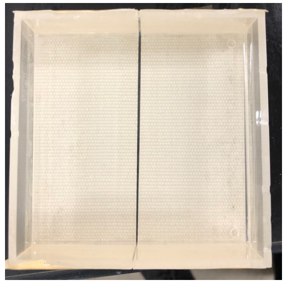
High speed camera
(FASTCAM SA1.1)
Flame rate = 10000 f sec^{-1}





ρ=7800 kg/m3
R= 3.0, 4.0, 5.0, 7.0, 8.0 mm



Polydimethylsiloxane

E ≃ 0.78 MPa
μ = 141 Pa·s

https://arxiv.org/abs/2203.00253

Dynamical impact of sphere on R = 8.0 mm

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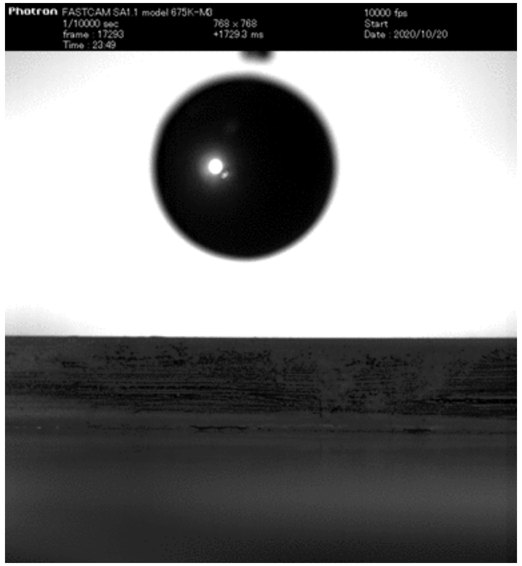

