Theory of Mpemba effect after a quench in an asymmetric double-well potential

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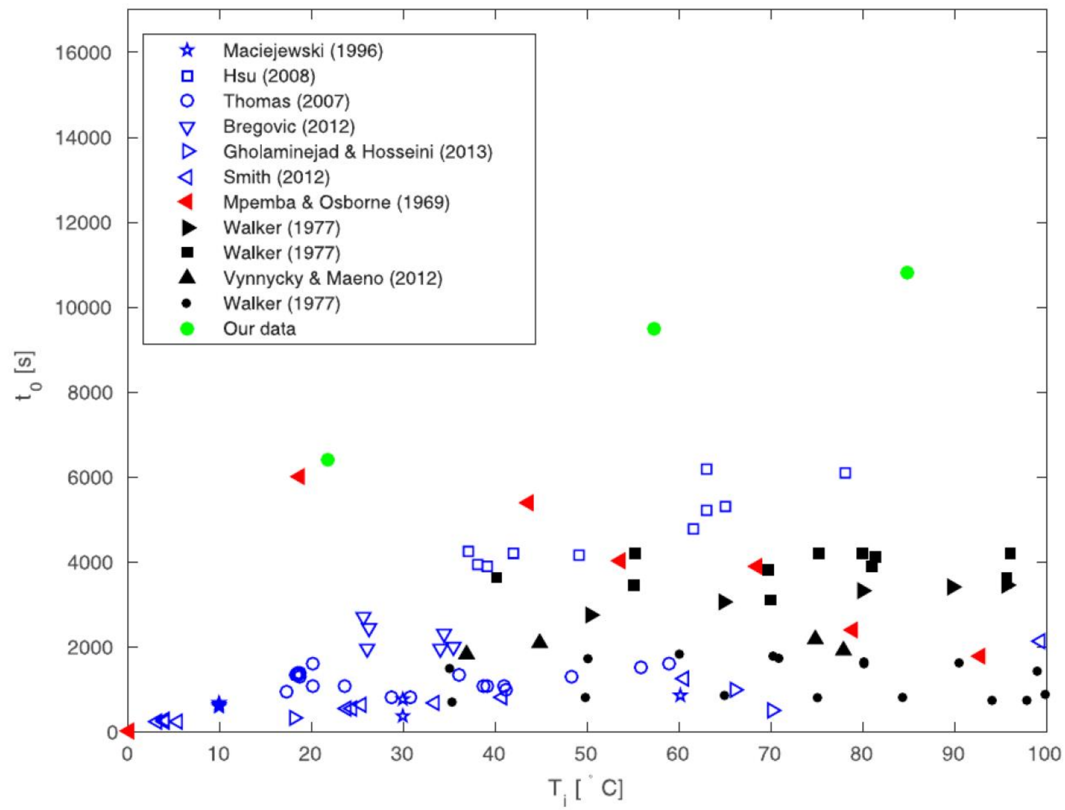
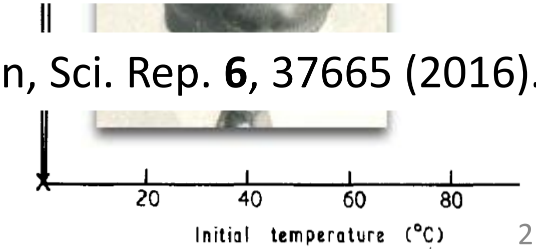
Statphys 28 at Tokyo, August 10th, 2023 [T2b-10A-02] in Room 233

What is Mpemba effect?

- What is Mpemba e
  - Erasto B. Mpemba
    that some hot sus
    (*ice cream mix*) ca
    faster than cold (1
  - With the help of D
    Osborne he has p
    scientific paper (1
  - Even in 2016, an a
    Mpemba paper has been
    published.

