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"Theoretical studies of non-equilibrium driven-dissipative systems“
@ YITP, Kyoto Univ.

TUAT Mechanical Systems Engineering

Stress propagation in a two-dimensional elastic circular disk under diametric loads

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Introduction: fracture

<u>Fracture</u>

Widely observed
in many situations

Understanding leads to
・Improved efficiency
・Stabilization of grain size

Applicable to civil eng.,
earthquakes, etc.

Mining machine

Pepper mill

crater

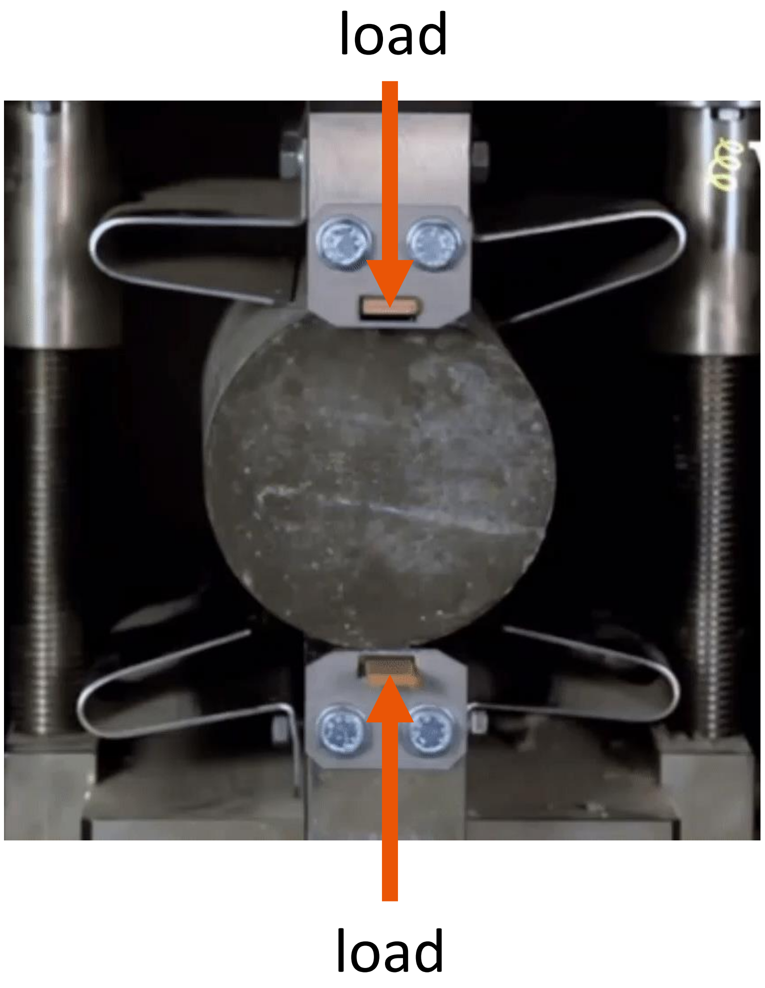
To understand fracture phenomena,
we need to know what happens inside the system.

Introduction: fracture under quasi-static condition

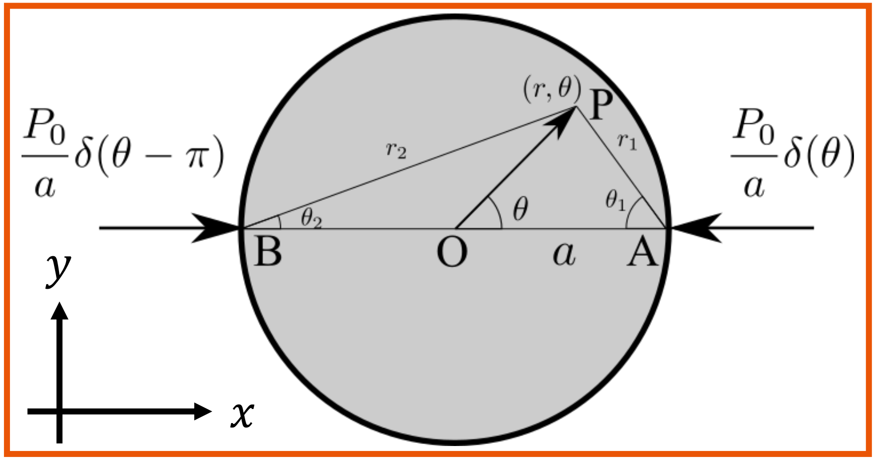
Brizilian test

- Strength test in which loads are applied to concrete
- Under quasi-static loads, fracture occurs on the load axis

How is the stress distributed in the system?