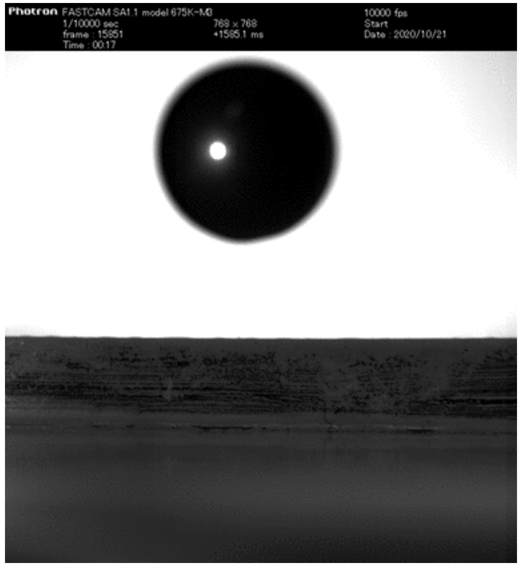

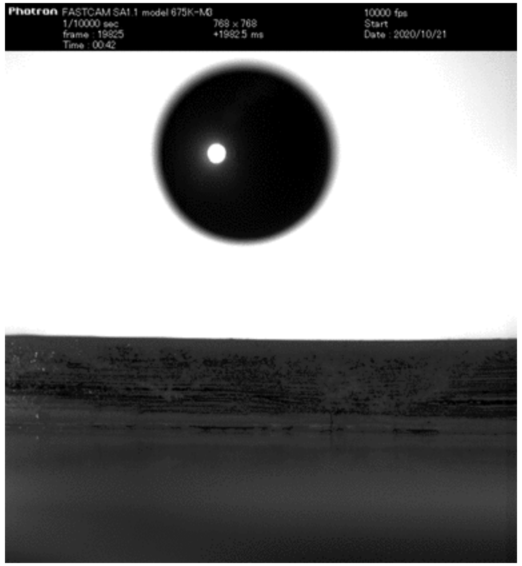

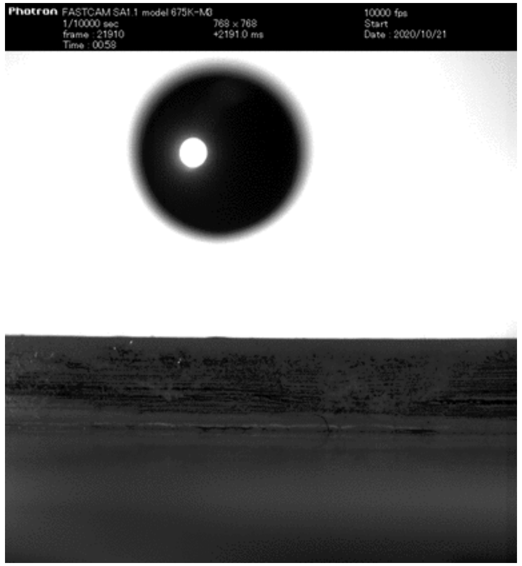

R = 8.0 mm
Vi = 390 mm/s
ρ=7800 kg/m3

Mettalic ball

R = 8.0 mm
Vi = 750 mm/s
ρ=7800 kg/m3

Mettalic ball

R = 8.0 mm
Vi = 1290 mm/s
ρ=7800 kg/m3

Mettalic ball

R = 8.0 mm
Vi = 2302 mm/s
ρ=7800 kg/m3

Mettalic ball

Dynamical impact of sphere on vi = 400 mm/s

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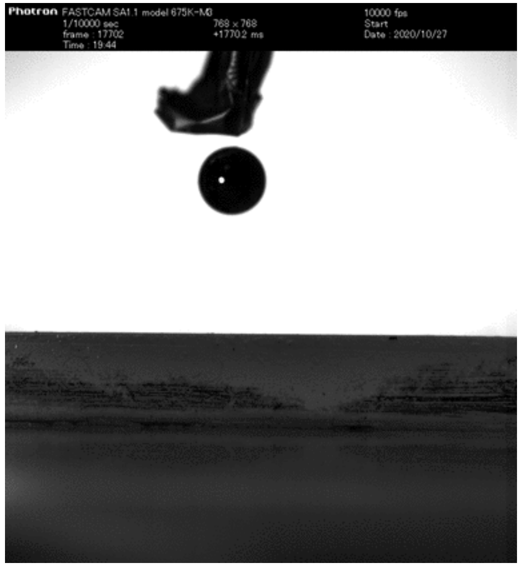
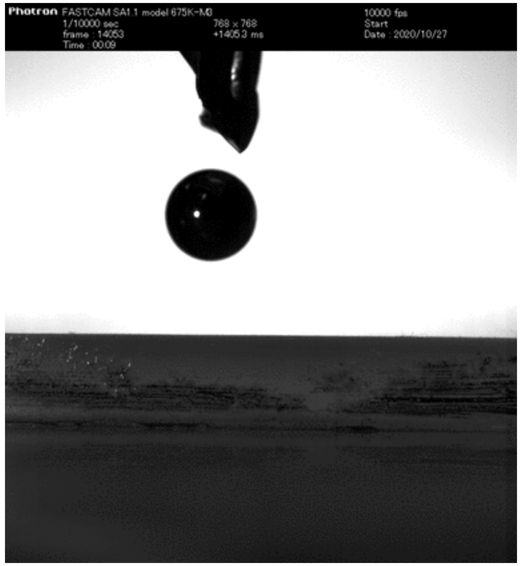


