Sound damping near jamming

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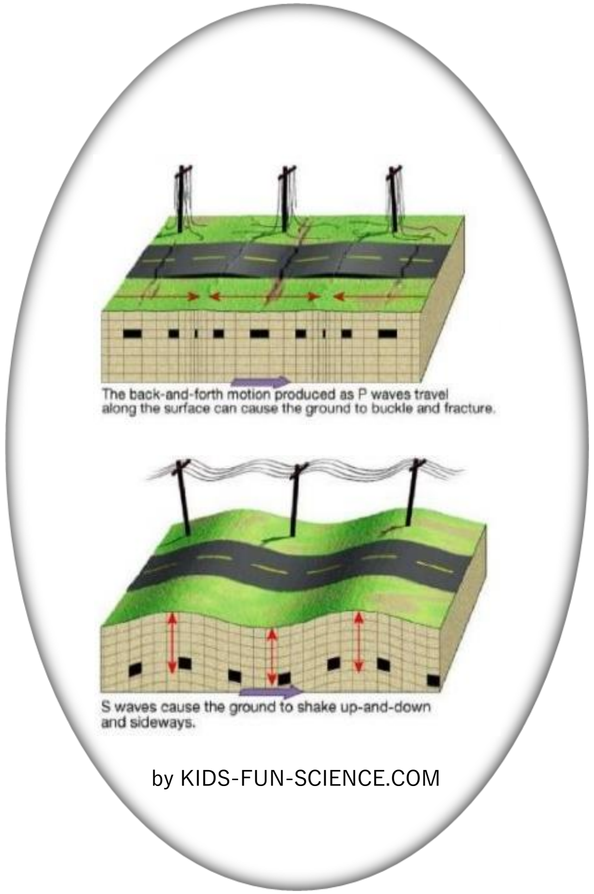
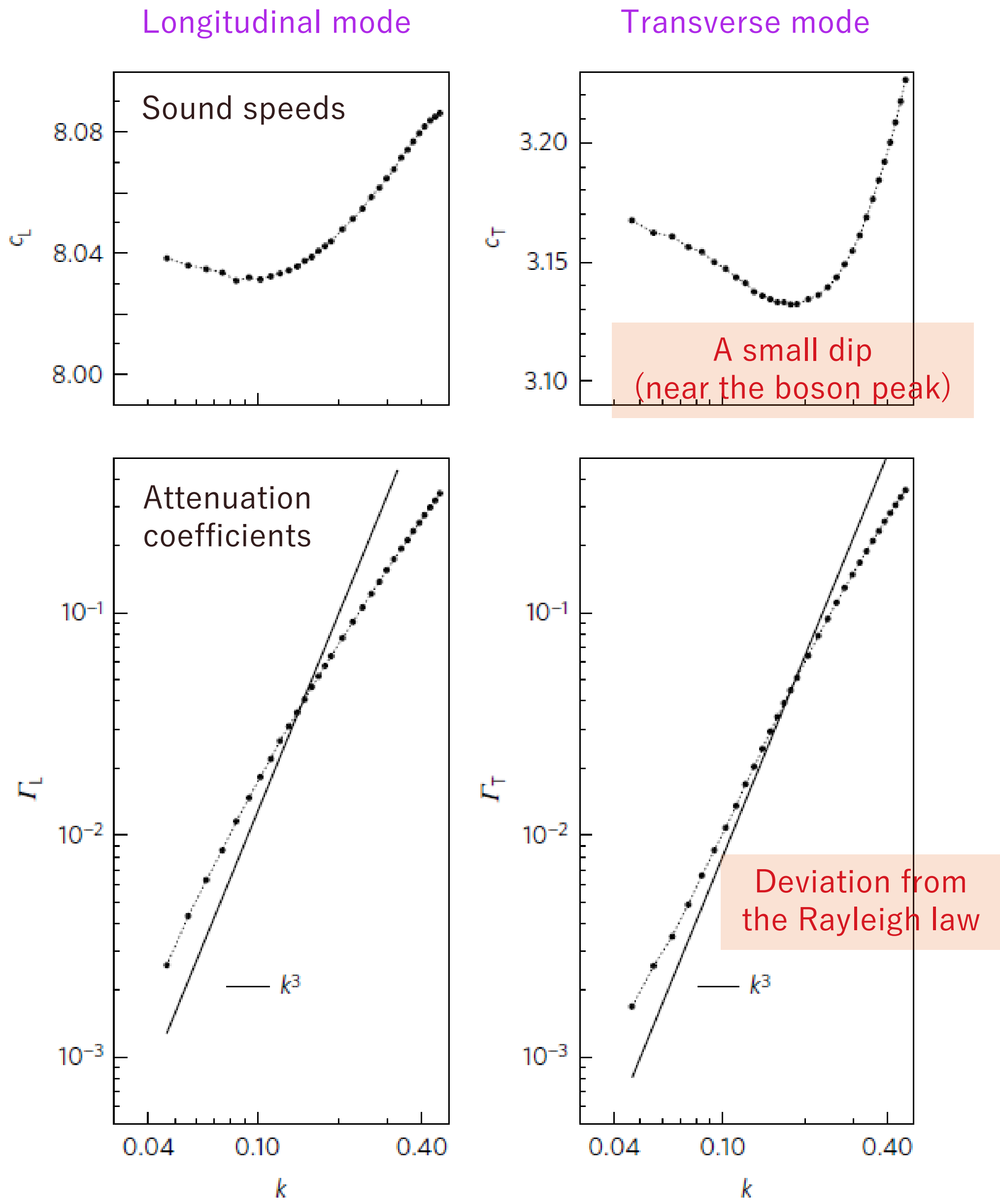
Introduction

- **Sound** in granular media, e.g. *earthquakes*.
- Useful for the measurements of **elastic moduli**, e.g. K and G .
- Anomalous properties of sound in **amorphous solids.**





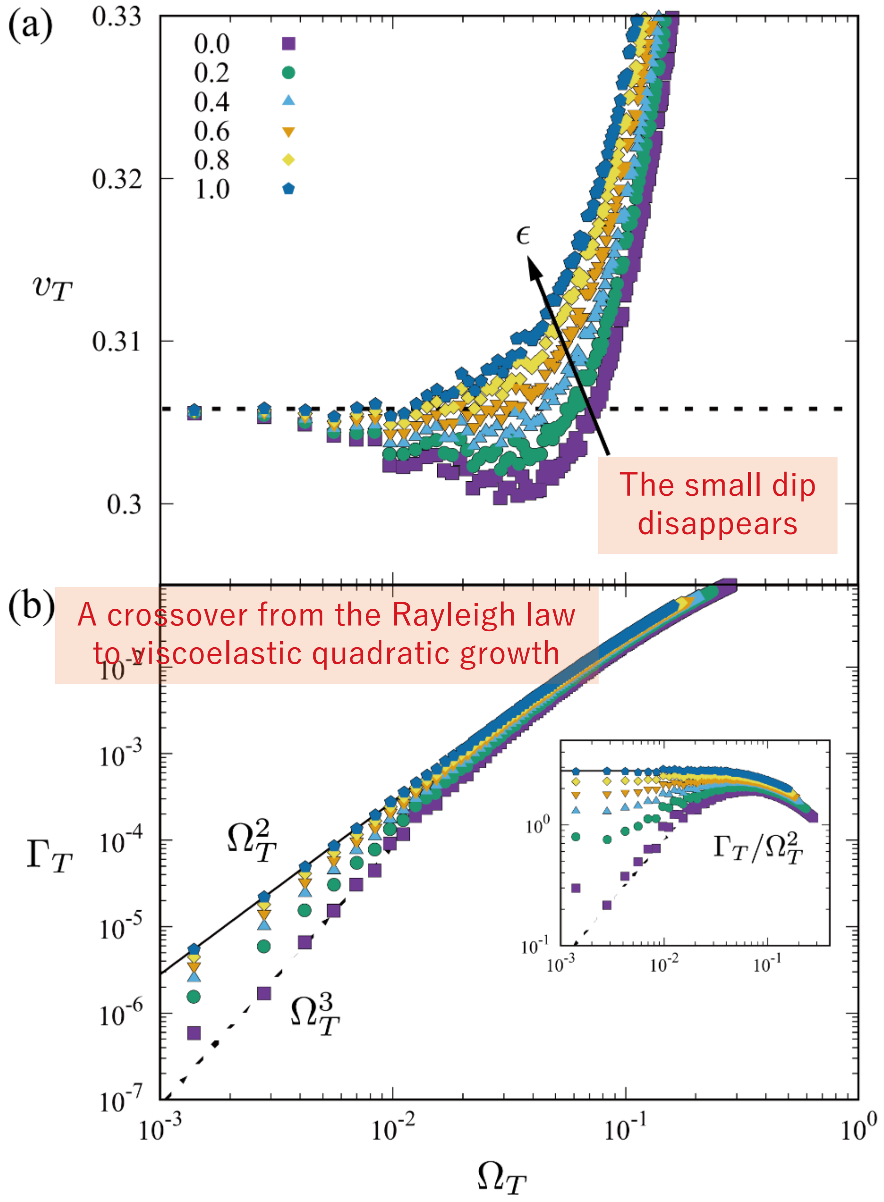
S. Gelin, H. Tanaka, and A. Lemaitre, *Nat. Materials* **15**, 1177 (2016).
A. Marruzzo, W. Schirmacher, A. Fratalocchi, and G. Ruocco, *Sci. Rep.* **3**, 1407 (2013).
G. Monaco and S. Mossa, *PNAS* **106**, 16907 (2009).