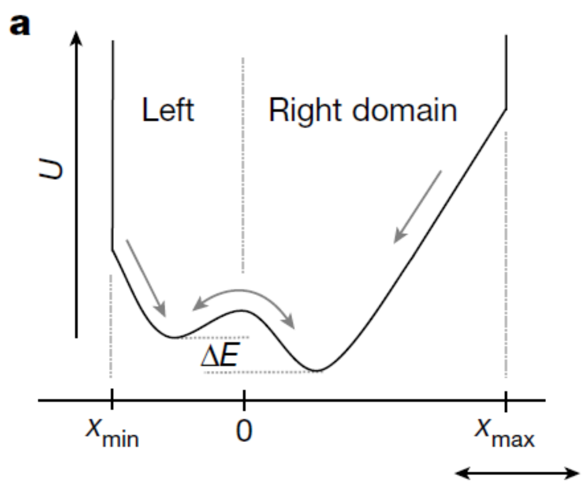


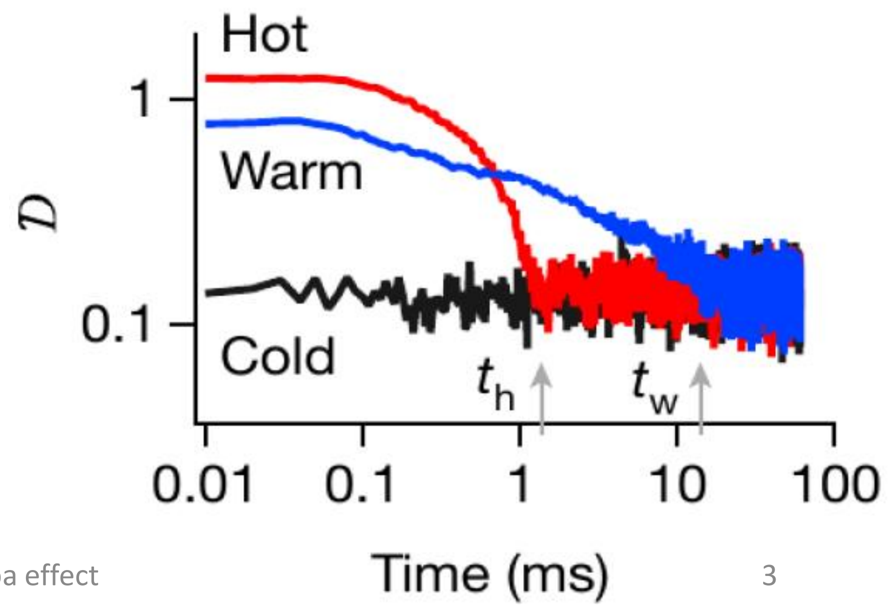
Burridge and Linden, Sci. Rep. **6**, 37665 (2016).



Experimental confirmation

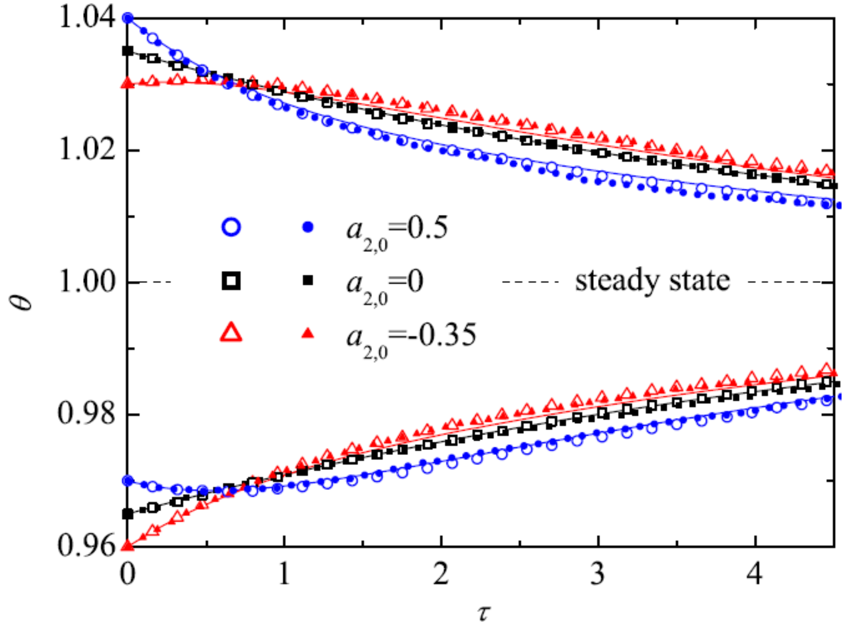
- Kumar & Bechhoeffer, Nature **584**, 64 (2020).
- They have analyzed trapped colloids in a double well potential.
- They observed the distance between the distribution and equilibrium one.

