Vortex solutions on membranes

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Introduction

What is string theory?

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- The theory of strong interaction

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- The theory of everything

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What is string theory?

- The theory of strong interaction
- The theory of everything
- AdS/CFT: Logical completion of QFT
  (incl. QCD, Condensed Matter Systems...)

AdS/CFT correspondence

(Some) strongly coupled field theories have gravity duals [Maldacena]

- Conformal group $SO(d,2) \leftrightarrow AdS_{d+1}$
  (Poincaré $M_{\mu\nu}$, P_μ, Dilatation D, SCG K_μ)

  Poincaré coordinates are

$$ds^2 = \rho^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{d\rho^2}{\rho^2}$$

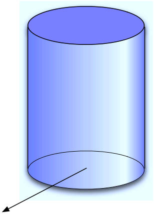
  Dilatation is $D : x^\mu \mapsto \lambda x^\mu, \rho \mapsto \rho/\lambda$

- Dictionary: [Gubser, Klebanov, Polyakov] [Witten]

$$\langle e^{\int \phi_0 \mathcal{O}} \rangle = Z_{AdS}[\phi_0] \sim e^{-S[\phi_0]}$$

  Boundary conditions $\leftrightarrow$ states
  Variation of boundary conditions $\leftrightarrow$ correlation functions

- Radial coordinate $\leftrightarrow$ energy scale ('holographic renormalization')



- A realisation of holographic principle
- Canonical example:
  $AdS_5 \times S^5$ in Type II B string theory $\leftrightarrow$ $\mathcal{N} = 4$ SYM in 4 dim
  $SO(4,2)_{conf} \times SO(6)_R$: isometries of the spacetime

$\lambda = g_{YM}^2 N$: 't Hooft coupling
3 versions of AdS/CFT

- weak: large λ
- strong: any finite λ but $N \to \infty$ and $g_s = g_{YM}^2 \to 0$
  (exact in α' but for small g_s only)
- strongest: any g_s and N
  (α' and g_s)

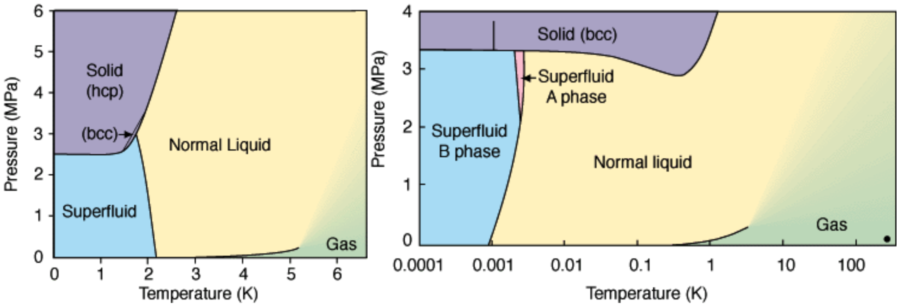
Condensed matter applications of AdS/CFT

Application to QCD relatively successful
– why not Condensed Matter Theory?

- AdS/CFT in laboratory
  - Strongly correlated systems near criticality in low dimensions
  - Rich examples
  - Good control
- A new computational tool in CMT
  - Beyond perturbative field theory and lattice
- A new arena of string theory
  - Typically, non-relativistic $(\Delta \sim (q - q_c)^{\nu z}, \xi \sim (q - q_c)^{-\nu})$
  - $D : t \mapsto \lambda^z t,\ x^i \mapsto \lambda x^i$
  - NR AdS/CFT duality: limited technology – a new challenge

- Examples of condensed matter systems
  - Superconductivity
  - Superfluidity
  - Quantum Hall systems
- AdS/CFT involves large N SYM.
  - The AdS is realised by some other fields – CMT is assumed to be a probe on a background geometry
  - Supersymmetry – broken by finite T (?)
  - Phenomenological approach – be maximally optimistic and use as a computational tool
- Questions
  - Phase transition, solitonic excitation (brane excitations)
  - Especially, vortices play important roles in above examples
  - ABJM: M-theory on $AdS_4 \times S^7/\mathbb{Z}_k \Longleftrightarrow$ CFT on M2-branes