Introduction: static solution

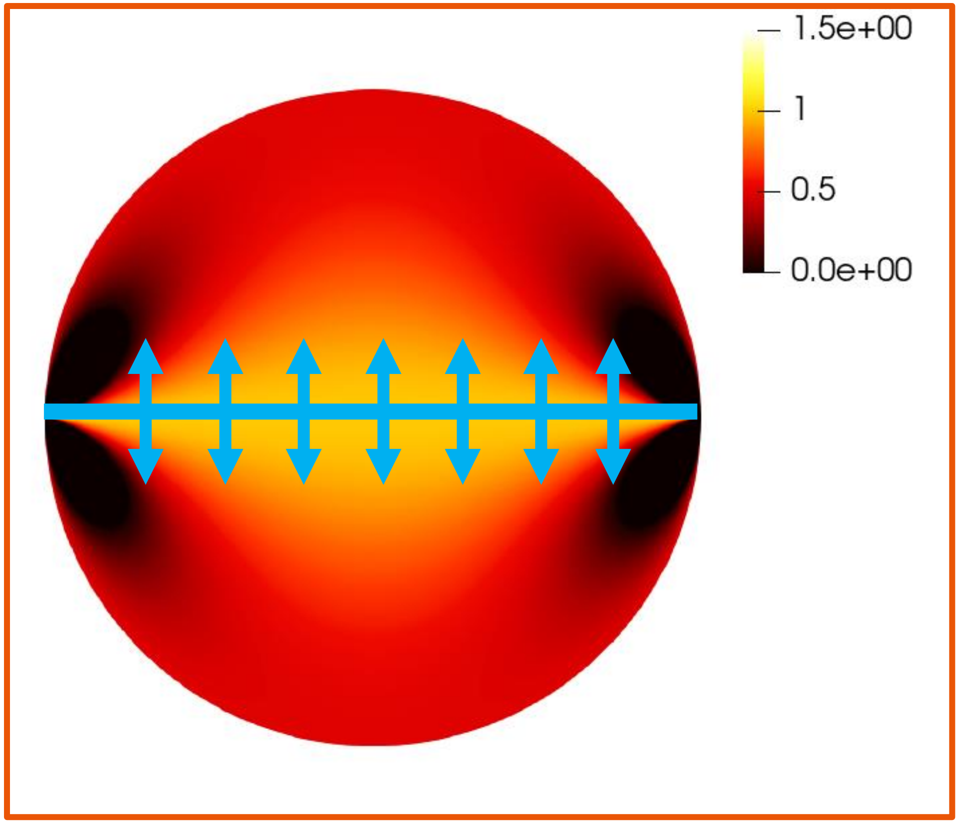


$$\sigma_{yy} = \frac{P_0}{\pi a} - \frac{2P}{\pi}\left(\frac{\cos\theta_1 \sin^2\theta_1}{r_1} + \frac{\cos\theta_2 \sin^2\theta_2}{r_2}\right)$$

Extensional stress (σ_{yy})

- is constant $\left(\frac{P_0}{\pi a}\right)$
- becomes maximum

on this line.



Fracture under dynamic loading

Shock propagation

Wave phenomena

- Rapid wave propagation inside the system
- Amplification by superposition of reflected and traveling waves

Different from under static load

Analysis of impact propagation inside an object is important

This study

Stress propagation in elastic media

What we solve in this study

Previous study

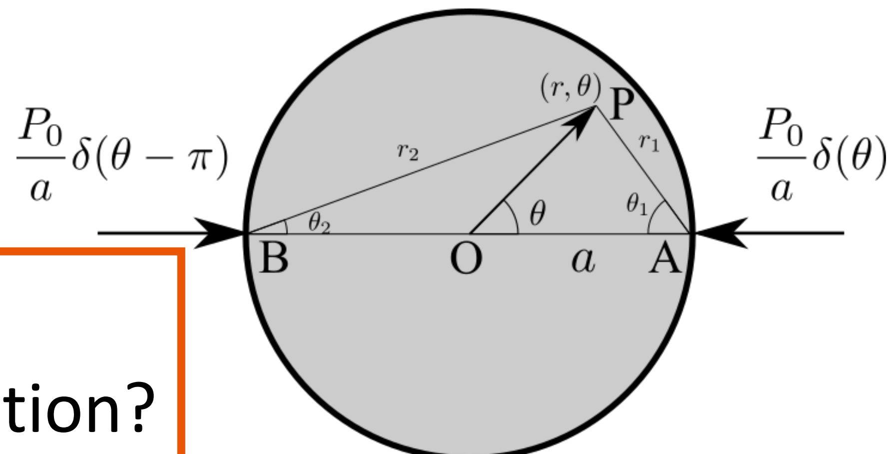
semi-infinite space

- P-, S-, and surface waves
- Convergence to static sol.

[redacted]

This study

Elastic disk (finite size)



Motivation

Can we obtain a solution?