Motivation

- The role of **contact damping** (see figures).

K. Saitoh and H. Mizuno, *Soft Matter* **17**, 4204 (2021).

- **How does the jamming transition affect the sound characteristics?**
- **What is the link to viscoelasticity, e.g. $G'(\omega)$ and $G''(\omega)$?**

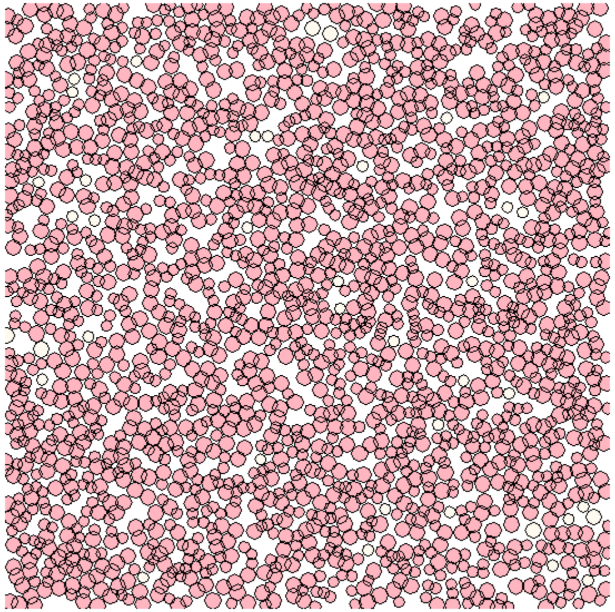


Methods

We use a binary mixture of frictionless soft particles in two dimensions, where the size ratio is $R_L/R_S = 1.4$ and the number of particles is $N = 2097152$.

- Displacements around mechanical equilibrium, $|q(t)\rangle \equiv \left(\{\boldsymbol{u}_i(t)\}_{i=1,\cdots,N}\right)^{\mathrm{T}}$
- Linear equations of motion, $m|\ddot{q}(t)\rangle = -D|q(t)\rangle - B|\dot{q}(t)\rangle$
- **Dynamical matrix (Hessian),** $D \equiv \left[\frac{\partial^2 E(0)}{\partial q_k \partial q_l}\right]_{k,l=1,\cdots,2N}$
- **Damping matrix,** $B \equiv \left[\frac{\partial^2 R(0)}{\partial \dot{q}_k \partial \dot{q}_l}\right]_{k,l=1,\cdots,2N}$
- Elastic energy, $E(t) = \frac{k_n}{2}\sum_{i<j} \xi_{ij}(t)^2$
- Dissipation function, $R(t) = \frac{\eta}{2}\sum_{i<j} \dot{\boldsymbol{u}}_{ij}(t)^2$

cf. *Durian's bubble model* or *balanced contact damping*.



Static packing made by the FIRE.

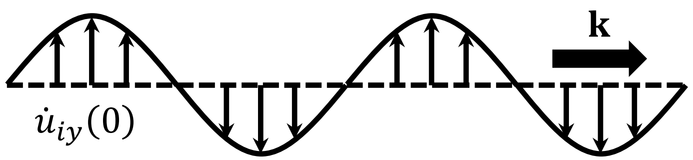
Numerical simulations

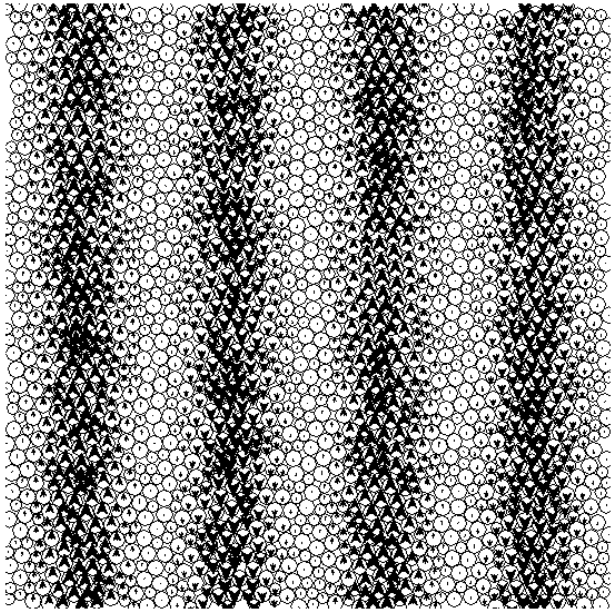
We numerically solve the linear equations of motion with initial velocities,

$$\dot{\boldsymbol{u}}_i(0) = \boldsymbol{A}\sin[\boldsymbol{k}\cdot\boldsymbol{r}_i(0)] \quad (i = 1, \cdots, N)$$

- Amplitude, $|\mathbf{A}| = 10^{-3}\, d_0/t_0$, where $d_0 \equiv R_L + R_S$ and $t_0 \equiv \sqrt{m/k_n}$.
- Wave number, $k \equiv |\boldsymbol{k}| = \frac{2\pi}{L} n \quad (n = 1,2,3,\cdots)$
- $\boldsymbol{A}\cdot\boldsymbol{k} = Ak$ for the analysis of longitudinal (L) mode.
- $\boldsymbol{A}\cdot\boldsymbol{k} = 0$ for the analysis of transverse (T) mode.
- Inelasticity, $\epsilon \equiv \frac{t_d}{t_0} = \frac{\eta/k_n}{\sqrt{m/k_n}} = \frac{\eta}{\sqrt{mk_n}}$, where we examine $\epsilon = 1$ and 0.1.