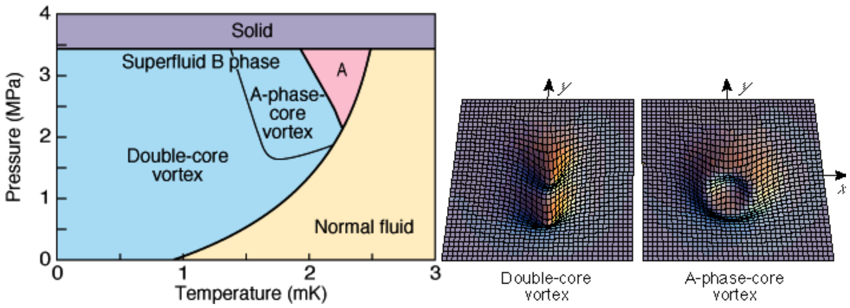
Superfluidity in He



^{4}He phase diagram (left) and ^{3}He phase diagram (right)

Source: Low Temperature Laboratory, TKK, Finland

Vortices in ^{3}He superfluidity



A closer look at the low temperature ^{3}He phase diagram (left) and two types of vortices in the superfluid B phase (right)

Source: Low Temperature Laboratory, TKK, Finland

Our focus: vortex solutions on the CFT side.
Solitonic solutions in ABJM

- Abelian vortices (Jackiw-Lee-Weinberg type) and domain walls [Arai, Montonen, Sasaki 2008]
- Non-abelian vortices [Kim, Kim, Kwon, Nakajima 2009] [Auzzi, Kumar 2009]
- Abelian, non-relativistic vortices [Kawai, Sasaki 2009]

The ABJM model

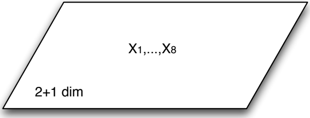
The ABJM model [Aharony, Bergman, Jafferis, Maldacena (2008)]

$$S^{\rm bos}_{\rm ABJM} = \int d^3x \Big(\mathcal{L}^{\rm bos}_{\rm kin} + \mathcal{L}_{\rm CS} - V^{\rm bos}_D - V^{\rm bos}_F \Big),$$

$$\begin{aligned}
\mathcal{L}^{\rm bos}_{\rm kin} &= -\,{\rm Tr}\left[(D_\mu Z^{\hat{A}})^\dagger (D^\mu Z^{\hat{A}}) + (D_\mu W_{\check{A}})^\dagger (D^\mu W_{\check{A}})\right], \\
\mathcal{L}_{\rm CS} &= \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\,{\rm Tr}\left[A_\mu \partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3}\hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda\right], \\
V^{\rm bos}_D &= \frac{4\pi^2}{k^2}\,{\rm Tr}\left[\left|Z^{\hat{B}} Z^\dagger_{\hat{B}} Z^{\hat{A}} - Z^{\hat{A}} Z^\dagger_{\hat{B}} Z^{\hat{B}} - W^{\dagger\check{B}} W_{\check{B}} Z^{\hat{A}} + Z^{\hat{A}} W_{\check{B}} W^{\dagger\check{B}}\right|^2\right. \\
&\qquad \left. + \left|W^{\dagger\check{B}} W_{\check{B}} W^{\dagger\check{A}} - W^{\dagger\check{A}} W_{\check{B}} W^{\dagger\check{B}} - Z^{\hat{B}} Z^\dagger_{\hat{B}} W^{\dagger\check{A}} + W^{\dagger\check{A}} Z^\dagger_{\hat{B}} Z^{\hat{B}}\right|^2\right], \\
V^{\rm bos}_F &= \frac{16\pi^2}{k^2}\,{\rm Tr}\left[\left|\epsilon_{\hat{A}\hat{C}}\epsilon^{\check{B}\check{D}} W_{\check{B}} Z^{\hat{C}} W_{\check{D}}\right|^2 + \left|\epsilon^{\check{A}\check{C}}\epsilon_{\hat{B}\hat{D}} Z^{\hat{B}} W_{\check{C}} Z^{\hat{D}}\right|^2\right].
\end{aligned}$$

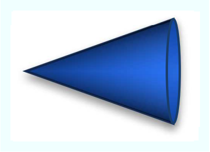
A_μ, $\hat{A}_\mu \cdots U(N)\times U(N)$ gauge fields, and $Z^{\hat{A}}$, $W^{\dagger\check{A}}$ ($\hat{A}=1,2$, $\check{A}=3,4$) $\cdots$ complex scalars in $U(N)\times U(N)$ bi-fundamental $(\mathbf{N}, \bar{\mathbf{N}})$ rep