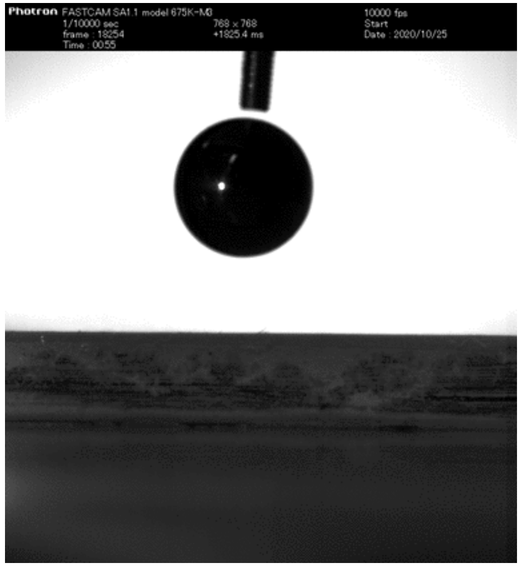
R = 3.0 mm
Vi = 471 mm/s
ρ=7800 kg/m3

Mettalic ball



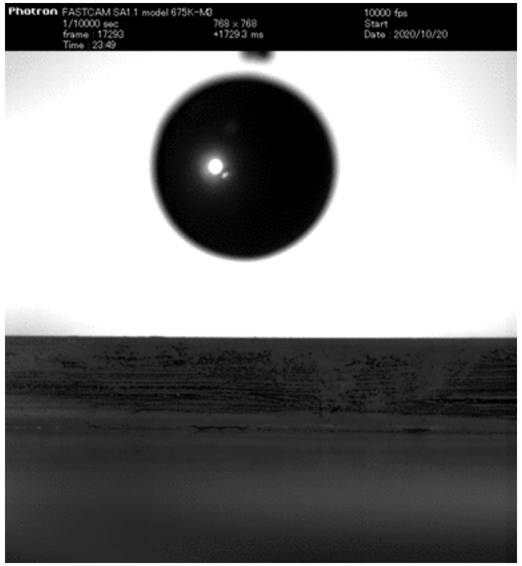
R = 4.0 mm
Vi = 430 mm/s
ρ=7800 kg/m3

Mettalic ball



R = 6.0 mm
Vi = 369 mm/s
ρ=7800 kg/m3

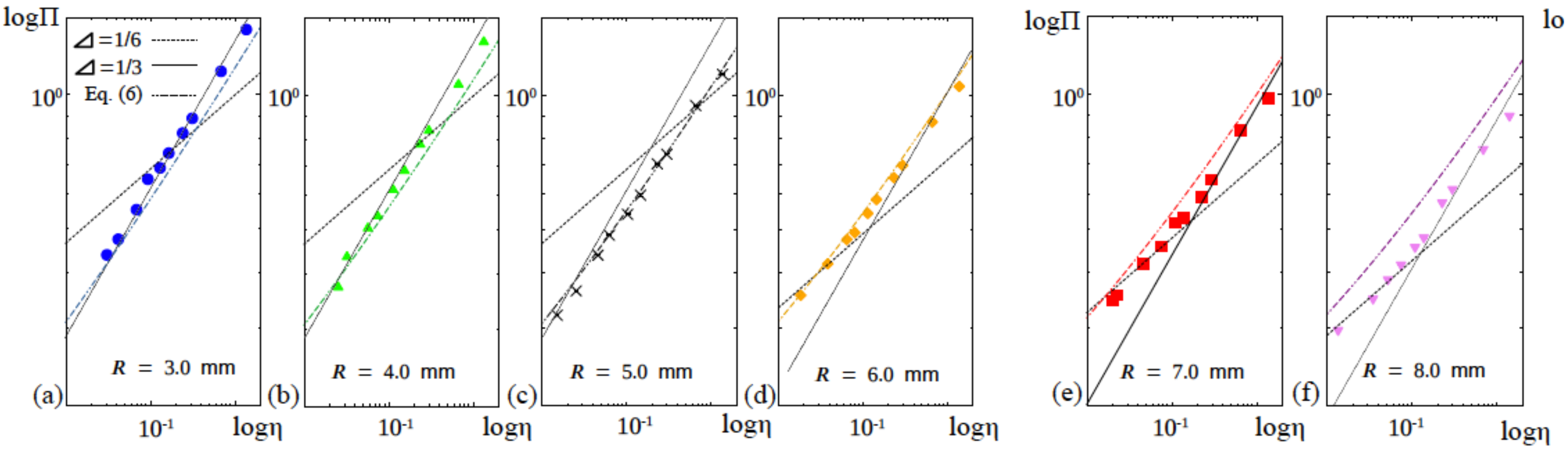
Mettalic ball



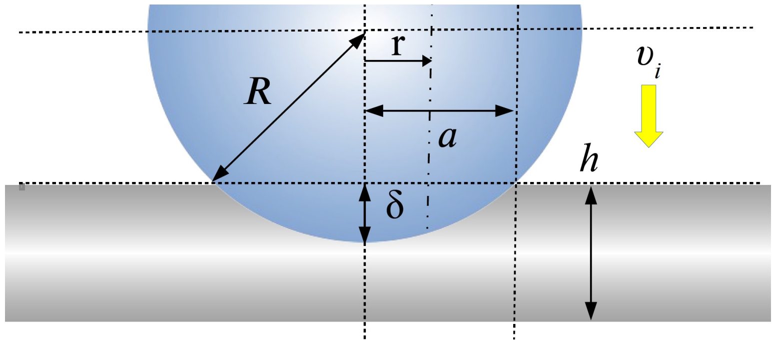
R = 8.0 mm
Vi = 390 mm/s
ρ=7800 kg/m3

Mettalic ball

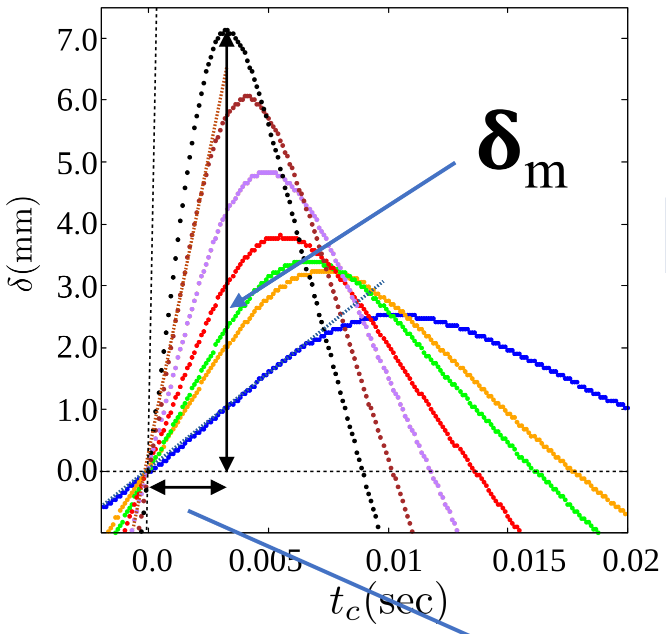
The maximum deformations and the impact velocities

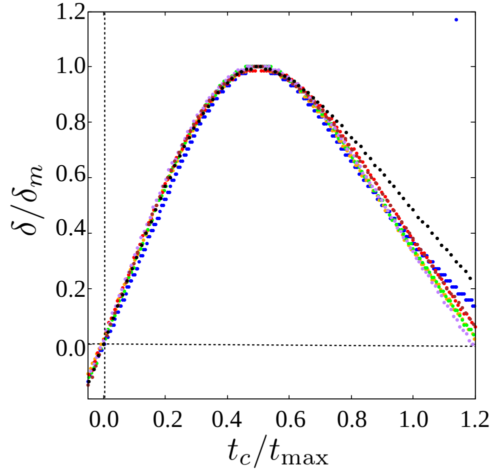


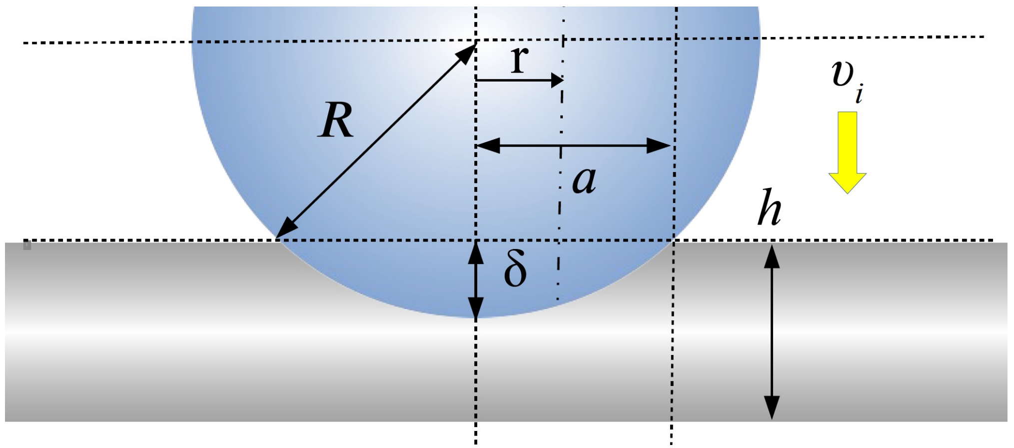
$$\Pi = \frac{\delta_m}{R} \quad \eta = \frac{\rho v_i^2}{E}$$



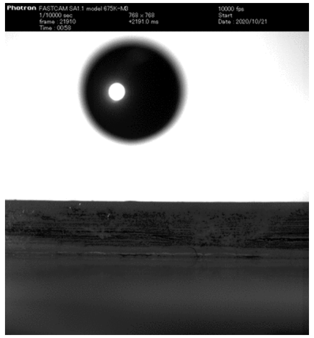
Time evolution of deformation on the impacts