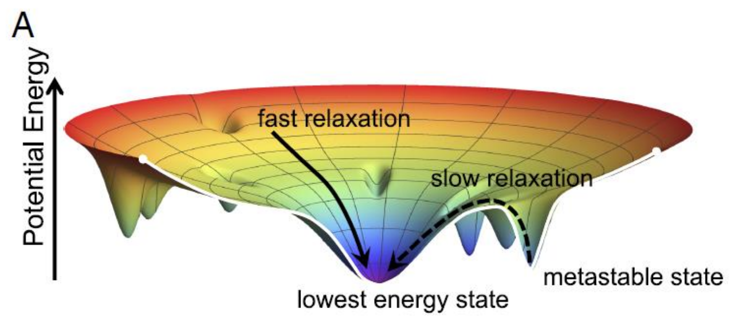




Some theoretical studies

- Lasanta et al. PRL **119**, 148001 (2017) found that a granular gas can have both ME and the inverse ME by controlling *kurtosis*.
  - Along this line, there are some papers such as Takada et al, PRE **103**, 032901 (2021 & Santos' talk [T2a-07B-04].
- Lu & Raz, PNAS **114**, 5083 (2017) indicated that the slow relaxation can take place by trapping at local minima.





Lu & Raz (PNAS2017)

- They have analyzed the master equation:

$$\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = \sum_j R_{ij}(T_b)p_j(t) \quad \text{for } i = 1, 2, \cdots, n.$$

- They are interested in the slowest relaxation mode=>approach to the equilibrium state:

$$\vec{p}(t) = \vec{\pi}(T_b) + e^{\lambda_2 t} a_2 \vec{v}_2 + \ldots \qquad \pi_i(T_b) = \frac{e^{-E_i/k_B T_b}}{\sum_i e^{-E_i/k_B T_b}}$$

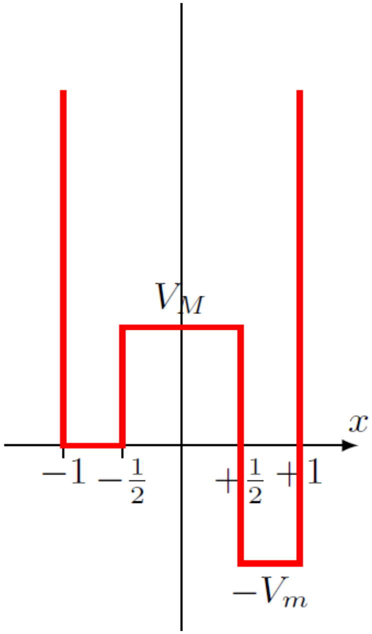
- The condition for Markovian Mpemba effect:

$$|a_2^c| > |a_2^h|$$

Purpose of this talk

- We analyze the relaxation process in a double well potential to clarify the validity of the scenario by Lu & Raz (2017).
  - There are some counter examples in kinetic theory of granular gases, inertial suspensions and quantum Mpemba effect.
  - Rush to [T2a-10A-02] from 1:30 PM in Koshiba Hall to know *quantum Mpemba effect*.