Chern-Simons-matter theory on 2+1 dimensions



$$(Z^{\hat{A}}, W^{\dagger\check{A}}) \to e^{\frac{2\pi i}{k}}(Z^{\hat{A}}, W^{\dagger\check{A}})$$

M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$
$N, k \to \infty$, $\lambda = \frac{N}{k}$ fixed ('t Hooft limit): IIA on $AdS_4 \times \mathbb{CP}^3$



We consider:

The ABJM model (Chern-Simons-matter theory)

$\downarrow$ massive deformation

Mass-deformed ABJM

$\downarrow$ non-relativistic limit

NR, mass-deformed ABJM

$\downarrow$ solving BPS eqns

BPS vortex solutions

A bigger picture:

ABJM CSm-th $\Longleftrightarrow$ M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$

$\downarrow$ ABJM CSm-th → Massive ABJM → NR Massive ABJM → NR BPS vortices

$\downarrow$ M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ → Fuzzy spheres → NR → gravity dual of NR BPS vortices?

NR BPS vortices $\Leftrightarrow$ gravity dual of NR BPS vortices?

Relativistic mass-deformed ABJM

Before taking the NR limit we take massive deformation

[Hosomichi, Lee^3, Park 2008], [Gomis, Rodríguez-Gómez, Van Raamsdonk, Verlinde 2008]

$\cdots$ introducing a scale that is necessary for the solitonic solutions

Maximally supersymmetric ($\mathcal{N} = 6$) massive deformation

- $SO(8)_R \to SU(2) \times SU(2) \times U(1) \times \mathbb{Z}_2$
- The scalars acquire equal masses

The change of (the bos. part of) the Lanrangian:

$$\delta\mathcal{L} = \mathrm{Tr}\Big[- m^2 Z^\dagger_{\hat{A}} Z^{\hat{A}} - m^2 W^{\dagger\check{A}} W_{\check{A}}$$
$$+ \frac{4\pi m}{k} \big((Z^{\hat{A}} Z^\dagger_{\hat{A}})^2 - (W^{\dagger\check{A}} W_{\check{A}})^2 - (Z^\dagger_{\hat{A}} Z^{\hat{A}})^2 + (W_{\check{A}} W^{\dagger\check{A}})^2 \big) \Big]$$

The non-relativistic limit

Now we consider the NR limit [Nakayama, Sakaguchi, Yoshida 2009], [Lee[3] 2009]

1. Write down the action with c and $\hbar$ explicitly
2. Decompose: $Z^{\hat{A}} = \frac{\hbar}{\sqrt{2m}} \left(e^{-i\frac{mc^2 t}{\hbar}} z^{\hat{A}} + e^{i\frac{mc^2 t}{\hbar}} \hat{z}^{*\hat{A}} \right)$ etc.
   ($z^{\hat{A}}$, $\hat{z}^{*\hat{A}}$ are non-relativistic scalar fields)
   Keep the particle DoF ($z^{\hat{A}}, w^{\dagger\check{A}}$) & drop the antiparticles
3. Send c, $m \longrightarrow \infty$ and look at the leading orders

The resulting (bos. part of the) Lagrangian:

$$\begin{aligned}
\mathcal{L}^{\mathrm{NR,bos}}_{\mathrm{ABJM}} &= \frac{k\hbar c}{4\pi}\epsilon^{\mu\nu\lambda}\mathrm{Tr}\left[A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda - \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right] \\
&+\mathrm{Tr}\left[\frac{i\hbar}{2}\left(-z^\dagger_{\hat{A}} D_t z^{\hat{A}} + D_t z^{\hat{A}}\cdot z^\dagger_{\hat{A}}\right) - \frac{\hbar^2}{2m}D_i z^{\hat{A}} D_i z^\dagger_{\hat{A}}\right. \\
&\quad + \frac{i\hbar}{2}\left(-w_{\check{A}} D_t w^{\dagger\check{A}} + D_t w^{\dagger\check{A}}\cdot w_{\check{A}}\right) - \frac{\hbar^2}{2m}D_i w^{\dagger\check{A}} D_i w_{\check{A}} \\
&\quad \left. + \frac{\pi\hbar^2}{km}\left\{(z^{\hat{A}} z^\dagger_{\hat{A}})^2 - (z^\dagger_{\hat{A}} z^{\hat{A}})^2 - (w^{\dagger\check{A}} w_{\check{A}})^2 + (w_{\check{A}} w^{\dagger\check{A}})^2\right\}\right].
\end{aligned}$$