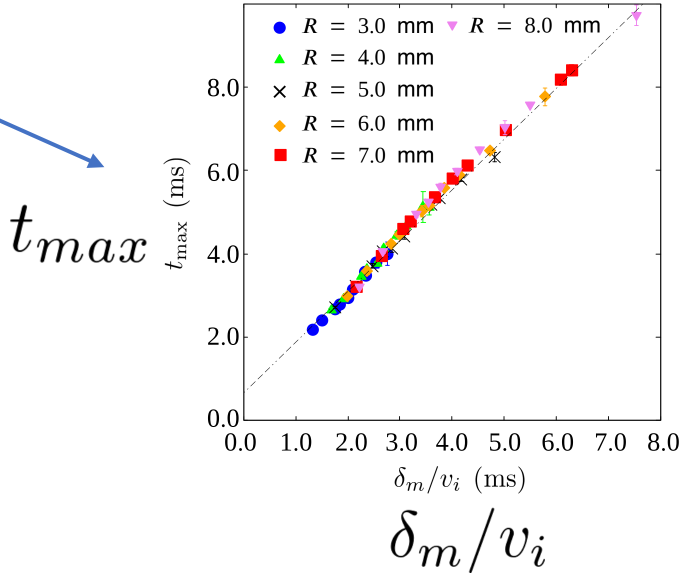




$$\Pi = \frac{\delta}{R},\ \kappa = \frac{h}{R},\ \eta = \frac{\rho v^2}{E},\ \theta = \frac{\mu}{E^{1/2}\rho^{1/2}R},$$

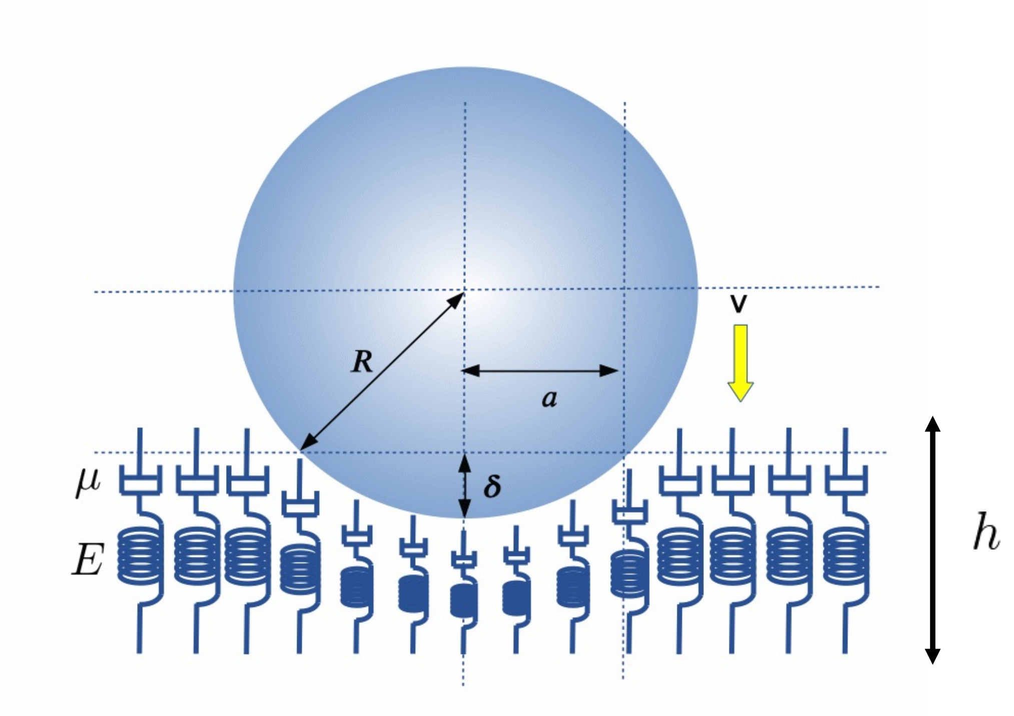


R = 8.0 mm
Vi = 2302 mm/s



- The similarities of the attractors of deformations is found.
- A scaling relation between the contact times and the impact velocities.

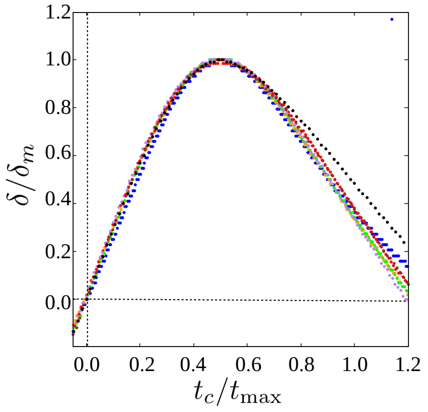
Maxwell Viscoelastic Foundation model



$$\sigma + \frac{\mu}{E}\frac{d\sigma}{dt} = \mu\frac{d\epsilon}{dt}$$

σ : stress
ϵ : strain

$$\frac{d\delta}{dt} = v_i$$



$$E_{MVF} = \frac{\phi\mu\pi R\delta_m^2}{h}\frac{d\delta}{dt}\left[1 - e^{-\frac{E}{\mu}t_c}\right]$$

https://arxiv.org/abs/2203.00253

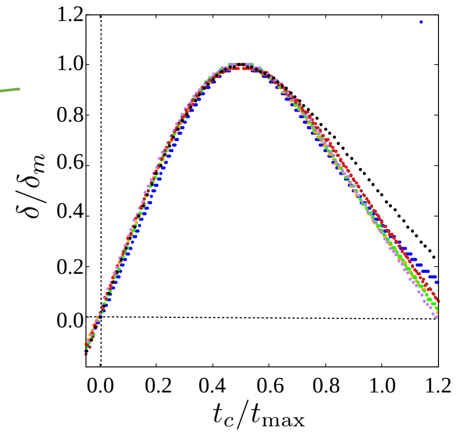
Maxwell Viscoelastic Foundation model

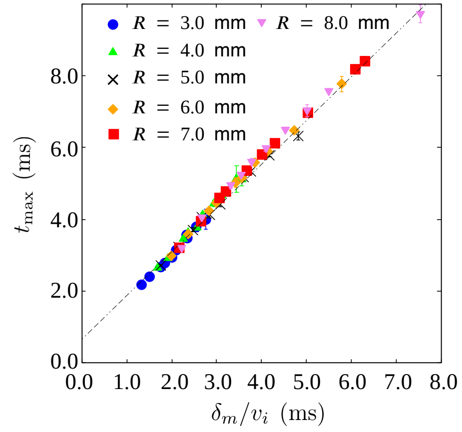
$$E_{MVF} = \frac{\phi\mu\pi R\delta_m^2}{h}\frac{d\delta}{dt}\left[1 - e^{-\frac{E}{\mu}t_c}\right]$$