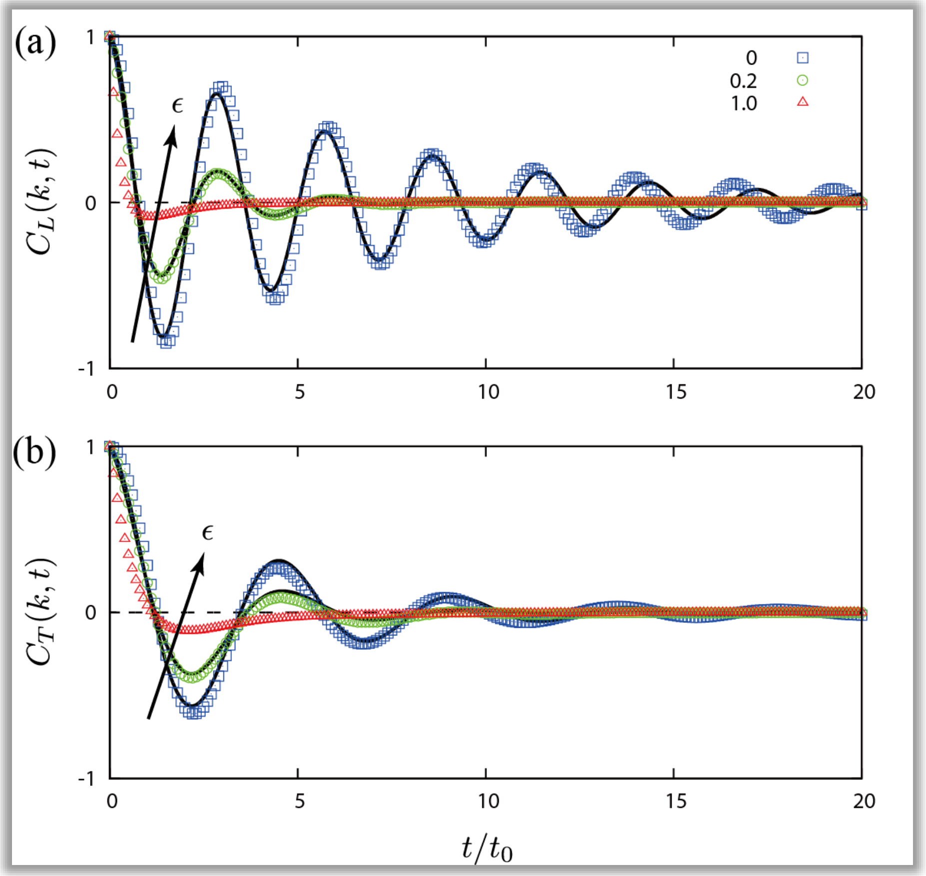
Initial standing wave





Numerical analyses

- Fourier transform, $\dot{\boldsymbol{u}}(\boldsymbol{k},t) = \sum_{i=1}^{N} \dot{\boldsymbol{u}}_i(t) e^{-I\boldsymbol{k}\cdot\boldsymbol{r}_i(t)}$
- L mode, $\dot{\boldsymbol{u}}_L(\boldsymbol{k},t) \equiv \{\dot{\boldsymbol{u}}(\boldsymbol{k},t)\cdot\hat{\boldsymbol{k}}\}\hat{\boldsymbol{k}}$, where $\hat{\boldsymbol{k}} \equiv \boldsymbol{k}/k$.
- T mode, $\dot{\boldsymbol{u}}_T(\boldsymbol{k},t) \equiv \dot{\boldsymbol{u}}(\boldsymbol{k},t) - \dot{\boldsymbol{u}}_L(\boldsymbol{k},t)$
- Autocorrelation function, $C_\alpha(k,t) = \langle \dot{\boldsymbol{u}}_\alpha(\boldsymbol{k},t)\cdot\dot{\boldsymbol{u}}_\alpha(-\boldsymbol{k},0)\rangle$, for $\alpha = L, T$.
- Fitting to a **damped oscillation**, $C_\alpha(k,t) \sim e^{-\Gamma_\alpha(k)t}\cos\Omega_\alpha(k)t$



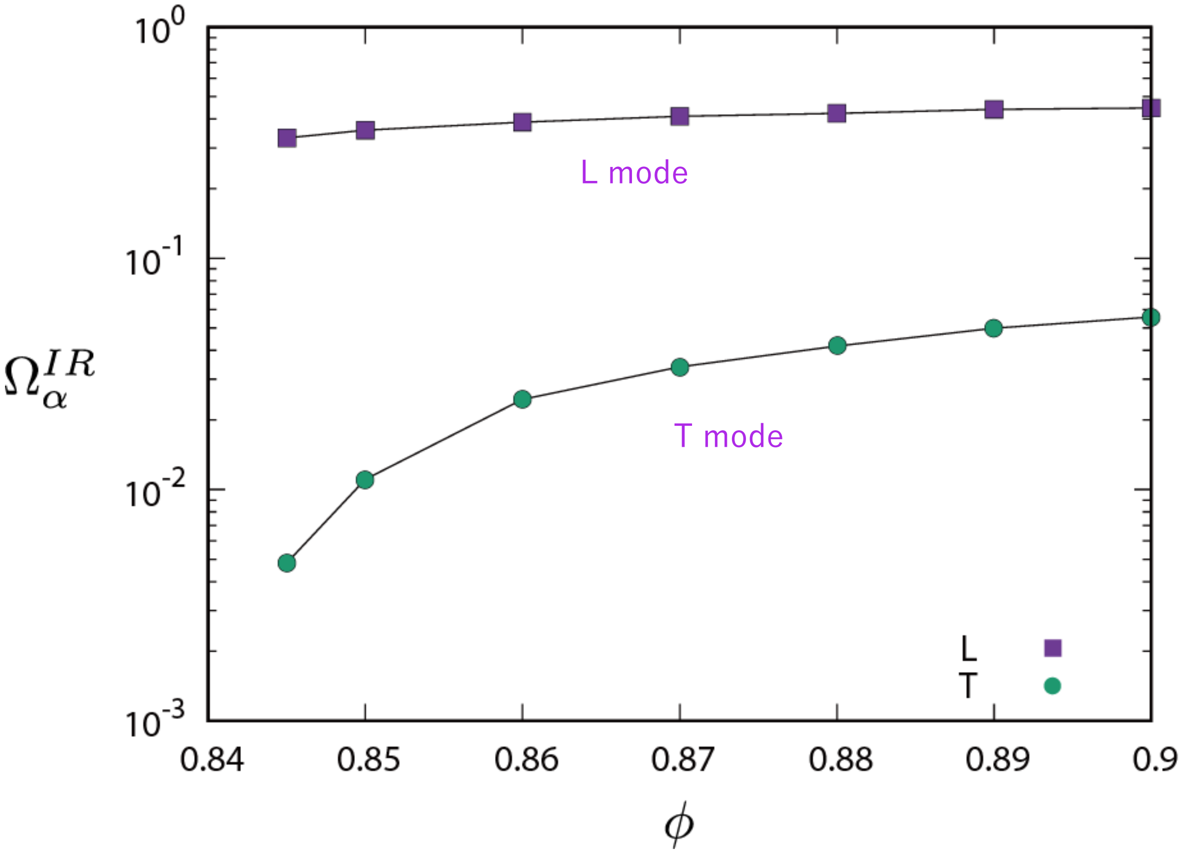
We analyze the dependence of the **dispersion relation** $\Omega_\alpha(k)$ and **attenuation coefficient** $\Gamma_\alpha(k)$ on the proximity to jamming.

The Ioffe-Regel limits

- Note that $C_\alpha(k,t)$ oscillates only if
$$\frac{\pi\Gamma_\alpha(k)}{\Omega_\alpha(k)} < 1$$
- Otherwise, $C_\alpha(k,t)$ is overdamped.

- The ratio is an increasing function of k and less than unity if $k < k_\alpha^{IR}$.

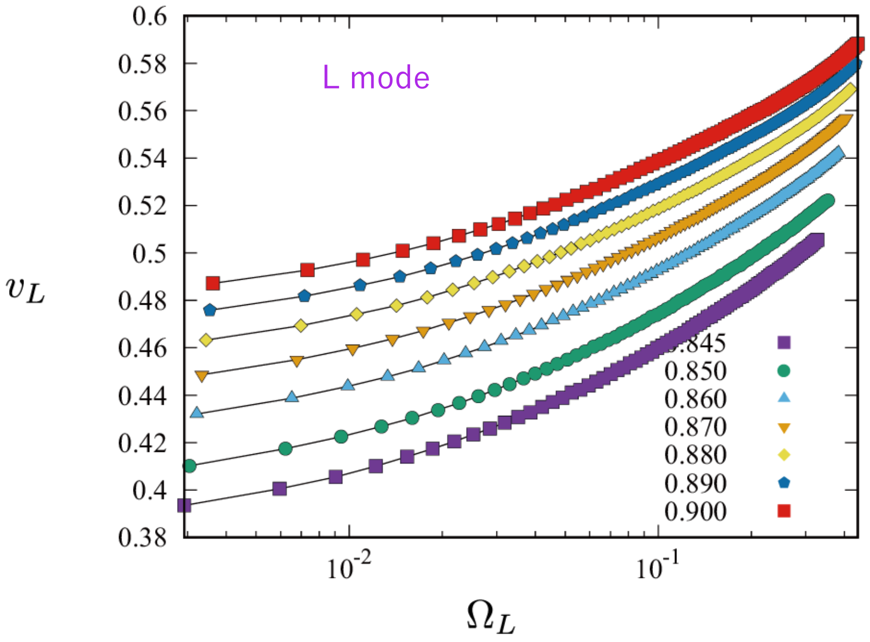
- The **Ioffe-Regel limit** is defined as
$$\Omega_\alpha^{IR}(k_\alpha^{IR}) \equiv \pi\Gamma_\alpha(k_\alpha^{IR})$$