The equations of motion

- The scalar part: nonlinear Schrödinger equations

$$i\hbar D_t z^{\hat{A}} = -\frac{\hbar^2}{2m} D_i^2 z^{\hat{A}} - \frac{2\pi\hbar^2}{km}(z^{\hat{B}} z^\dagger_{\hat{B}} z^{\hat{A}} - z^{\hat{A}} z^\dagger_{\hat{B}} z^{\hat{B}}),$$
$$i\hbar D_t w^{\dagger\check{A}} = -\frac{\hbar^2}{2m} D_i^2 w^{\dagger\check{A}} + \frac{2\pi\hbar^2}{km}(w^{\dagger\check{B}} w_{\check{B}} w^{\dagger\check{A}} - w^{\dagger\check{A}} w_{\check{B}} w^{\dagger\check{B}}).$$

- The gauge field part: Gauss-law constraints
- The fermionic part:

$$i\hbar D_t \psi_{-A} + 2mc^2 \delta_A^{\hat{A}} \psi_{-\hat{A}} - i\hbar c D_- \psi_{+A} = 0,$$
$$i\hbar D_t \psi_{+A} + 2mc^2 \delta_A^{\check{A}} \psi_{+\check{A}} - i\hbar c D_+ \psi_{-A} = 0.$$

(due to these NR Dirac eqns 1/2 of the fermionic DoF drop)

Non-relativistic SUSY

Apply the same procedure of NR limit to SUSY trfn rules
$\longrightarrow$ non-relativistic SUSY (super Schrödinger symmetry)
14 supercharges:

- 10 kinematical SUSY: $(\tilde{\omega}_{+\hat{A}\hat{B}},\ \tilde{\omega}_{-\check{A}\check{B}},\ \tilde{\omega}_{\pm\hat{A}\check{B}})$
- 2 dynamical SUSY: $(\tilde{\omega}_{-\hat{A}\hat{B}},\ \tilde{\omega}_{+\check{A}\check{B}})$
- 2 conformal SUSY: $(\xi_{\hat{A}\hat{B}},\ \xi_{\check{A}\check{B}})$

SUSY parameters defined by

$\omega_{AB} = \epsilon_i \Gamma^i_{AB}\ (i = 1, 2, \dots, 6),\quad \omega = \frac{1}{\sqrt{2}}\begin{pmatrix} \tilde{\omega}_- + \tilde{\omega}_+ \\ -i\tilde{\omega}_- + i\tilde{\omega}_+ \end{pmatrix},\quad \tilde{\omega}^{AB}_{\pm} = (\tilde{\omega}_{\pm AB})^\dagger = \frac{1}{2}\epsilon^{ABCD}\tilde{\omega}_{\pm CD}$

Conformal SUSY $\sim$ Special conformal charge $\times$ Dynamical SUSY,

$$S = i[K, Q_D]$$

The BPS equations

The Hamiltonian density (Noether charge of the time translation)

$$\mathcal{H} = \mathrm{Tr}\left[\frac{\hbar^2}{2m}|D_i z^{\hat{A}}|^2 + \frac{\hbar^2}{2m}|D_i w^{\dagger\check{A}}|^2 - \frac{\pi\hbar^2}{km}\left\{(z^{\hat{A}}z^\dagger_{\hat{A}})^2 - (z^\dagger_{\hat{A}}z^{\hat{A}})^2 - (w^{\dagger\check{A}}w_{\check{A}})^2 + (w_{\check{A}}w^{\dagger\check{A}})^2\right\}\right].$$

Using $D_\pm \equiv D_1 \pm iD_2$ and Bogomol'nyi completion $\Longrightarrow$ BPS bound:

$$E = \int d^2x\, \mathcal{H} = \int d^2x\, \mathrm{Tr}\left[\frac{\hbar^2}{2m}\left|D_- z^{\hat{A}}\right|^2 + \frac{\hbar^2}{2m}\left|D_+ w^{\dagger\check{A}}\right|^2\right] \geq 0$$

saturated by BPS equations

$$D_- z^{\hat{A}} = 0, \quad D_+ w^{\dagger\check{A}} = 0,$$

Recall: $z^{\hat{A}}$ and $w^{\dagger\check{A}}$ are matrix-valued ($N \times N$)