Formulation of linearized elastodynamics

Starting point:

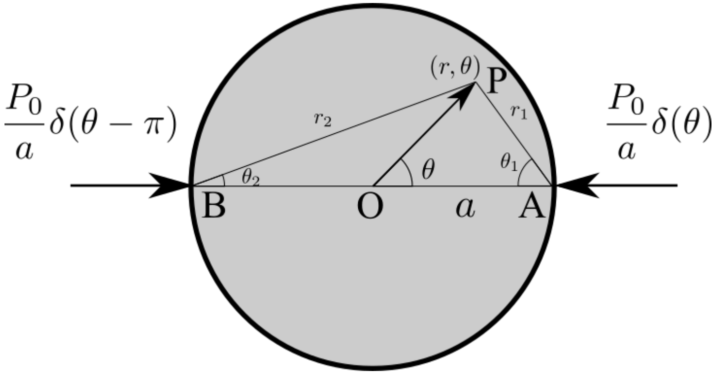
Navier-Cauchy equation

$$\rho \frac{\partial^2}{\partial t^2}\boldsymbol{u} = \nabla^2 \boldsymbol{u} + G\,\frac{1+\nu}{1-\nu}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u})$$

ρ: density, $\boldsymbol{u}$: displacement,
ν: Poisson ratio,
G: shear modulus

Constitutive relation

$$\sigma_{\alpha\beta} = \lambda\delta_{\alpha\beta}\varepsilon_{\gamma\gamma} + 2G\varepsilon_{\alpha\beta} \quad \left(\varepsilon = \frac{1}{2}(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T)\right)$$



+ Initial condition for $t \leq 0$

$$\boldsymbol{u} = \boldsymbol{0}$$

+ Boundary condition at $r = a$

$$\sigma_{rr} = -P(\theta)\Theta(\mathrm{t}), \quad \sigma_{r\theta} = 0$$

Formulation of linearized elastodynamics

Starting point:

Navier-Cauchy equation

$$\rho \frac{\partial^2}{\partial t^2}\boldsymbol{u} = \nabla^2 \boldsymbol{u} + G\frac{1+\nu}{1-\nu}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u})$$

Introduction of two potentials: $\phi, \boldsymbol{A}$

$$\boldsymbol{u} = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\times\boldsymbol{A}$$

Wave equations (d'Alembertian)

$$\square_L \phi = 0, \square_T A = 0$$

Laplace transform

$$\mathcal{L}[\phi](s) = \sum_m a_m(s)\cos(m\theta)\, I_m\left(\frac{sr}{v_\mathrm{L}}\right)$$

$$\mathcal{L}[A](s) = \sum_m b_m(s)\sin(m\theta)\, I_m\left(\frac{sr}{v_\mathrm{T}}\right)$$

$I_m(r)$: modified Bessel function of the first kind

$v_\mathrm{L} \equiv \sqrt{\frac{2}{1-\nu}\frac{G}{\rho}}$: speed of P−wave

$v_\mathrm{T} \equiv \sqrt{\frac{G}{\rho}}$: speed of S−wave

$a_m(s)$ and $b_m(s)$: determined by the boundary condition

Formulation of linearized elastodynamics

Inverse Laplace transform

$$\phi = \mathcal{L}^{-1}\left[\sum_m a_m(s)\cos(m\theta)\, I_m\left(\frac{sr}{v_\mathrm{L}}\right)\right] = \int_\mathrm{Br}\sum_m a_m(s)\cos(m\theta)\, I_m\left(\frac{sr}{v_\mathrm{L}}\right)e^{st}ds$$

$$A = \mathcal{L}^{-1}\left[\sum_m b_m(s)\sin(m\theta)\, I_m\left(\frac{sr}{v_\mathrm{T}}\right)\right] = \int_\mathrm{Br}\sum_m b_m(s)\sin(m\theta)\, I_m\left(\frac{sr}{v_\mathrm{T}}\right)e^{st}ds$$

Conversion to contour integration

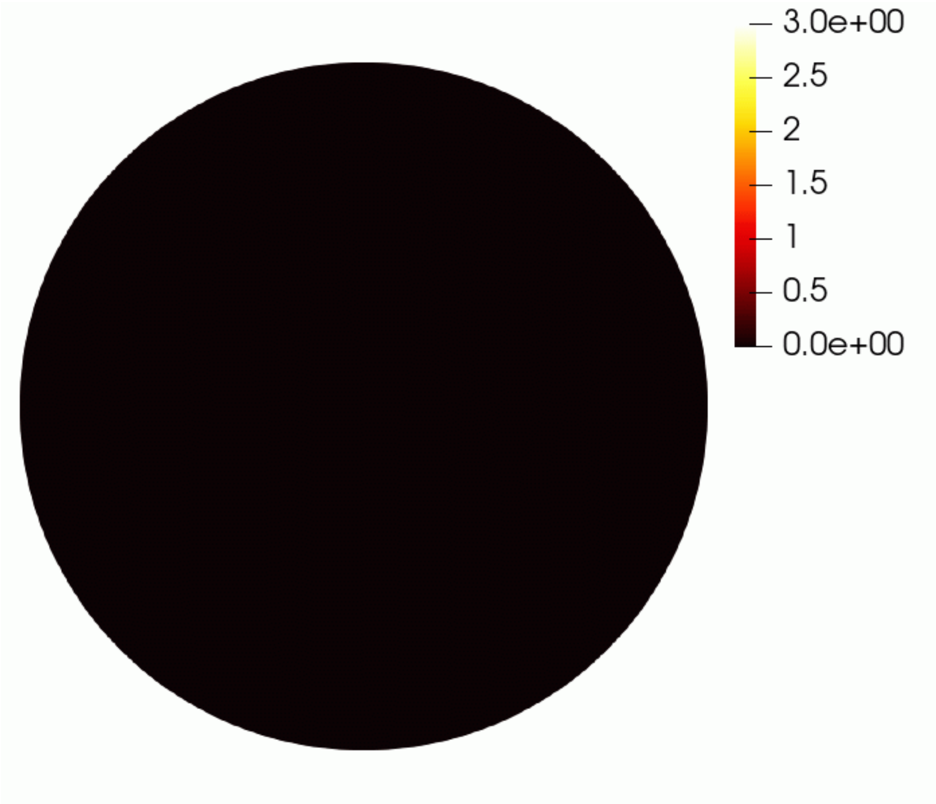
Final result:

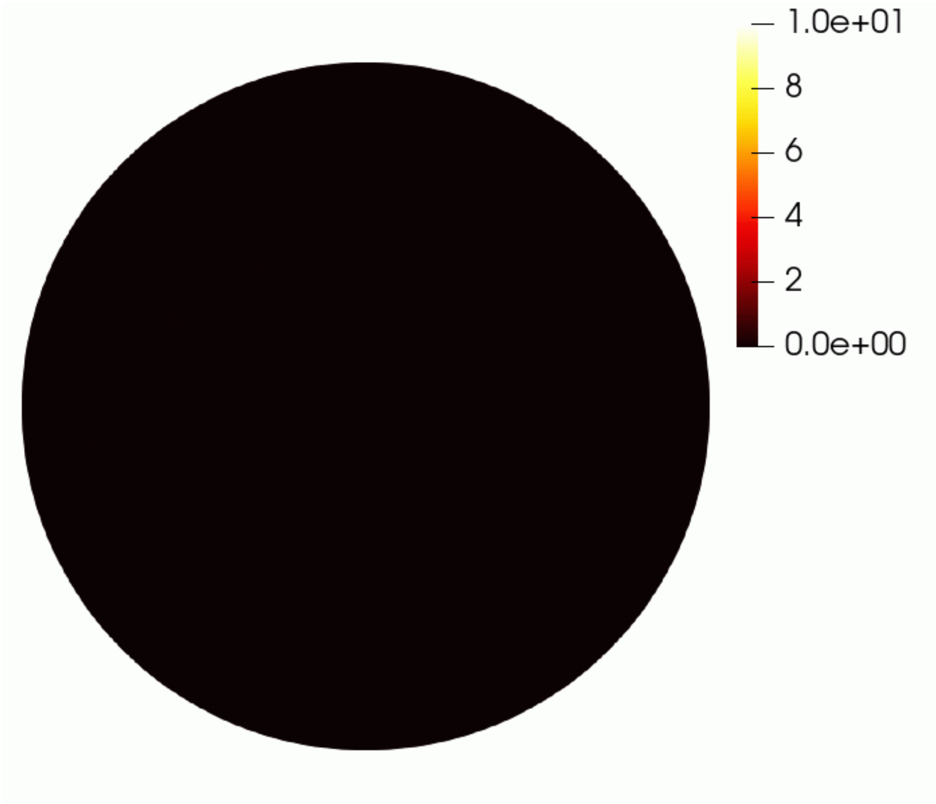
$$\tilde{u}_r = \Theta_\mathrm{P}(1-\Theta_\mathrm{S})\tilde{u}_r^{(\mathrm{tr})}(r^*,\theta,t^*) + \Theta_\mathrm{S}\tilde{u}_r^{(\mathrm{st})}(r^*,\theta)$$
$$+ \sum_{m=0,2,4,\cdots}\sum_{n=1}^{\infty}\left[\Theta_\mathrm{P}\tilde{u}_{r,\mathrm{P}}^{(m)}(r^*,\omega_{m,n}^*) + \Theta_\mathrm{S}\tilde{u}_{r,\mathrm{S}}^{(m)}(r^*,\omega_{m,n}^*)\right]\cos(m\theta)\cos(\omega_{m,n}^*t^*)$$

$\tilde{u}_r^{(\mathrm{tr})}$	$\frac{1}{2}\frac{1-\nu}{1+\nu}r^* + \frac{1-\nu}{1+\nu}\frac{1-r^{*2}}{2r^*r_1^*r_2^*}\cos(\theta_1-\theta_2) - \frac{1-\nu}{1+\nu}\frac{2r^*t^{*2}}{r_1^{*2}r_2^{*2}}\cos[2(\theta+\theta_1-\theta_2)]$ $-\frac{2(3-\nu)+(1-\nu^2)r^{*2}}{4(1+\nu)^2}\frac{1}{r^{*2}}\left[\cos\theta\log\frac{r_2^*}{r_1^*}+(\theta_1+\theta_2)\sin\theta\right]$ $-\frac{5-2\nu+\nu^2}{2(1+\nu)^2}\frac{1}{r^{*3}}[\cos(2\theta)\log(r_1^*r_2^*)-(\theta_1-\theta_2)\sin(2\theta)]$		
$\tilde{u}_r^{(\mathrm{st})}$	$\frac{1}{2}\frac{1-\nu}{1+\nu}r^* + \frac{1}{2}\sin\theta_1\sin(\theta+\theta_1) + \frac{1}{2}\sin\theta_2\sin(\theta-\theta_2) - \frac{1}{1+\nu}\cos\theta\log\frac{r_2^*}{r_1^*} - \frac{1}{2}\frac{1-\nu}{1+\nu}(\theta_1+\theta_2)\sin\theta$		
$\tilde{u}_{r,\mathrm{P}}^{(m)}$	$\begin{cases}\frac{f_{0,0}(r^*\omega^*)}{r^*\omega^*g_{0,1}(\omega^*)} & (m=0)\\ \frac{2f_{m,0}(r^*\omega^*)f_{m,3}(\mu\omega^*)}{r^*\omega^*d_m(\omega^*)} & (m\geq 2)\end{cases}$	$\tilde{u}_{r,\mathrm{S}}^{(m)}$	$\begin{cases}0 & (m=0)\\ -\frac{2mJ_m(\mu r^*\omega^*)f_{m,2}(\omega^*)}{r^*\omega^*d_m(\omega^*)} & (m\geq 2)\end{cases}$