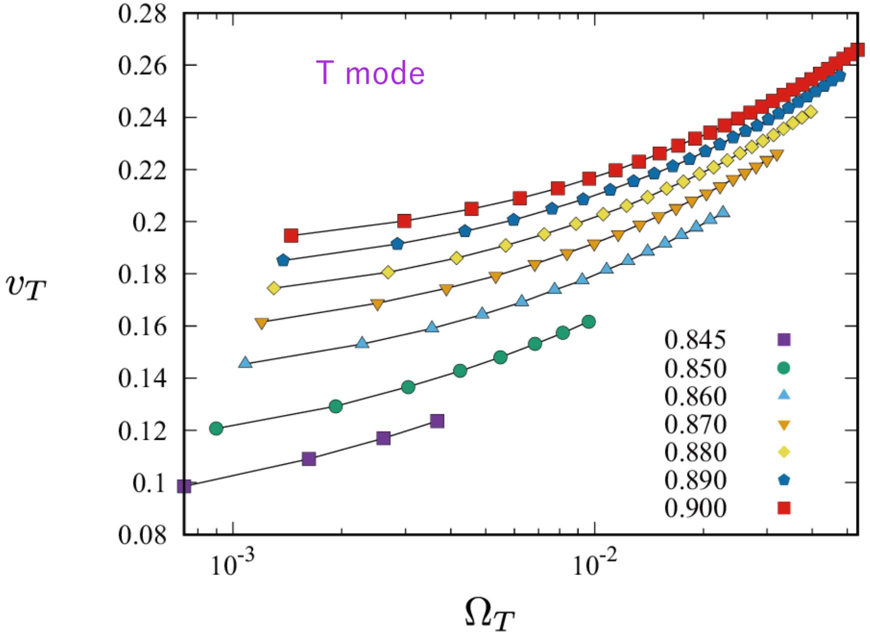


H. Mizuno, S. Mossa, and J.-L. Barrat, *PNAS* **111**, 11949 (2014).

Sound speeds

- Defined as $v_\alpha(k) \equiv \Omega_\alpha(k)/k$ with $\alpha = L, T$.
- Parametric plots of $v_\alpha(k)$ and $\Omega_\alpha(k)$.
- Dependence on the packing fraction ϕ (as listed).
- Inelasticity is $\epsilon = 1$.

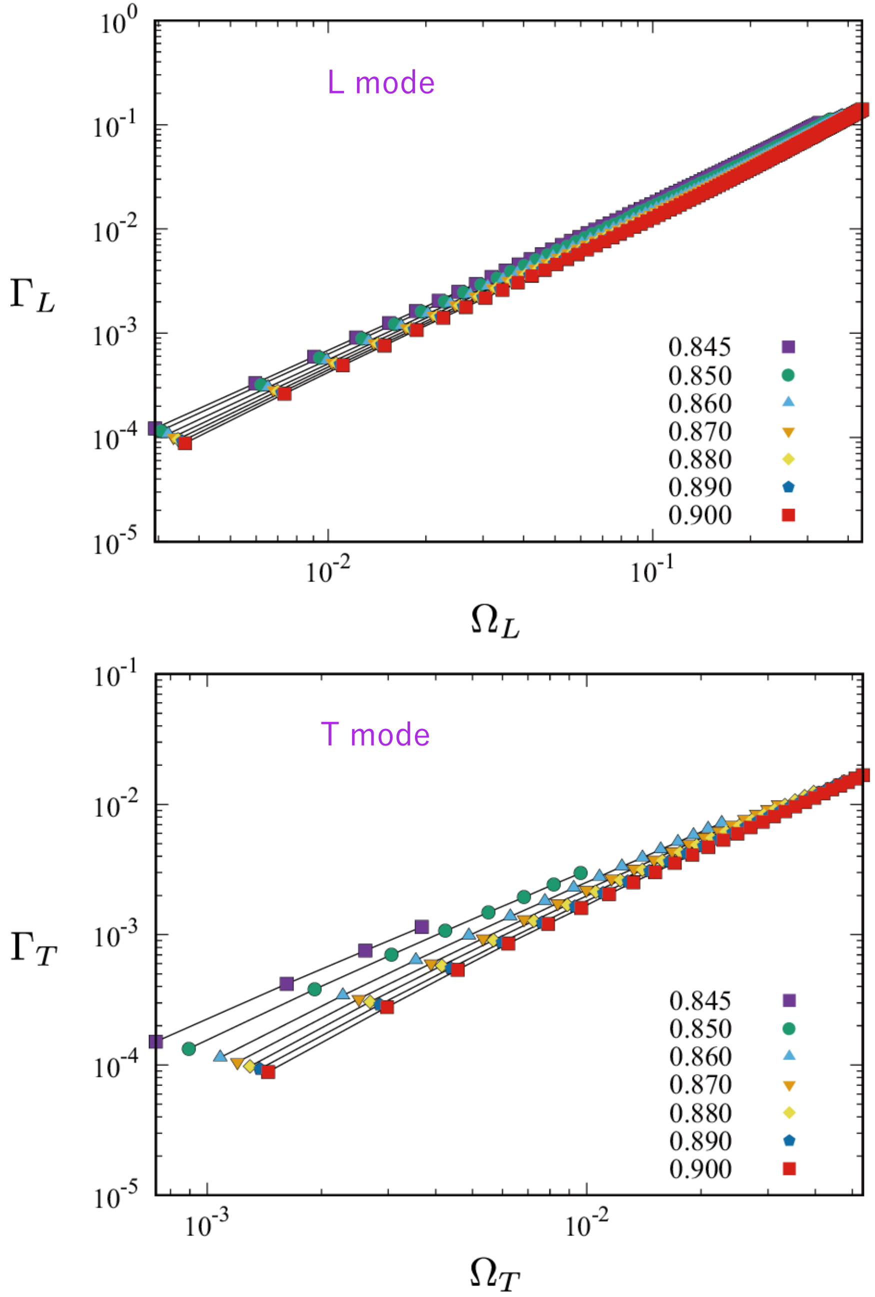




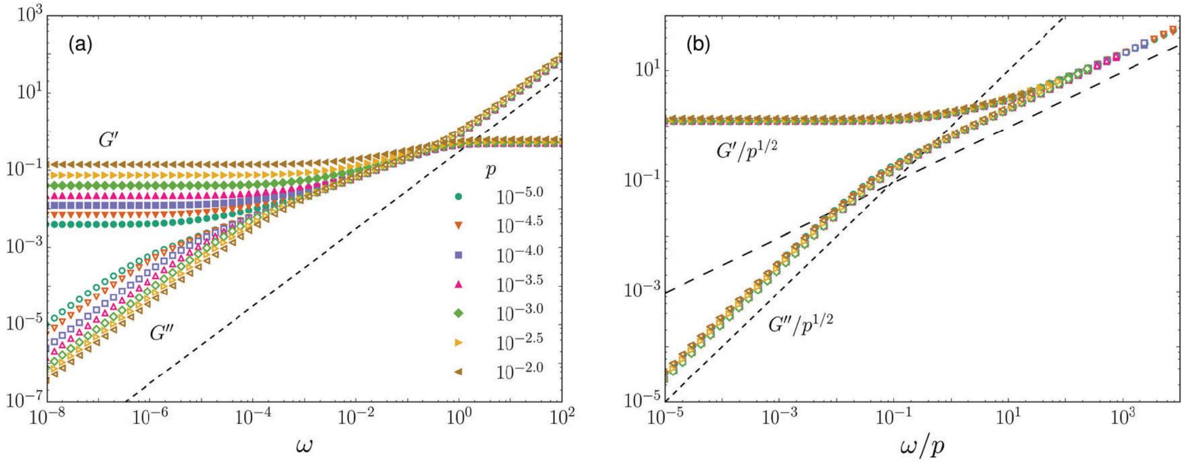
Attenuation coefficients

- Dependence on ϕ (as listed).
- Inelasticity is $\epsilon = 1$.
- We show the data below the Ioffe-Regel limit,

$$\Omega_\alpha < \Omega_\alpha^{IR}$$



Viscoelasticity near jamming



- Complex moduli

$$\frac{1}{K^*(\omega)} \sim \int_0^\infty \frac{|\Xi(\omega')|^2 D(\omega')}{\omega'^2 + i\omega\mu_n} d\omega'$$

$$\frac{1}{G^*(\omega)} \sim \int_0^\infty \frac{|\Lambda(\omega')|^2 D(\omega')}{\omega'^2 + i\omega\eta_n} d\omega'$$