Setup



- We solve the Fokker-Planck equation describing a particle confined in a square-box double well potential.
- The continuity condition

$$j = -\frac{1}{\beta} e^{-\beta V} \partial_x \left[e^{\beta V} p \right]$$

$$-\beta \int_{x_0^-}^{x_0^+} dx\, e^{\beta V(x)} j(x,t) = 0 = e^{\beta V(x_0^+)} p(x_0^+, t) - e^{\beta V(x_0^-)} p(x_0^-, t)$$

- The continuity of current at $x = \pm 1, \pm 1/2$ and $e^{\beta V} p$ at $x = \pm 1/2$ => 6 conditions.

Our analysis

- Eigenvalue expansion

$$p(x,t) = \sum_\lambda \mathrm{e}^{-\lambda t} r_\lambda(x) \int \mathrm{d}x' \ell_\lambda(x') p_i(x')$$

- If a_2 is only dominant, $p(x,t)$ can be given by

$$p(x,t) = p_{\mathrm{eq}}(x,\beta) + \mathrm{e}^{-\lambda_2 t} r_2(x) \underbrace{\int \mathrm{d}x' \ell_2(x') p_i(x')}_{=a_2(\beta_i,\beta)}$$

with the initial equilibrium state:

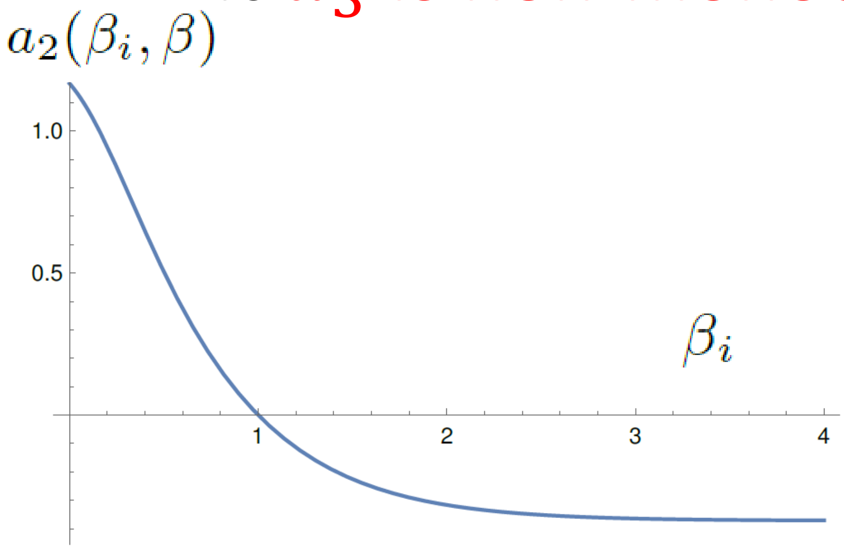
$$p_{\mathrm{eq}}(x,\beta) = \frac{\mathrm{e}^{-\beta V(x)}}{Z(\beta)}$$