$$\frac{d\delta}{dt} = v_i \quad t_c = \frac{\delta_m}{v_i}$$

$$E_{MVF} \underset{Et_c/\mu \ll 1}{\longrightarrow} \frac{\phi\mu\pi R\delta_m^2}{h} v_i \cdot \frac{E\delta_m}{\mu v_i} \simeq \frac{\phi E\pi R\delta_m^3}{h} \simeq E_{el}$$

$$Z = \frac{Et_c}{\mu}$$





- Z << 1 corresponds to elastic energy estimation
- Z ~ 1 corresponds to viscoelastic energy estimation

A framework for crossover of scaling law

Step 1

- Start from a scaling law valid in a idealized region:

$$y = Ax^{\alpha}$$

Step 2

- Define the dimensionless number composed of the scaling law:

$$\Pi = \frac{y}{Ax^{\alpha}}$$

Step 3

- Construct a self-similar solution by identifying an interfering dimensionless number:

$$\Pi = \Phi\left(\theta\right)$$

Dimensionless number of a scaling law in idealized region: Π

Interfering dimensionless number θ

$$\theta = \frac{z}{x^{\beta}}$$

The procedure to construct a self-similar solution

Step 1 : Start from a scaling law in a idealized region

CGR solution

$$\Pi = \mathrm{const} \left(\frac{\kappa}{\phi} \right)^{1/3} \eta^{1/3}$$

$$\Pi = \frac{\delta_m}{R}, \quad \kappa = \frac{h}{R}, \quad \eta = \frac{\rho v_i^2}{E}, \quad \theta = \frac{\mu}{E^{1/2} \rho^{1/2} R}$$

Chastel et al., J. Fluid Mech., 2016.

Step 2 : define a dimensionless number composed of the scaling.

$$\Psi = \frac{\phi}{\kappa} \frac{\Pi^3}{\eta}$$

The procedure to construct a self-similar solution

$$\frac{2}{3} = \Pi^2 \theta \frac{\phi}{\kappa} \frac{1}{\eta^{1/2}} \left[1 - \exp\left(- \frac{\Pi}{\theta \eta^{1/2}} \right) \right]$$

$$\Pi = \frac{\delta_m}{R}, \quad \kappa = \frac{h}{R}, \quad \eta = \frac{\rho v_i^2}{E}, \quad \theta = \frac{\mu}{E^{1/2} \rho^{1/2} R}$$

Step 3 : construct a self-similar solution by identifying an interfering dimensionless:

CGR sollution

$$\Psi = \frac{\phi}{\kappa} \frac{\Pi^3}{\eta}$$

$$Z = \frac{\Pi}{\theta \eta^{1/2}}$$

The interfering dimensionless number: *A inverse Deborah number.*

$$\Psi = \frac{2}{3} \frac{Z}{[1 - \exp(-Z)]}$$

The self-similar solution

The hierarchical structure of self-similarity

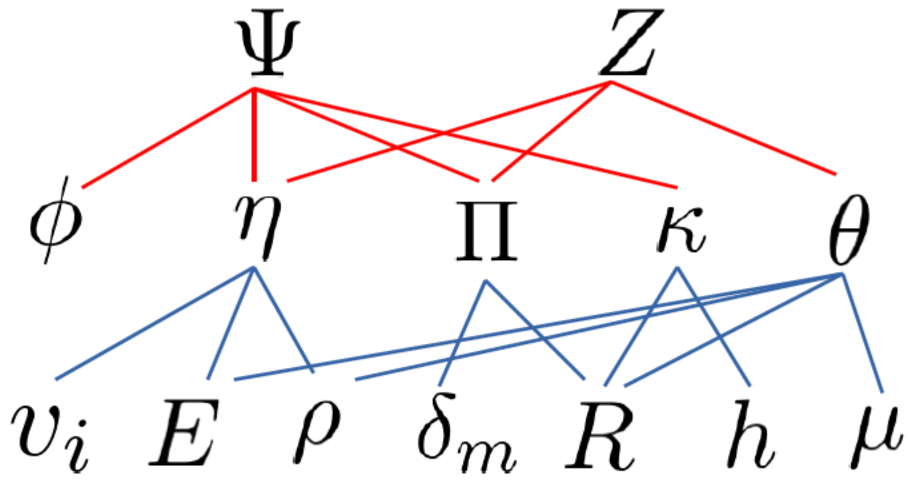
$$\Psi = \frac{\phi}{\kappa}\frac{\Pi^3}{\eta} = \frac{\delta^3 E\phi}{R^2\rho v^2 h}$$

$$Z = \frac{\Pi}{\theta\eta^{1/2}} = \frac{E\delta}{\mu v}$$

$$\Psi = \frac{2}{3}\frac{Z}{1-\exp(-Z)} \xrightarrow[Z\to 0]{} \frac{2}{3} \quad \Longleftrightarrow \quad \Pi = \left(\frac{2\kappa}{3\phi}\right)^{1/3}\eta^{1/3}$$