- $D(\omega)$ is the density of states

- Critical scaling near jamming $p \to 0$,

$$G'(\omega) \sim \begin{cases} p^{1/2} & (\omega < p) \\ \omega^{1/2} & (\omega > p) \end{cases}$$

$$G''(\omega) \sim \begin{cases} \omega/p^{1/2} & (\omega < p) \\ \omega^{1/2} & (\omega > p) \end{cases}$$

$$K' \sim K_0 + G'$$

$$K'' \sim G''$$

K. Baumgarten and B.P. Tighe, *Soft Matter* **13**, 8368 (2017).
K.S., T. Hatano, A. Ikeda, and B.P. Tighe, *Phys. Rev. Lett.* **124**, 118001 (2020).

Viscoelasticity and sound (1) L mode

- Longitudinal waves in viscoelastic media.

- The mass density ρ.

$$v_L{}^{-1} = \mathrm{Re}\left[\frac{\rho}{K^*(\omega) + G^*(\omega)}\right]$$

$$-\frac{\Gamma_L}{\omega} = \mathrm{Im}\left[\frac{\rho}{K^*(\omega) + G^*(\omega)}\right]$$

Y. C. Fung, "*Foundations of Solid Mechanics*" 2nd edition (1965)

- In long wave lengths ($\omega < p$),

$$G'(\omega) \sim p^{1/2}$$

$$\frac{G''(\omega)}{\omega} \sim \frac{1}{p^{1/2}}$$

- Near jamming $p \to 0$,

$$v_L \sim \sqrt{K' + G'} \sim \sqrt{1 + \chi p^{1/2}}$$

$$\Gamma_L \sim \frac{\omega(K'' + G'')}{(K' + G')^{3/2}} \sim \frac{\omega^2}{p^{1/2}}$$

$$\therefore \frac{\Gamma_L}{p^{1/2}} \sim \left(\frac{\omega}{p^{1/2}}\right)^2$$

quadratic

Viscoelasticity and sound (2) T mode

- Transverse waves in viscoelastic media.

$$ {v_T}^{-1} = \mathrm{Re}\left[\frac{\rho}{G^*(\omega)}\right] $$

$$ -\frac{\Gamma_T}{\omega} = \mathrm{Im}\left[\frac{\rho}{G^*(\omega)}\right] $$

- In long wave lengths ($\omega < p$),

$$ G'(\omega) \sim p^{1/2} $$

$$ \frac{G''(\omega)}{\omega} \sim \frac{1}{p^{1/2}} $$

- Near jamming $p \to 0$,

$$ v_T \sim \sqrt{G'} \sim p^{1/4} $$

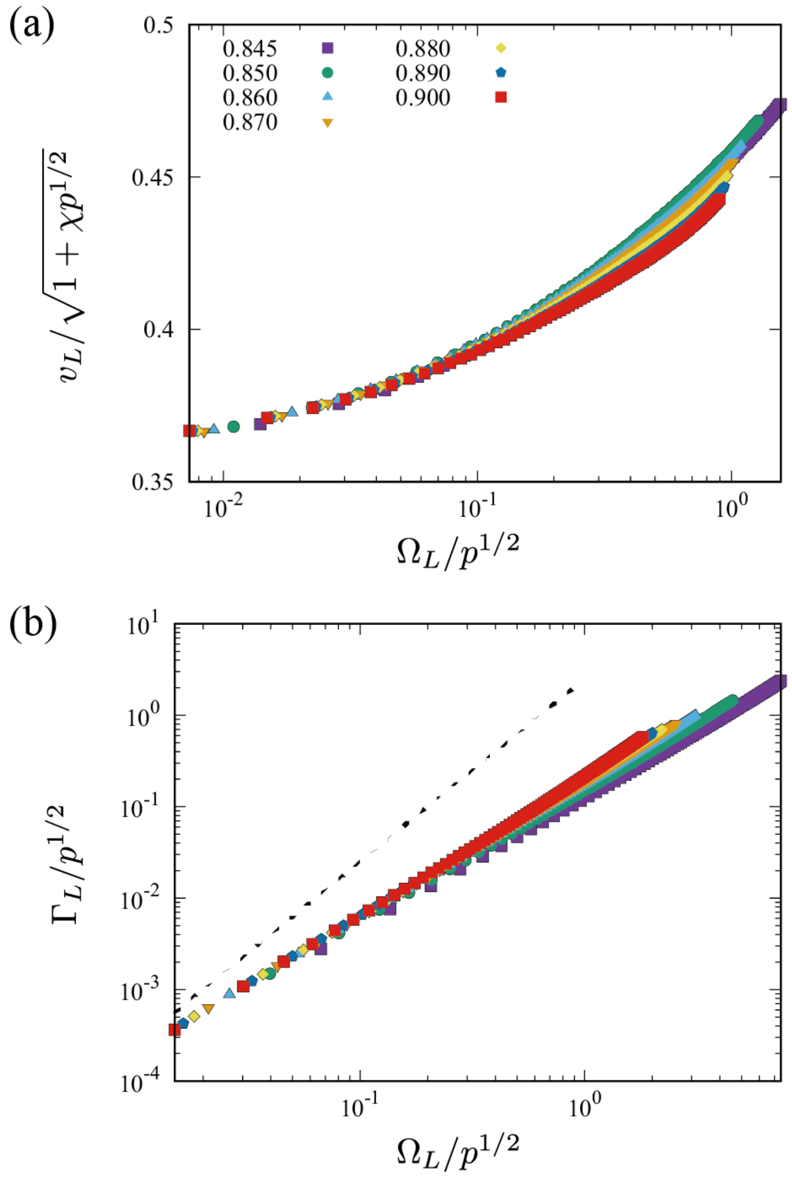
$$ \Gamma_T \sim \frac{\omega G''}{G'^{3/2}} \sim \frac{\omega^2}{p^{5/4}} $$