The fuzzy 3-sphere ansatz [Gomis, Rodríguez-Gómez, Van Raamsdonk, Verlinde 2008]

$$z^{\hat{A}}(x) = \psi_z(x) S^I, \; w^{\dagger \check{A}}(x) = \psi_w(x) S^I, \; A_i(x) = a_i(x) S^I S_I^\dagger, \; \hat{A}_i(x) = a_i(x) S_I^\dagger S^I.$$

Here, ψ_z, ψ_w, $a_i \in \mathbb{C}$, $(S_1^\dagger)_{mn} = \sqrt{m-1}\,\delta_{mn}$, $(S_2^\dagger)_{mn} = \sqrt{N-m}\,\delta_{m+1,n}$

$$\begin{aligned} S^I &= S^J S_J^\dagger S^I - S^I S_J^\dagger S^J, \\ S_I^\dagger &= S_I^\dagger S^J S_J^\dagger - S_J^\dagger S^J S_I^\dagger, \\ \mathrm{Tr}\, S^I S_I^\dagger &= \mathrm{Tr}\, S_I^\dagger S^I = N(N-1). \end{aligned}$$

Then the BPS eqns $\Longrightarrow$ the Jackiw-Pi vortex eqns [Jackiw and Pi 1990]

$$(\mathcal{D}_1 - i\mathcal{D}_2)\psi_z(x) = 0, \qquad (\mathcal{D}_1 + i\mathcal{D}_2)\psi_w(x) = 0 \qquad (\mathcal{D}_i \equiv \partial_i + i a_i)$$

J-P eqns allow exact solutions

The exact vortex solutions

Finding solutions:
Set $w^{\dagger \check{A}} = 0$ & solve for $z^{\hat{A}}$, A_i and $\hat{A}_i$ (BPS I), or
set $z^{\hat{A}} = 0$ & solve for $w^{\dagger \check{A}}$, A_i and $\hat{A}_i$ (BPS II)

The radius of the fuzzy S^3 (in the case of BPS I):

$$R^2 = \frac{2}{N T_{M2}} \mathrm{Tr} \left[Z^{\hat{A}} Z^{\dagger}_{\hat{A}} \right] = \frac{N-1}{T_{M2}} \frac{|\psi_z|^2}{m},$$

T_{M2}: the tension of an M2-brane

The BPS I solutions

1. Changing the variables $\psi_z(x) = e^{i\theta(x)}\rho^{\frac{1}{2}}(x), \quad (\theta, \rho \in \mathbb{R})$, the BPS eqns become

$$a_i(x) = -\partial_i\theta + \frac{1}{2}\epsilon_{ij}\partial^j \ln\rho.$$

2. Using the Gauss law constraint
$\Longrightarrow$ Liouville equation $\nabla^2 \ln\rho = -\frac{4\pi}{k}\rho$,
solved by

$$\rho(x) = \frac{k}{2\pi}\nabla^2 \ln\left(1 + |f(z)|^2\right)$$

($f(z)$: a holomorphic function of $z = x_1 + ix_2$)

3. θ is fixed by regularity of ψ_z at $z = 0$

Examples

Examples of BPS I solutions

Choose a profile: $f(z) = \left(\frac{z_0}{z}\right)^n$, $n \in \mathbb{Z}$, z_0: a complex const; yielding

$$\rho(x) = \frac{k}{2\pi}\,\frac{4n^2}{r_0^2}\,\frac{\left(\frac{r}{r_0}\right)^{2(n-1)}}{\left[1+\left(\frac{r}{r_0}\right)^{2n}\right]^2}, \qquad \theta = -(n-1)\arg z = -(n-1)\arctan(x_2/x_1)$$