$$\Theta_\mathrm{P} \equiv \Theta\big(t^* - (1-r^*)\big)$$
$$\Theta_\mathrm{S} \equiv \Theta\big(t^* - \mu(1-r^*)\big)$$

Result: displacement and principal stress difference





$$u = \sqrt{u_r^2 + u_\theta^2}$$

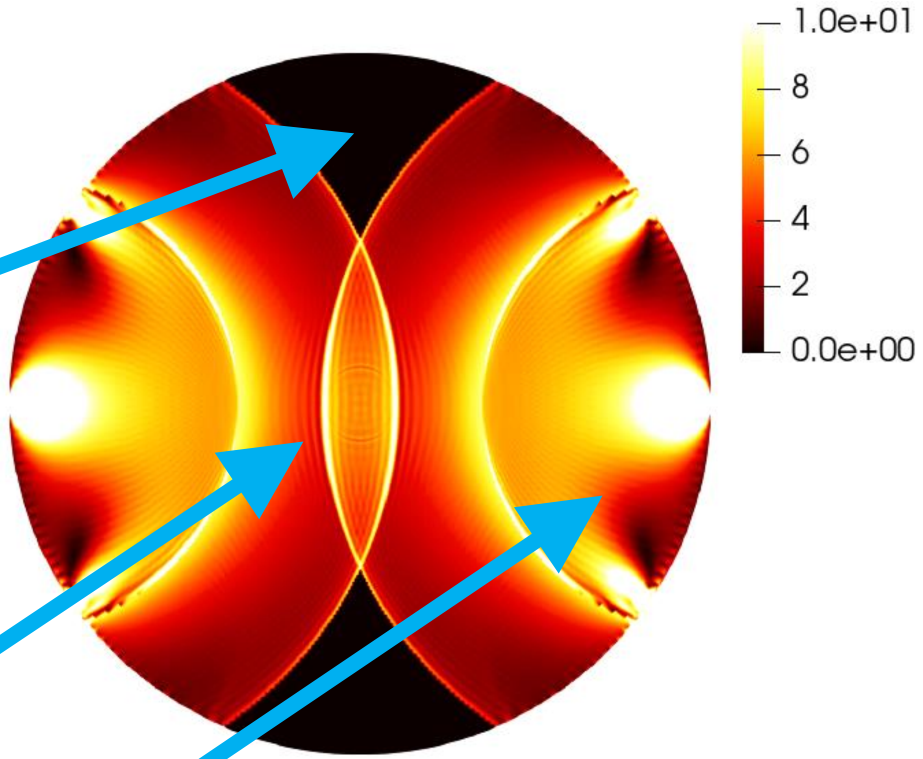
$$\sigma_{11} - \sigma_{22} = 2\sqrt{\left(\frac{\sigma_{rr} - \sigma_{\theta\theta}}{2}\right)^2 + \sigma_{r\theta}^2}$$