CGR solution

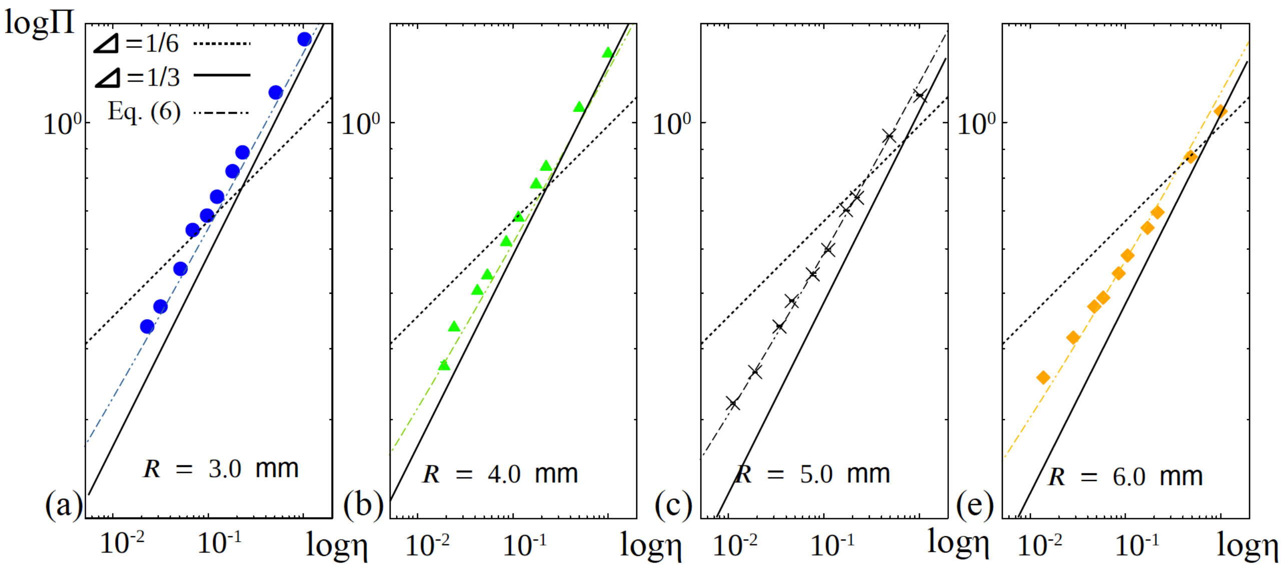
$$\frac{\text{elastic}}{\text{kinetic}} \Longleftrightarrow \frac{\text{viscous}}{\text{elastic}}$$



Self-similarity of the second class	$\Psi = \Phi(Z)$	Dynamics of element
Self-similarity of the first class	$\Pi = \Phi(\phi, \kappa, \theta, \eta)$	Dimensionless parameters
Physical quantity	$\delta_m = f(R, h, \phi, \rho, \mu, E, v_i)$	Physical quantity

https://arxiv.org/abs/2203.00253

Hierarchical structure of self-similarity



$$\Psi = \frac{\phi}{\kappa}\frac{\Pi^3}{\eta} = \frac{\delta^3 E\phi}{R^2\rho v^2 h} = \frac{\text{elastic}}{\text{kinetic}}$$

$$Z = \frac{\Pi}{\theta\eta^{1/2}} = \frac{E\delta}{\mu v} = \frac{\text{viscous}}{\text{elastic}}$$

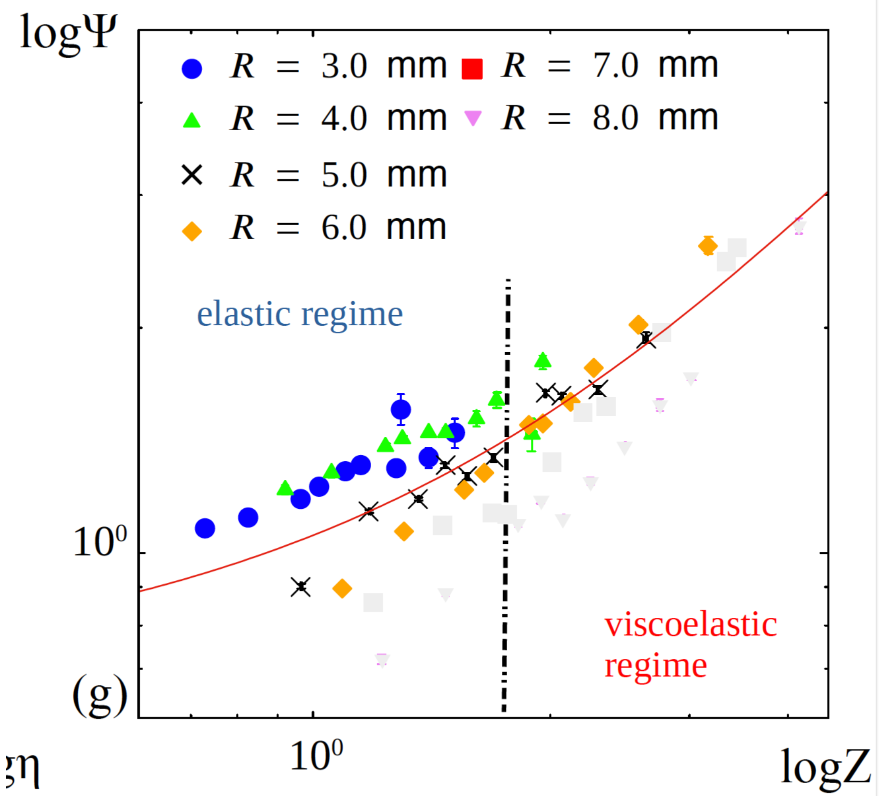
Z (small → large)