$$ \therefore \frac{\Gamma_T}{p^{3/4}} \sim \left(\frac{\omega}{p}\right)^2 $$

quadratic

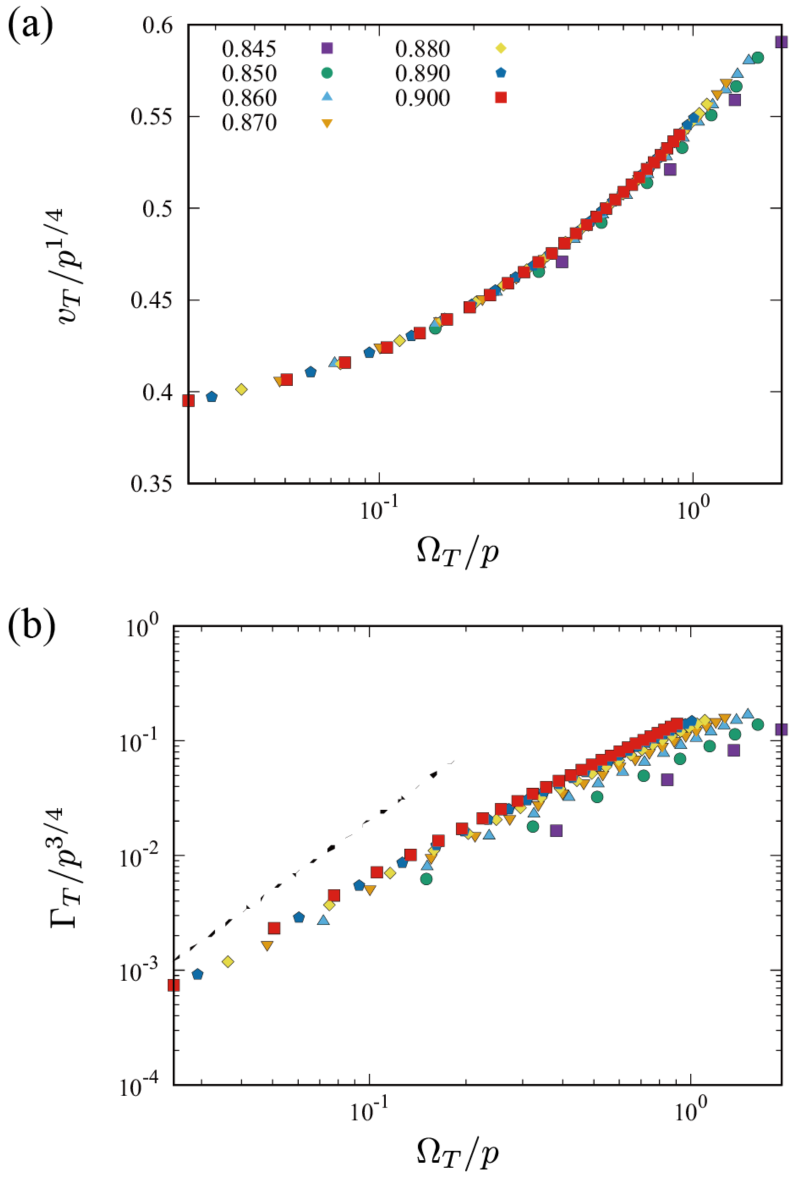
Data collapses (1) L mode

- Sound characteristics scaled by the pressure p.
- The parameter χ is independent of p.
- Inelasticity is $\epsilon = 1$.
- The **dashed line** is quadratic.



Data collapses (2) T mode

- Sound characteristics near jamming.
- Inelasticity is $\epsilon = 1$.
- The **dashed line** is quadratic.
- The collapse of Γ_T is NOT satisfactory.



Non-local constitutive relations

- The wave number dependent complex shear modulus.

- In long wave lengths,

$$\Omega_T(k) \propto \sqrt{G'(\omega)}k$$

$$\Gamma_T(k) \propto \frac{G''(\omega)}{\omega}k^2$$

H. Mizuno and R. Yamamoto, *Phys. Rev. Lett.* **110**, 095901 (2013)

- The sound speed and attenuation coefficient,

$$v_T(k) = \frac{\Omega_T(k)}{k} \propto \sqrt{G'(\omega)}$$

$$\Gamma_T(k) \propto \frac{G''(\omega)}{\omega}\left(\frac{\Omega_T(k)}{v_T(k)}\right)^2 \sim \frac{G''(\omega)}{\omega G'(\omega)}\Omega_T(k)^2$$

- Near jamming $p \to 0$,

$$v_T(k) \propto \sqrt{G'(\omega)} \sim p^{1/4}$$

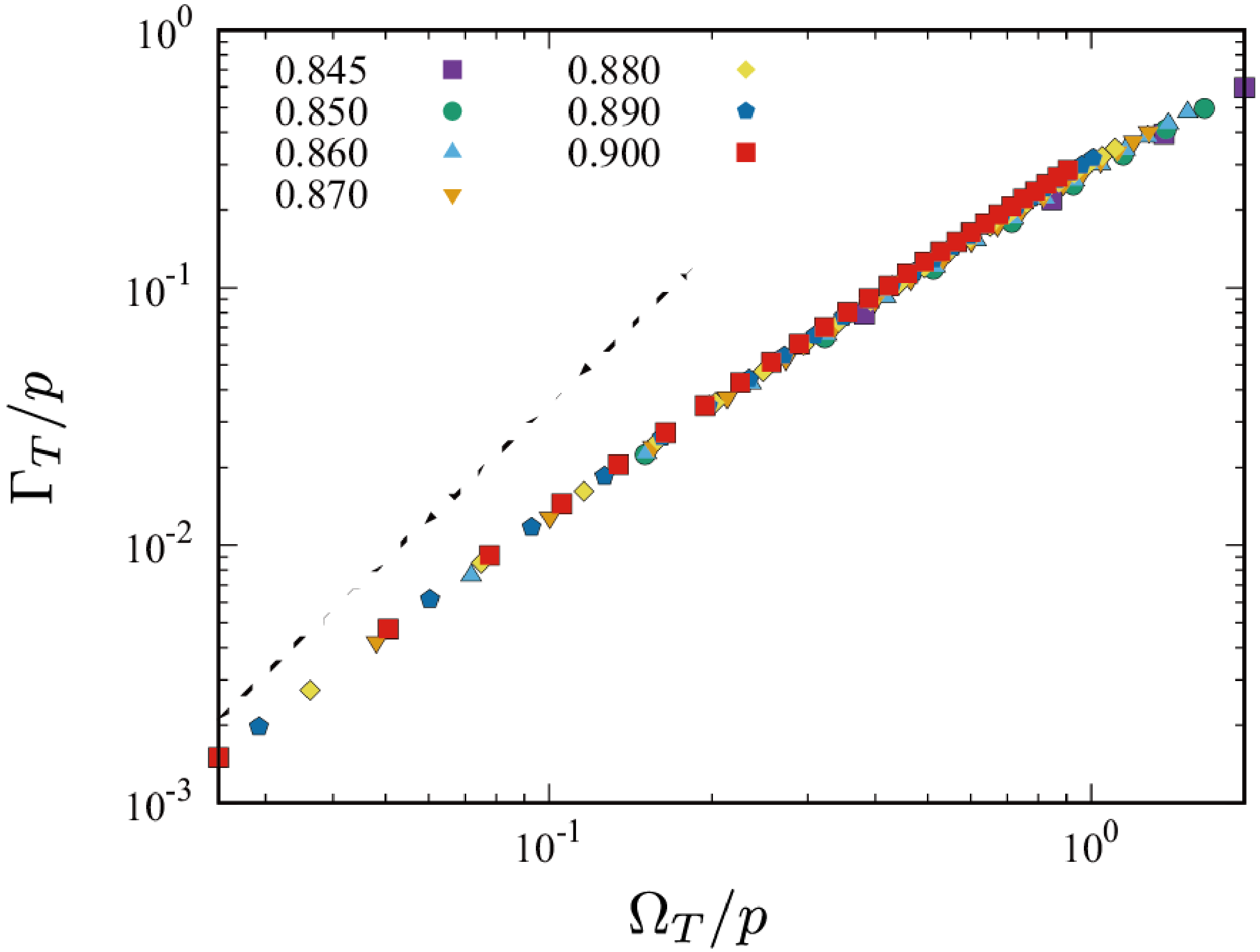
$$\Gamma_T(k) \propto \frac{G''(\omega)}{\omega G'(\omega)}\Omega_T(k)^2 \sim \frac{\Omega_T(k)^2}{p}$$

$$\therefore \frac{\Gamma_T(k)}{p} \sim \left(\frac{\Omega_T(k)}{p}\right)^2$$

quadratic

Data collapse – revised

- Attenuation coefficient of T mode.
- Inelasticity is $\epsilon = 1$.
- The **dashed line** is quadratic.
- The non-local constitutive relations well explain the data.