Result: three different regimes

Three different regimes

① No wave arrives.
No stress

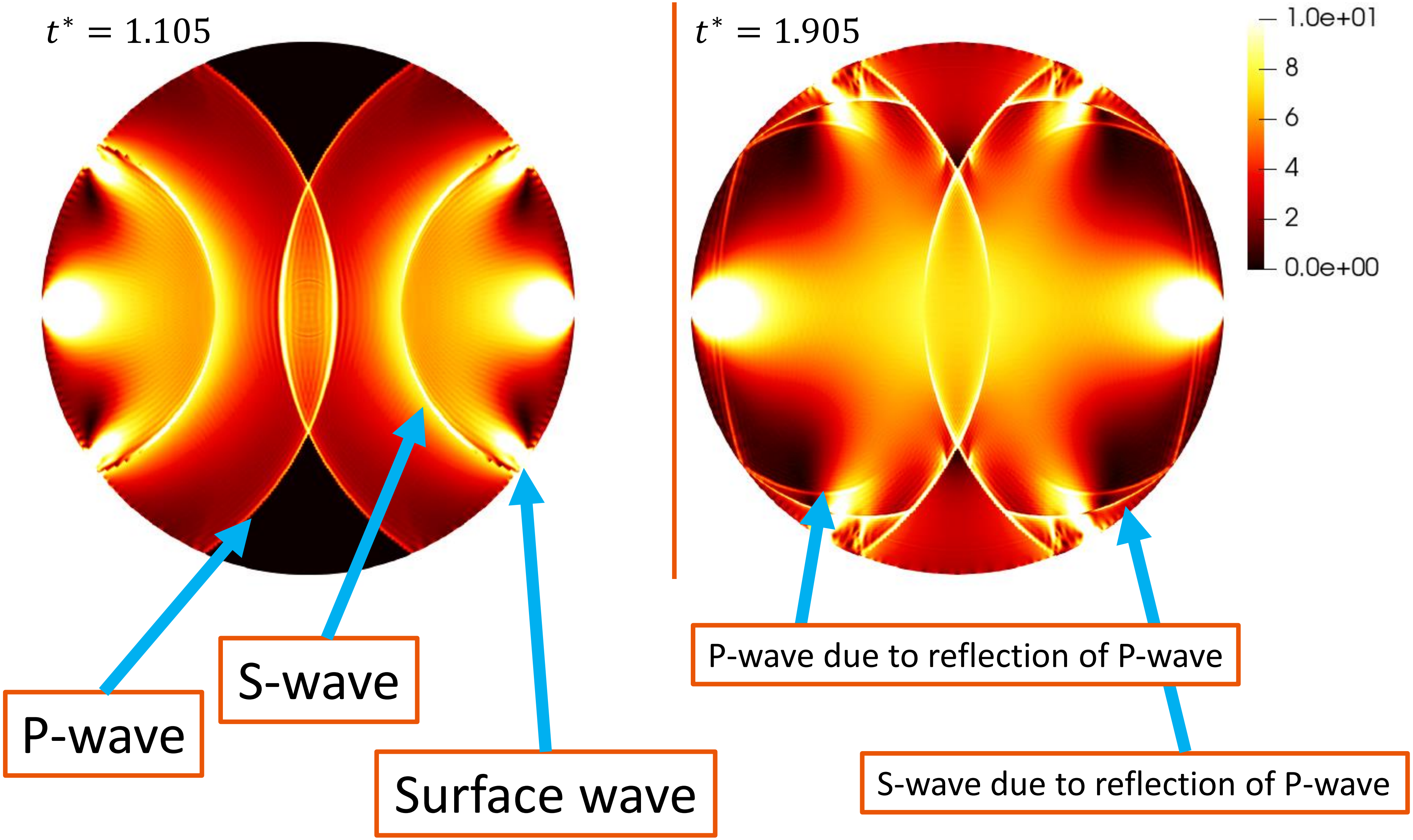
② Only P-wave arrives.



