# Simple Lie algebra

○: long root, •: short root, $(j; a_j; a_j^\vee)$: (label;mark;comark), green line: $-\theta$, $\theta$; highest root, $W$:Weyl group, $P$: weight lattice, $Q$ : root lattice, $Q^\vee$ : coroot lattice $[\lambda_1, \dots, \lambda_r]$: Dynkin label.

Definitions and relations

$$\alpha_j^\vee := \frac{2}{|\alpha_j|^2}\alpha_j, \qquad A_{ij} := (\alpha_i|\alpha_j^\vee),$$

$$\theta = \sum_{j=1}^r a_j\alpha_j = \sum_{j=1}^r a_j^\vee \alpha_j^\vee, \qquad a_j^\vee = \frac{|\alpha_j|^2}{2} a_j,$$

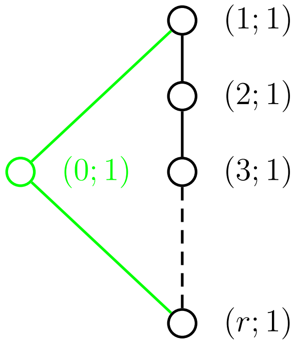
$$h := \sum_j a_j + 1 = \frac{|\Delta|}{r}, \qquad h^\vee := \sum_j a_j^\vee + 1,$$

$$(\omega_i|\alpha_j^\vee) = \delta_{ij}, \qquad \alpha_i = \sum_{j=1}^r A_{ij}\omega_j,$$

$$\omega_i = \sum_{j=1}^r (A^{-1})_{ij}\alpha_j, \qquad \rho := \sum_{j=1}^r \omega_j = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha,$$

$$\dim\lambda = \prod_{\alpha\in\Delta_+}\frac{(\lambda+\rho|\alpha)}{(\rho|\alpha)}, \qquad |\rho|^2 = \frac{h^\vee}{12}\dim G,$$

$A_r, \qquad r \geq 1, \qquad (\mathrm{SU(r+1)})$




$\dim = r^2 + 2r,$
$h = h^\vee = r + 1,$
$|W| = (r+1)!,$
$\theta = [1, 0, \ldots, 0, 1],$
$P/Q = \mathbb{Z}_{r+1},$
$\text{exponents} = \{1, 2, \ldots, r\}$

Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

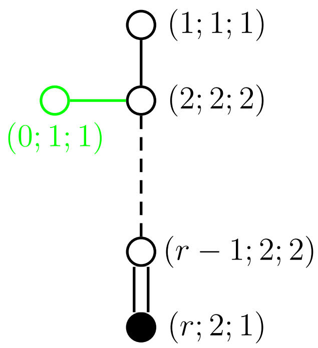
$$A^{-1} = \frac{1}{r+1} \begin{pmatrix} r & r-1 & r-2 & \cdots & 2 & 1 \\ r-1 & 2(r-1) & 2(r-2) & \cdots & 4 & 2 \\ r-2 & 2(r-2) & 3(r-2) & \cdots & 6 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(r-1) & r-1 \\ 1 & 2 & 3 & \cdots & r-1 & r \end{pmatrix}$$

Quadratic form matrix: $A^{-1}$
Realization with orthonormal basis $e_j, \quad j = 1, \ldots, r+1$

$$\alpha_j = e_j - e_{j+1}, \qquad j = 1, 2, \ldots, r$$

$$\Delta_+ = \{e_i - e_j \mid i, j = 1, \ldots, r+1, \qquad i < j\}$$

$B_r, \qquad r \geq 2, \qquad (\mathrm{SO(2r+1)})$




$\dim = 2r^2 + r,$
$h = 2r,$
$h^\vee = 2r - 1,$
$|W| = 2^r r!,$
$\theta = [0, 1, 0, \ldots, 0, 0],$
$P/Q = \mathbb{Z}_2,$
$\text{exponents} = \{1, 3, \ldots, 2r-1\}$

Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -2 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}, \quad A^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 4 & 4 & \cdots & 4 & 4 \\ 2 & 4 & 6 & \cdots & 6 & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(r-1) & 2(r-1) \\ 1 & 2 & 3 & \cdots & r-1 & r/2 \end{pmatrix}$$

Quadratic form matrix:

$$\frac{1}{2} \begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 1 \\ 2 & 4 & 4 & \cdots & 4 & 2 \\ 2 & 4 & 6 & \cdots & 6 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(r-1) & r-1 \\ 1 & 2 & 3 & \cdots & r-1 & r \end{pmatrix}$$

Realization with orthonormal basis $e_j, \quad j = 1, \ldots, r$

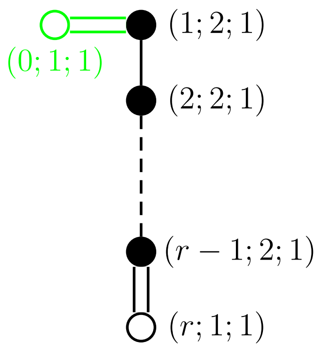
$$\alpha_j = e_j - e_{j+1}, \qquad j = 1, 2, \ldots, r-1,$$

$$\alpha_r = e_r$$

$$\Delta_+ = \{e_i \pm e_j \mid i, j = 1, \ldots, r+1, \qquad i < j\}$$

$$\cap \, \{e_i \mid i = 1, \ldots, r\}$$

$C_r, \qquad r \geq 2, \qquad (\mathrm{Sp(r)})$




$\dim = 2r^2 + r,$
$h = 2r,$
$h^\vee = r + 1,$
$|W| = 2^r r!,$
$\theta = [2, 0, \dots, 0],$
$P/Q = \mathbb{Z}_2,$
$\text{exponents} = \{1, 3, \dots, 2r - 1\}$

Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -2 & 2 \end{pmatrix}, \quad A^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 1 \\ 2 & 4 & 4 & \cdots & 4 & 2 \\ 2 & 4 & 6 & \cdots & 6 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(r-1) & r-1 \\ 2 & 4 & 6 & \cdots & 2(r-1) & r \end{pmatrix}$$

Quadratic form matrix:

$$\frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & r-1 & r-1 \\ 1 & 2 & 3 & \cdots & r-1 & r \end{pmatrix}$$

Realization with orthonormal basis $e_j, \quad j = 1, \dots, r$

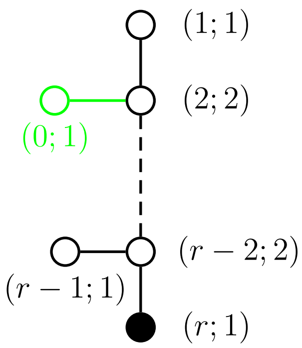
$$\alpha_j = (e_j - e_{j+1})/\sqrt{2}, \qquad j = 1, 2, \dots, r-1,$$

$$\alpha_r = \sqrt{2} e_r$$

$$\Delta_+ = \{(e_i \pm e_j)/\sqrt{2} \mid i, j = 1, \dots, r+1, \qquad i < j\}$$

$$\cap \{\sqrt{2} e_i \mid i = 1, \dots, r\}$$

$D_r, \qquad r \geq 4, \qquad (\mathrm{SO(2r)})$




$\dim = 2r^2 - r,$
$h = h^\vee = 2r - 2,$
$|W| = 2^{r-1} r!,$
$\theta = [0, 1, 0, \ldots, 0],$
$P/Q = \mathbb{Z}_4 \ (r : \text{odd}),$
$\quad = \mathbb{Z}_2 \times \mathbb{Z}_2 \ (r : \text{even}),$
$\text{exponents} = \{1, 3, \ldots, 2r - 3, r - 1\}$

Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 1 & 1 \\ 2 & 4 & 4 & \cdots & 4 & 2 & 2 \\ 2 & 4 & 6 & \cdots & 6 & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(r-2) & r-2 & r-2 \\ 1 & 2 & 3 & \cdots & r-2 & r/2 & (r-2)/2 \\ 1 & 2 & 3 & \cdots & r-2 & (r-2)/2 & r/2 \end{pmatrix}$$

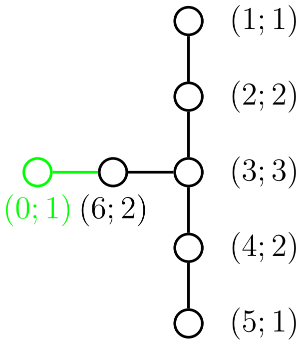
Quadratic form matrix: $A^{-1}$
Realization with orthonormal basis $e_j, \quad j = 1, \ldots, r$

$$\alpha_j = e_j - e_{j+1}, \qquad j = 1, 2, \ldots, r-1,$$

$$\alpha_r = e_{r-1} + e_r$$

$$\Delta_+ = \{e_i \pm e_j \mid i, j = 1, \ldots, r+1, \qquad i < j\}$$

$E_6$




dim = 78,
$h = h^\vee = 12$,
$|W| = 51840$,
$\theta = [0, 0, 0, 0, 0, 1]$,
$P/Q = \mathbb{Z}_3$
exponents = $\{1, 4, 5, 7, 8, 11\}$

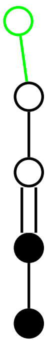
Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{3} \begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 3 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & 6 & 9 & 6 & 3 & 6 \end{pmatrix}$$

Quadratic form matrix: $A^{-1}$

$F_4$




dim = 52,
$h = 12$,
$h^\vee = 9$,
$|W| = 1152$,
$\theta = [1, 0, 0, 0]$,
$P/Q = \{0\}$,
exponents = $\{1, 5, 7, 11\}$

Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 6 & 8 & 4 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 2 \end{pmatrix}$$

Quadratic form matrix:

$$\begin{pmatrix} 2 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 3 & \frac{3}{2} \\ 1 & 2 & \frac{3}{2} & 1 \end{pmatrix}$$

Realization with orthonormal basis $e_j$, $\quad j = 1, 2, 3, 4$

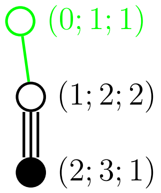
$$\alpha_1 = e_2 - e_3, \qquad \alpha_2 = e_3 - e_4, \qquad \alpha_3 = e_4, \qquad \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4).$$

$$\Delta_+ = \{(e_i \pm e_j) \mid i, j = 1, 2, 3, 4, \qquad i < j\}$$

$$\cap \{e_i \mid i = 1, 2, 3, 4\}$$

$$\cap \{(e_1 \pm e_2 \pm e_3 \pm e_4)/2\}$$

## $G_2$




dim $= 14$,
$h = 6$,
$h^{\vee} = 4$,
$|W| = 12$,
$\theta = [1, 0]$,
$P/Q = \{0\}$,
exponents $= \{1, 5\}$

Cartan matrix:

$$A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

Quadratic form matrix:

$$\frac{1}{3}\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}$$

Realization with orthonormal basis $e_j, \quad j = 1, 2, 3$

$$\alpha_1 = e_2 - e_3, \qquad \alpha_2 = \frac{1}{3}(e_1 - e_2 + 2e_3)$$

$$\Delta_+ = \left\{ e_2 - e_3,\ \frac{1}{3}(e_1 - e_2 + 2e_3),\ \frac{1}{3}(e_1 + 2e_2 - e_3),\ \frac{1}{3}(2e_1 + e_2 + e_3),\ e_1 + e_3,\ e_1 + e_2 \right\}$$