Summary

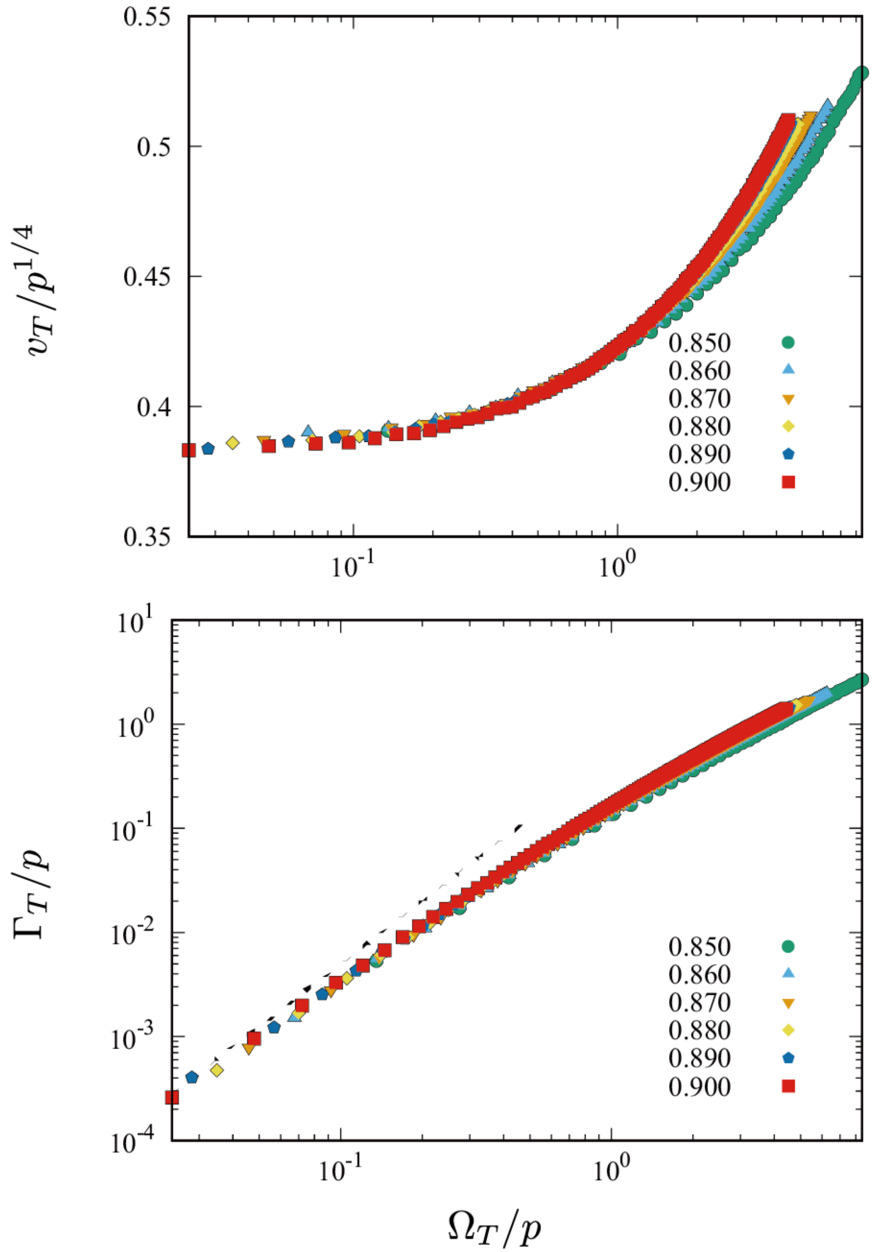
- We numerically investigated sound damping near jamming.
- The relation between sound characteristics and complex moduli near jamming has been clarified.
- **Critical scaling of Γ_α is NEW.**
- The non-local constitutive relations well explain the data of Γ_T.

Thank you so much!

Q&A

Effects of ϵ

- Data for $\epsilon = 0.1$.
- The same scaling seems to work.



Transverse mode

In a long wave-length limit ($k \ll 1$), $v_T \sim \sqrt{G'}$ $\Gamma_T \sim G''k^2$

By definition, $v_T \equiv \dfrac{\Omega_T}{k}$ $\therefore k = \dfrac{\Omega_T}{v_T} \sim \dfrac{\Omega_T}{\sqrt{G'}}$

Therefore, $\Gamma_T \sim G''k^2 \sim \dfrac{G''}{G'}{\Omega_T}^2$

If $\omega < p$, $G' \sim p^{1/2}$ $G'' \sim \omega/p^{1/2}$

Thus, $v_T \sim p^{1/4}$ $\Gamma_T \sim \dfrac{\omega}{p}{\Omega_T}^2$ ⇒ $\dfrac{\Gamma_T}{p} \sim \omega\left(\dfrac{\Omega_T}{p}\right)^2$

Longitudinal mode

In a long wave-length limit ($k \ll 1$), $v_L \sim \sqrt{K + G'}$ $\Gamma_L \sim ?$

By definition, $v_L \equiv \frac{\Omega_L}{k}$ $\therefore k = \frac{\Omega_L}{v_L} \sim \frac{\Omega_L}{\sqrt{K + G'}}$

Therefore, $\Gamma_L \sim ?$

If $\omega < p$, $G' \sim p^{1/2}$ $G'' \sim \omega / p^{1/2}$

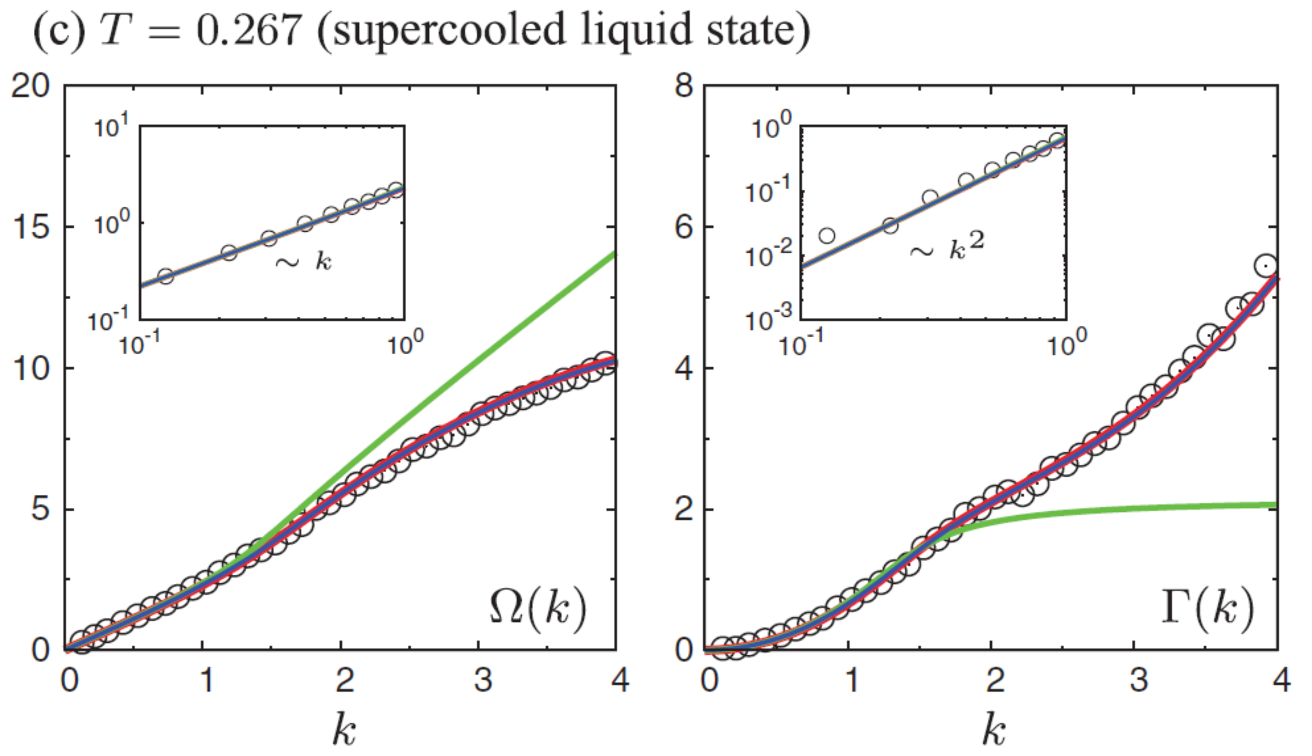
Thus, $v_L \sim \sqrt{K + p^{1/4}}$ $\Gamma_L \sim ?$