small — elastic

$$\Psi = \frac{2}{3}\frac{Z}{[1-\exp(-Z)]}$$

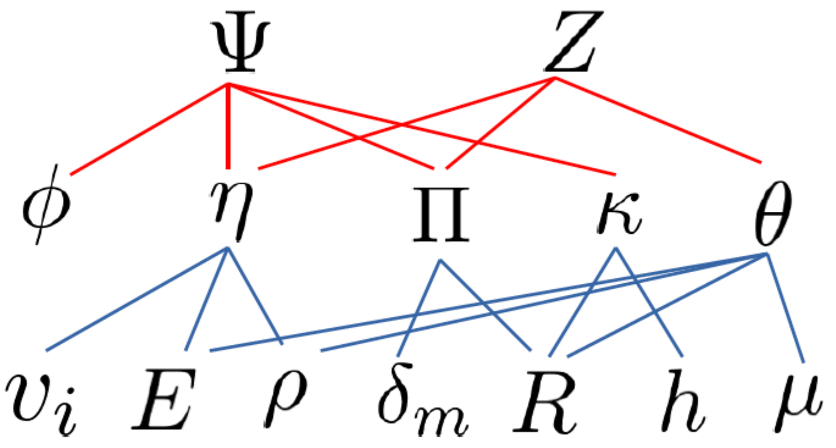
large — viscoelastic

$$\Pi = \frac{\kappa}{54\phi\theta^2} + \left(\frac{\kappa^2}{486\phi^2\theta^3}\right)^{\frac{1}{3}}\eta^{\frac{1}{6}} + \left(\frac{2\kappa}{3\phi}\right)^{\frac{1}{3}}\eta^{\frac{1}{3}}$$

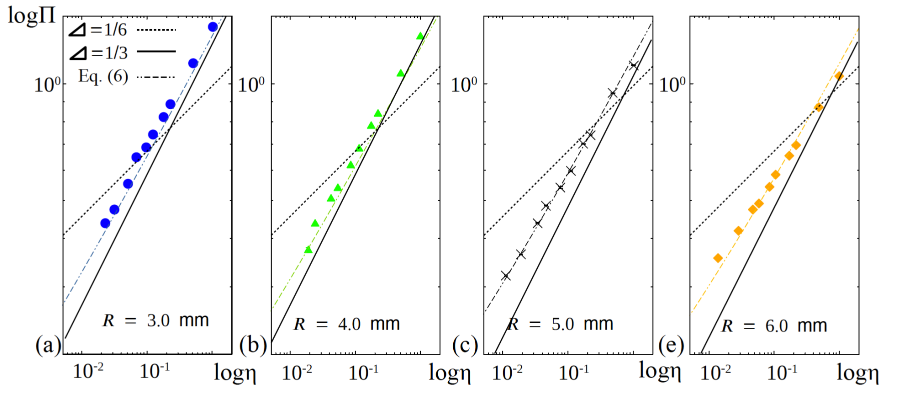


Scale invariance of dimensionless numbers

Scale transformation $x' = A^{\alpha} x$

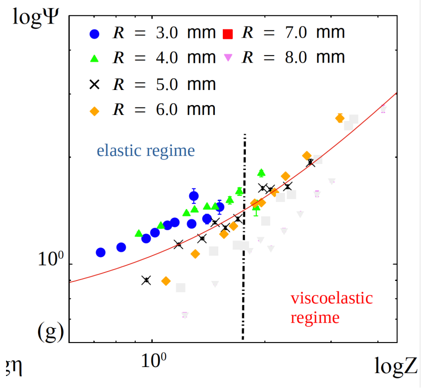


Self-similar solution	Scale invariance
$\Psi = \frac{2}{3}\frac{Z}{1-\exp(-Z)}$	×
$\frac{\Pi^3\phi}{\kappa\eta} = \frac{2}{3}\frac{\frac{\Pi}{\theta\eta^{1/2}}}{1-\exp\left(-\frac{\Pi}{\theta\eta^{1/2}}\right)}$	○
$\frac{\delta_m^3 E\phi}{R^2 h\rho v_i^2} = \frac{2}{3}\frac{\frac{E\delta_m}{\mu v_i}}{1-\exp\left(-\frac{E\delta_m}{\mu v_i}\right)}$	○



the first class

- Scaling behavior

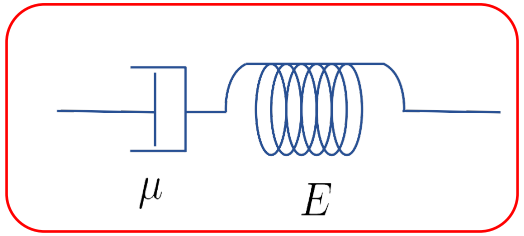


the second class

- Dynamical balance
- Scale dependence

Energy transition

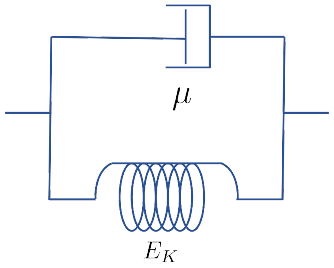
Maxwell model



$$\sigma + \frac{\mu}{E}\frac{d\sigma}{dt} = \mu\frac{d\epsilon}{dt}$$

$$E_{MVF} = \frac{\pi\mu\phi R\delta^2}{h}\frac{d\delta}{dt}\left[1 - \exp\left(-\frac{Et_c}{\mu}\right)\right]$$

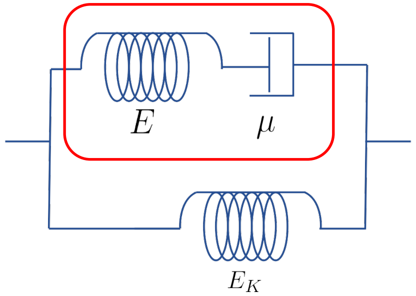
Kelvin Voigt model



$$\sigma = E_K\epsilon + \mu\frac{d\epsilon}{dt}$$

$$E_{KV} = \frac{\phi\pi E_K R\delta^3}{3h} + \frac{\phi\mu\pi R\delta^2}{h}\frac{d\delta}{dt}$$