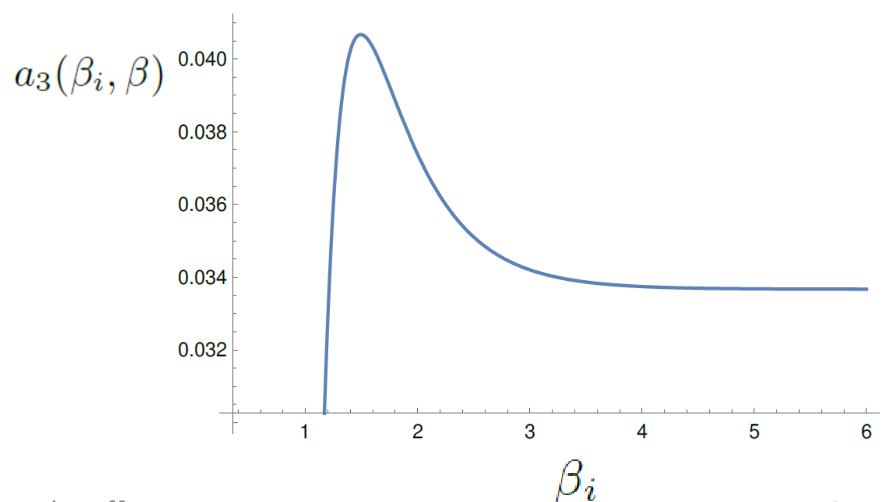
Result

$$[1+\mathrm{e}^{\beta V_m}+2\mathrm{e}^{-\beta V_M}]^{1/2}(\mathrm{e}^{\beta V_M}-\mathrm{e}^{\beta(V_m+V_M)}-\mathrm{e}^{\beta_i V_M}+\mathrm{e}^{\beta_i(V_m+V_M)}+\mathrm{e}^{(\beta+\beta_i)V_M})(-\mathrm{e}^{\beta V_m}+\mathrm{e}^{\beta_i V_m})$$

$$a_3(\beta_i,\beta)=\frac{2\sqrt{2}}{\pi}\mathrm{e}^{-\frac{1}{2}(\beta-2\beta_i)(V_m+V_M)}\sqrt{\frac{2+\mathrm{e}^{\beta V_M}+\mathrm{e}^{\beta(V_m+V_M)}}{1+\mathrm{e}^{\beta_i V_M}+\mathrm{e}^{\beta_i(V_m+V_M)}}}\frac{1+\mathrm{e}^{(\beta-\beta_i)V_m}-2\mathrm{e}^{(\beta-\beta_i)(V_m+V_M)}}{2+\mathrm{e}^{\beta_i V_M+\mathrm{e}^{\beta_i(V_m+V_M)}}}$$

- We found a_2 is a monotonic function of the initial temperature
- But a_3 is non-monotonic.





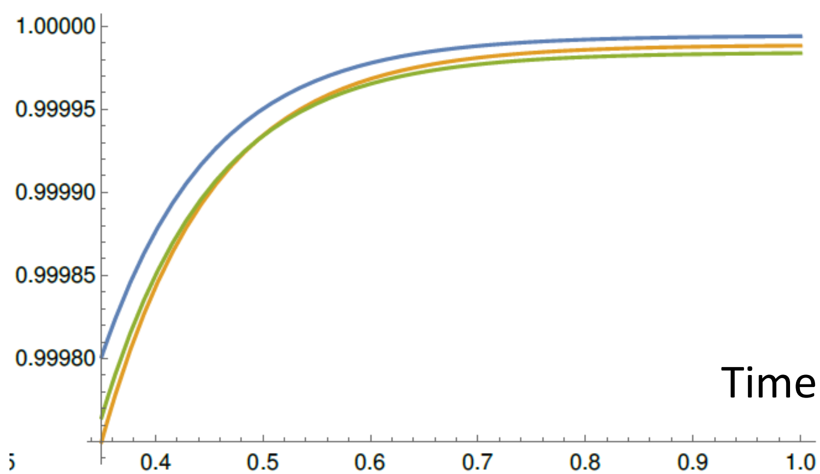
Inverse Mpemba effect

- We also found the existence of inverse-Mpemba effect.

$$\langle\theta\rangle(t) = \int \mathrm{d}x x V' p(x,t) = -\beta^{-1}\int \mathrm{d}x \underbrace{\frac{\mathrm{d}e^{-\beta V}}{\mathrm{d}x}}_{\text{sum of }\delta}\left[x\,\underbrace{e^{\beta V}p(x,t)}_{\text{continuous}}\right]$$