Relations to viscoelastic properties

$$\Omega_T(k) \approx \sqrt{\frac{G'}{\rho}}\,k\,, \qquad \Gamma_T(k) \approx \frac{G''}{2\rho}k^2$$

Sound speeds in 2D, $v_L = \sqrt{\frac{K+G'}{\rho}}$, $v_T = \sqrt{\frac{G'}{\rho}}$

H. Mizuno and R. Yamamoto, Phys. Rev. Lett. 110, 095901 (2013)

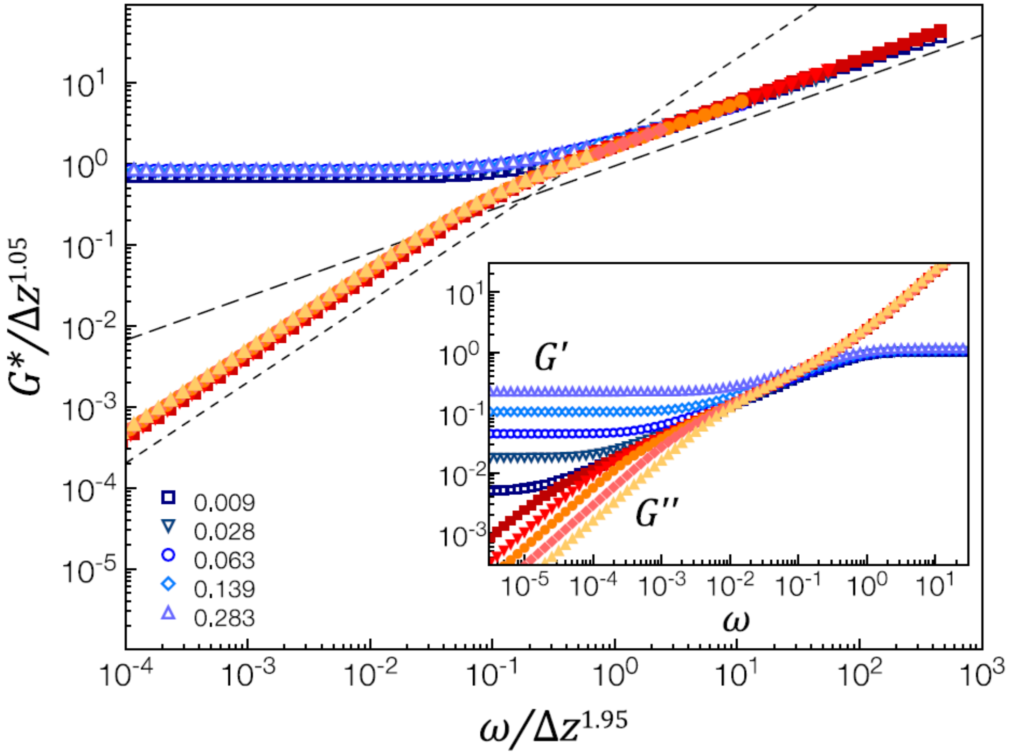


Viscoelasticity

$$G' \sim \begin{cases} \Delta z \\ \omega^{1/2} \\ 1 \end{cases}, \qquad G'' \sim \begin{cases} \omega/\Delta z & (\omega < s^*) \\ \omega^{1/2} & (s^* < \omega < 1) \\ \omega & (1 < \omega) \end{cases}$$

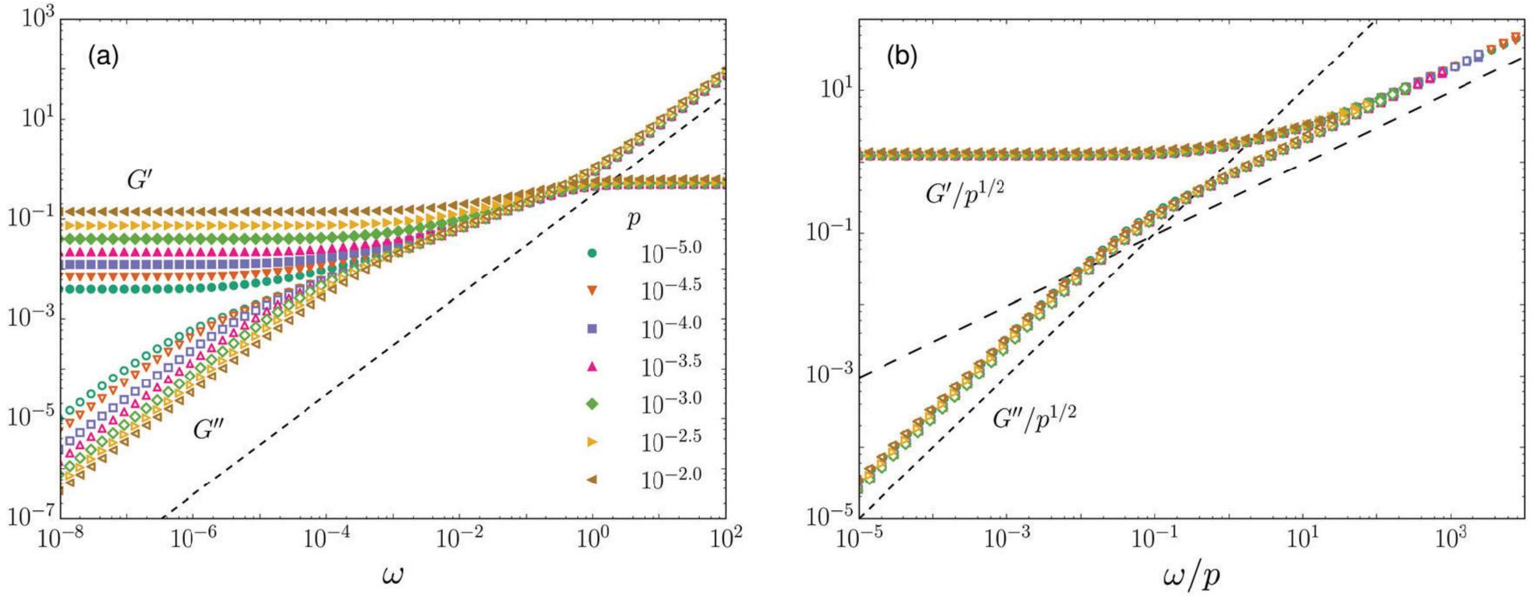
$$s^* \sim \Delta z^2$$

B.P. Tighe, Phys. Rev. Lett. 107, 158303 (2011)



$$G' \sim \begin{cases} p^{1/2} \\ \omega^{1/2} \end{cases}$$

$$G'' \sim \begin{cases} \omega/p^{1/2} & (\omega < p) \\ \omega^{1/2} & (\omega > p) \end{cases}$$



K. Baumgarten and B.P. Tighe, Soft Matter 13, 8368 (2017)

Viscoelasticity near jamming

- Near jamming, $p \to 0$

- Critical scaling,

$$G'(\omega) \sim \begin{cases} p^{1/2} & (\omega < p) \\ \omega^{1/2} & (\omega > p) \end{cases}$$

$$G''(\omega) \sim \begin{cases} \omega/p^{1/2} & (\omega < p) \\ \omega^{1/2} & (\omega > p) \end{cases}$$

K. Baumgarten and B.P. Tighe, *Soft Matter* **13**, 8368 (2017)

In long wave lengths ($\omega < p$),

$$G'(\omega) \sim p^{1/2} \qquad \frac{G''(\omega)}{\omega} \sim \frac{1}{p^{1/2}}$$

Therefore,

$$v_T(k) \propto \sqrt{G'(\omega)} \sim p^{1/4}$$

$$\Gamma_T(k) \propto \frac{G''(\omega)}{\omega G'(\omega)} \Omega_T(k)^2 \sim \frac{\Omega_T(k)^2}{p}$$

$$\therefore \frac{\Gamma_T(k)}{p} \sim \left(\frac{\Omega_T(k)}{p}\right)^2$$ ⋯ quadratic