Zener model



$$\frac{\mu}{E}\frac{d\sigma}{dt} = -\sigma + E_K\epsilon + \mu\left(\frac{E_K}{E} + 1\right)\frac{d\epsilon}{dt}$$

$$E_Z = \frac{\phi\pi E_K R\delta^3}{3h} + \frac{\phi\mu\pi R\delta^2}{h}\frac{d\delta}{dt}\left[1 - \exp\left(-\frac{Et_c}{\mu}\right)\right]$$

Perturbation solution

crossover

$$\Pi = \frac{\kappa}{54\phi\theta^2} + \left(\frac{\kappa^2}{486\phi^2\theta^3}\right)^{\frac{1}{3}}\eta^{\frac{1}{6}} + \left(\frac{2\kappa}{3\phi}\right)^{\frac{1}{3}}\eta^{\frac{1}{3}}$$ ○

$$\Pi = \sqrt{\frac{2\kappa}{3\phi\theta}}\eta^{\frac{1}{4}} - \frac{\kappa}{9\phi\theta^2}$$ ×

$E \ll E_K$ ✗ $\qquad E \gg E_K$

$$\Pi = (3K^3 - 2K^2)\frac{\kappa}{54\phi\theta^2} + K^{\frac{5}{3}}\left(\frac{\kappa^2}{486\phi^2\theta^3}\right)^{\frac{1}{3}}\eta^{\frac{1}{6}} + K^{\frac{1}{3}}\left(\frac{2\kappa}{3\phi}\right)^{\frac{1}{3}}\eta^{\frac{1}{3}}$$ ○

Conclusion

Conclusions for method

- There exits a self-similar solution on crossover of scaling law : the crossover is generated by another dimensionless number.
- A scale-variant function Φ drives the scale dependent behavior : the hierarchical structure is essential for crossover to estimate the stability of intermediate asymptotics.

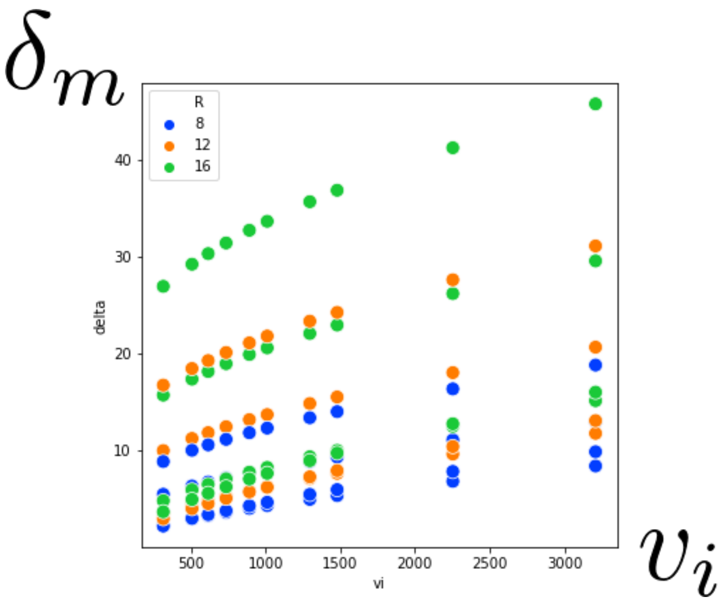
Conclusion for the phenomenon

- From the experimental observations, we found that the similar attractor of deformation : it gives the scaling law $d\delta/dt = v_i$ and $t_c = \delta_{m/}v_i$, which simplifies the calculation.
- Time-dependently, the energy E_{MVF} changes its form, giving rise to a crossover
- Maxwell element, serial connection of a dash pot and a spring, is necessary for crossover of scaling law

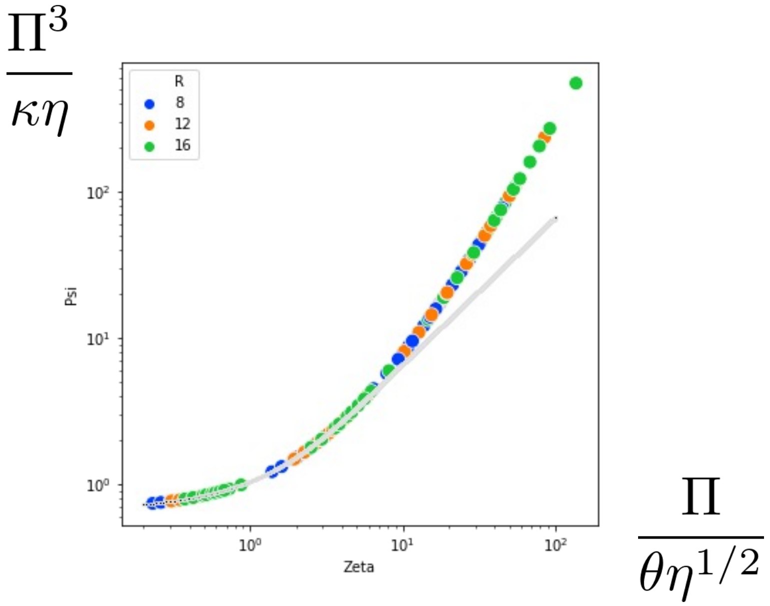
https://arxiv.org/abs/2203.00253

Vision

- Do we have a data driven algorithm to find a data collapse?



?



$$\delta_m = f\left(R, h, \phi, \rho, \mu, E, v_i\right)$$

$$\frac{\Pi^3}{\kappa\eta} = \Phi\left(\frac{\Pi}{\theta\eta^{1/2}}\right)$$

$$\frac{\Pi^3}{\kappa\eta} = \Phi\left(\frac{\Pi}{\theta^{\alpha_1}\kappa^{\alpha_2}\eta^{\alpha_3}}\right) \qquad I = I\left(\alpha_1, \alpha_2, \alpha_3\right)$$