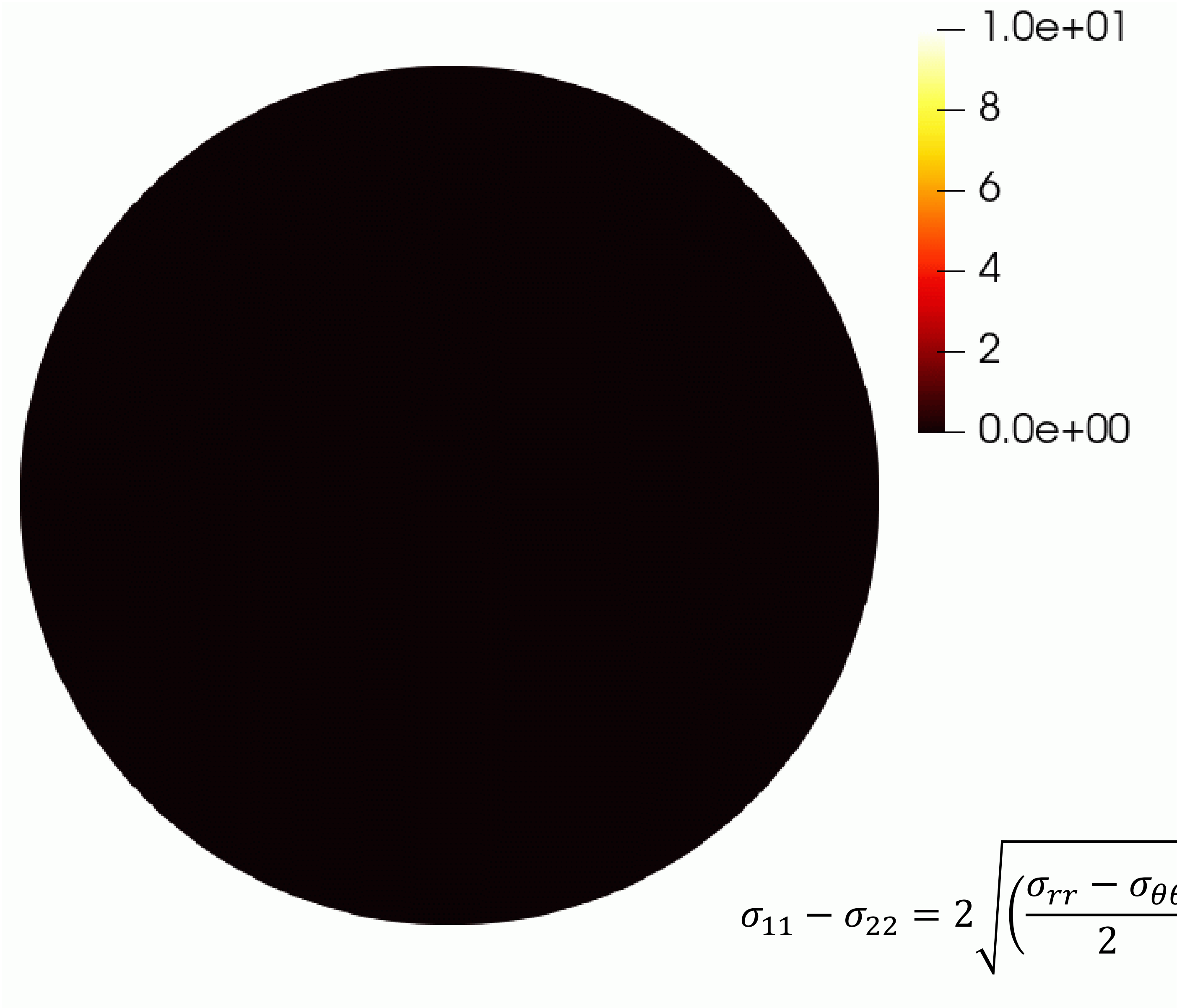
$t = 1.105$

③ S-wave also arrives.
Solution contains both P- and S-waves.

Result: snapshots



Result: principal stress difference



$$\sigma_{11} - \sigma_{22} = 2\sqrt{\left(\frac{\sigma_{rr} - \sigma_{\theta\theta}}{2}\right)^2 + \sigma_{r\theta}^2}$$

Long time limit

$$\tilde{u}_r = \Theta_{\mathrm{P}}(1-\Theta_{\mathrm{S}})\tilde{u}_r^{(\mathrm{tr})}(r^*,\theta,t^*) + \Theta_{\mathrm{S}}\tilde{u}_r^{(\mathrm{st})}(r^*,\theta)$$
$$+\sum_{m=0,2,4,\cdots}\sum_{n=1}^{\infty}\left[\Theta_{\mathrm{P}}\tilde{u}_{r,\mathrm{P}}^{(m)}(r^*,\omega_{m,n}^*) + \Theta_{\mathrm{S}}\tilde{u}_{r,\mathrm{S}}^{(m)}(r^*,\omega_{m,n}^*)\right]\cos(m\theta)\cos(\omega_{m,n}^* t^*)$$

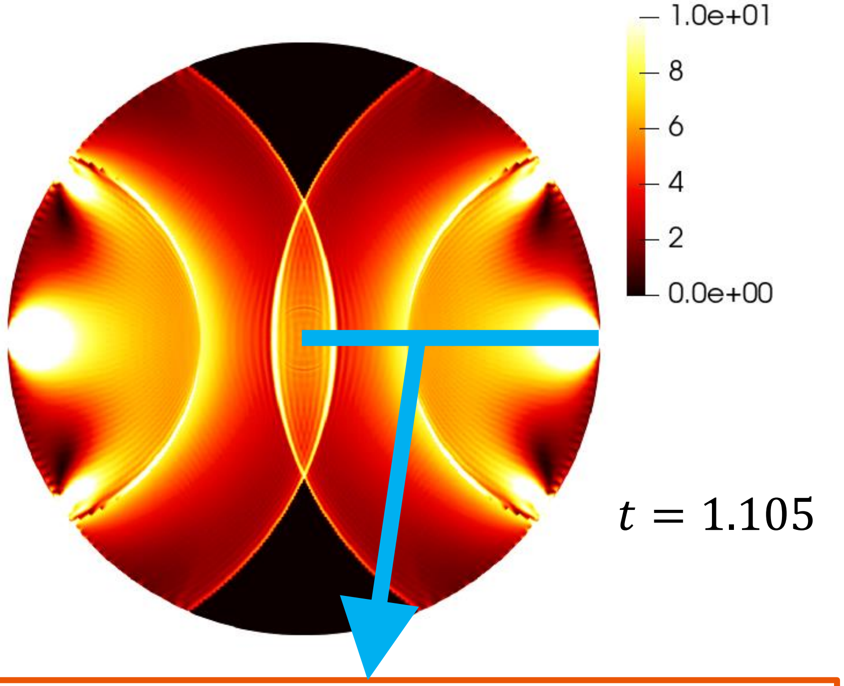
① Evaluated from the final value theorem

$$\lim_{t\to\infty}\sigma_{\alpha\beta}(\boldsymbol{r},t) = \lim_{s\to 0} s\mathcal{L}[\sigma_{\alpha\beta}](\boldsymbol{r},s)$$