$$\langle\theta\rangle(t)$$

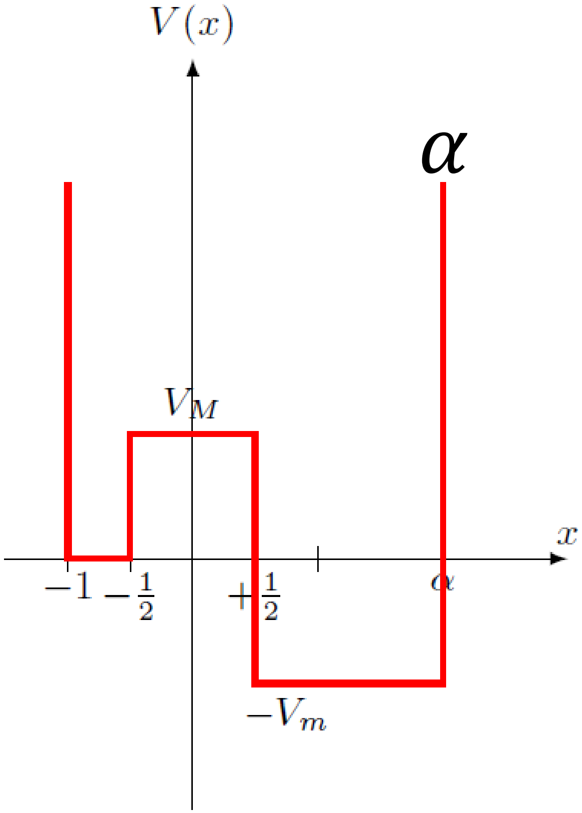
$$\theta = xV'$$



β is the inverse temperature for t>0.

What happens when we control α?

- We can modify the potential shape as the right figure.
- α is a new parameter.
- If $\alpha < 1$, we observe a non-monotonous a_2.
- Now the analysis is in progress.



Potential for $\alpha > 1$

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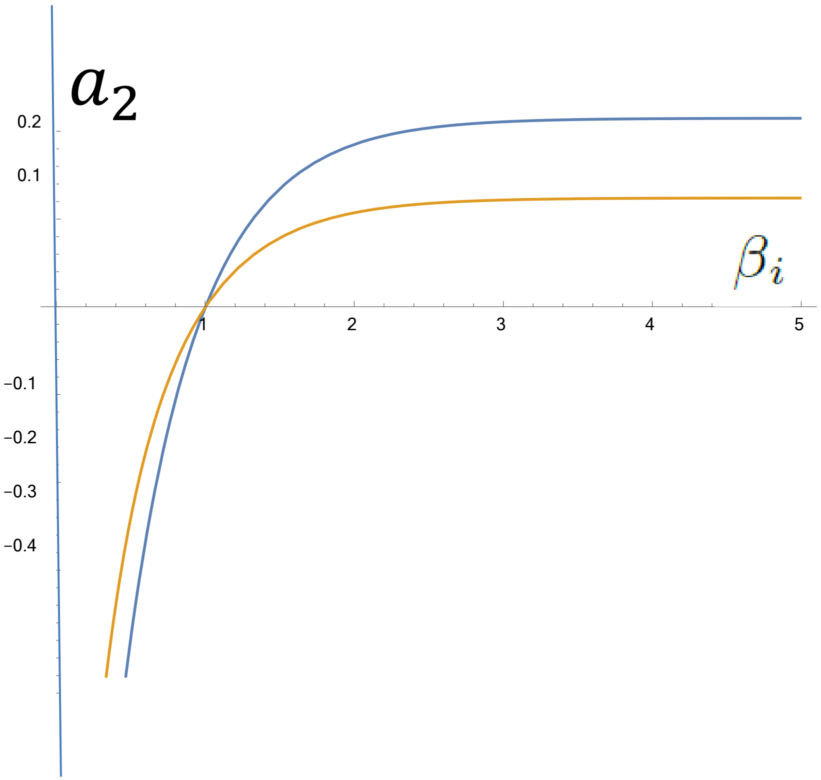
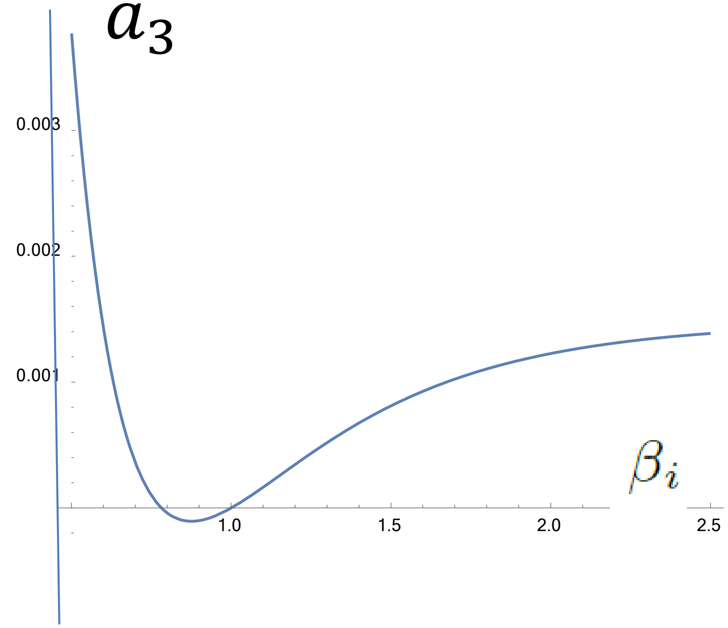
Some results