These are non-topological vortices since $|\psi_z| \to 0$ as $|z| \to \infty$

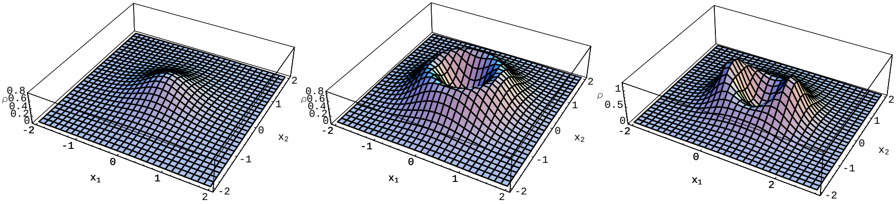


Figure: $|\psi_z|^2$ shown for $f(z) = \frac{1}{z}$, $\frac{1}{z^2}$ and $\frac{1}{z(z-1)}$, with $k = 1$

Examples of BPS II solutions

Setting $z^{\hat{A}} = 0$ we find similar solutions for $w^{\dagger \check{A}}$:

$$\begin{aligned}
\psi_w(x) &= e^{i\theta(x)} \rho^{\frac{1}{2}}(x), \\
\rho(x) &= -\frac{k}{2\pi} \nabla^2 \ln \left(1 + |f(z)|^2\right), \\
\theta &= (n-1) \arctan(x_2/x_1).
\end{aligned}$$

Preserved supersymmetry

Check the SUSY trfns against the BPS eqns (in the BPS-I case)

$$w^{\dagger\check{A}} = 0, \quad D_- z^{\hat{A}} = 0.$$

The fermion transformation rules:

$$\begin{aligned}
\delta_K \psi_{+\hat{A}} &= -\omega_{+\hat{A}\hat{B}} z^{\hat{B}}, & \delta_K \psi_{-\check{A}} &= +\omega_{-\check{A}\hat{B}} z^{\hat{B}}, \\
\delta_D \psi_{+\hat{A}} &= \frac{i}{2m} \omega_{-\hat{A}\hat{B}} D_+ z^{\hat{B}}, & \delta_D \psi_{-\check{A}} &= 0, \\
\delta_S \psi_{+\hat{A}} &\sim \xi_{\hat{A}\hat{B}} z^{\hat{B}}, & \delta_S \psi_{-\check{A}} &\sim 0.
\end{aligned}$$

Hence $\delta\psi = 0 \Longrightarrow \omega_{+\hat{A}\hat{B}} = \omega_{-\hat{A}\hat{B}} = \omega_{-\check{A}\hat{B}} = \xi_{\hat{A}\hat{B}} = 0$
This means that the BPS-I solutions break 5 kinematical, 1 dynamical and 1 conformal SUSYs (i.e. exactly 1/2).

The BPS II case is similar.
Summarising the results,

Type of SUSY	Kinematical				Dynamical		Conformal	
	$\omega_{+\hat{A}\check{B}}$	$\omega_{+\hat{A}\hat{B}}$	$\omega_{-\check{A}\check{B}}$	$\omega_{-\check{A}\hat{B}}$	$\omega_{-\hat{A}\hat{B}}$	$\omega_{+\check{A}\check{B}}$	$\xi_{\hat{A}\hat{B}}$	$\xi_{\check{A}\check{B}}$
BPS I	◯	×	◯	×	×	◯	×	◯
BPS II	×	◯	×	◯	◯	×	◯	×

Table: ◯: preserved, ×: broken

Summary of our solutions

- We find exact solutions of abelian vortices by solving BPS equations in the non-relativistic ABJM model: Jackiw-Pi combined with fuzzy 3-sphere
- These solutions preserve half of the super Schrödinger symmetry
- Any relevance in real physics?
  – more realistic, parity broken models with external fields desirable.

Vortex solutions in AdS

- Vortex line
    - Pure AdS [Dehghani Ghezelbash Mann, 2001]
    - AdS-Sch [Dehghani Ghezelbash Mann, 2001]
- with boundary magnetic field
    - [Albash Johnson 2009]
    - [Montull Pomarol Silva 2009]
    - [Maeda Natuume Okamura 2009]
- with vanishing magnetic field on the boundary

  [Keränen Keski-Vakkuri Nowling Yogendran 2009]

Unsorted list of problems

- Non-relativistic AdS/CFT exists at all?
- In $2+1$ dim, spontaneous breaking of continuous symmetry is not possible at finite temperature (Mermin-Wagner). However, such a phase transition is found in holographic superconductor. How do we interpret? Large N artefact?
- Berezinskii-Kosterlitz-Thouless transition in AdS?