② Consistent with the static solution

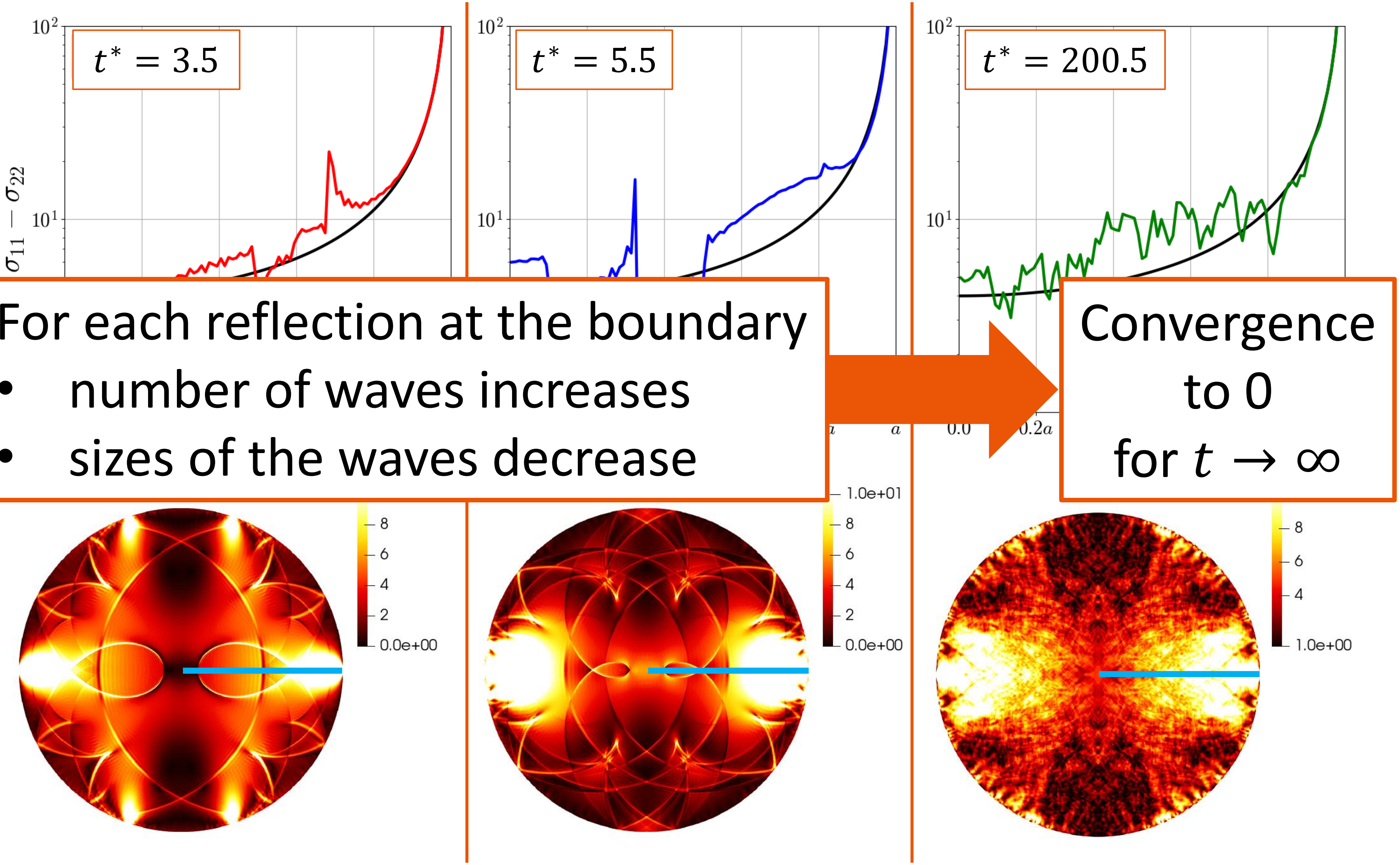
- P-wave term
- S-wave term

Converge to zero



Check the time evolution of the stress profile

Convergence



For each reflection at the boundary

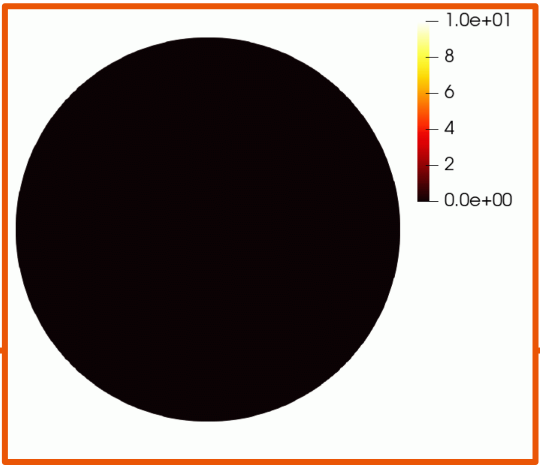
- number of waves increases
- sizes of the waves decrease

Convergence to 0 for $t \to \infty$

Summary

Purpose

Derivation of the stress propagation in a finite elastic disk



Result

Success to derive analytical solution

- Contains P-, S-, and surface waves
- Converges to static solution

- After S-wave arrives, (static)+(P-wave) +(S-wave)

Future work

- Application to elliptic or spherical elastic media