We have used $\beta =$ 1, $V_M =$ 4, $V_m =$ 2 and $\alpha =$ 2 (blue) and $\alpha =$ 5 (orange).

$\alpha = 5$

Related studies

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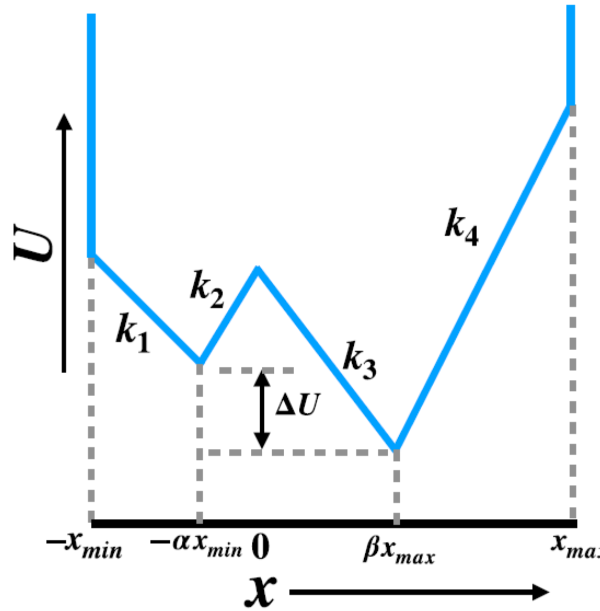
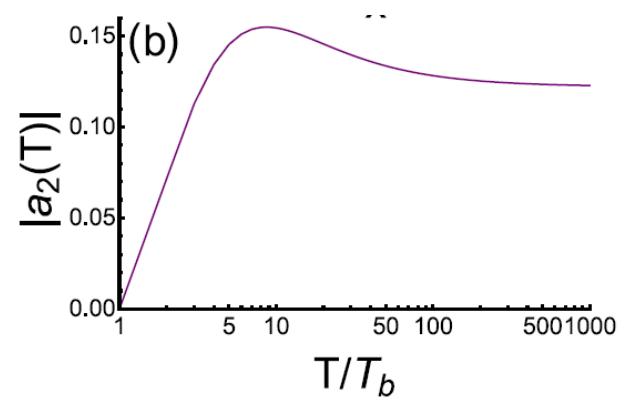
- Biswas, Rajesh & Pal, JCP **159**, 044120 (2023) .
- Biswas & Rajesh, arXiv:2305.06613.
- They analyzed a piece-wise linear double well potential and found the Mpemba effect.





Summary

- In general, we cannot understand Mpemba effect only based on the slowest mode.
  - The second slowest modes play major roles in some cases.
  - Most of our analysis follows such a situation.
  - In some case ($\alpha < 1$), the slowest mode is sufficient.
- To know the quantum Mpemba effect, please go to Koshiba Hall now.

Quantum Mpemba

Talk by A. K. Chatterjee [T2a-10A-02] from 1:30 PM in Koshiba Hall

See arXiv:2304.02411 (to be published in PRL)