Vortex solutions on membranes

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APCTP YongPyong Winter School, February 2010
Vortex solutions on membranes

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Introduction

What is string theory?
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- The theory of strong interaction
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What is string theory?

- The theory of strong interaction
- The theory of everything
What is string theory?

- The theory of strong interaction
- The theory of everything
- AdS/CFT: Logical completion of QFT (incl. QCD, Condensed Matter Systems...)
(Some) strongly coupled field theories have gravity duals [Maldacena]

- Conformal group $SO(d, 2) \leftrightarrow AdS_{d+1}$
  (Poincaré $M_{\mu\nu}, P_\mu$, Dilatation $D$, SCG $K_\mu$)

Poincaré coordinates are

$$ds^2 = \rho^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{d\rho^2}{\rho^2}$$

Dilatation is $D : x^\mu \mapsto \lambda x^\mu, \rho \mapsto \rho/\lambda$

- Dictionary: [Gubser, Klebanov, Polyakov] [Witten]

$$\langle e^{\int \phi_0 O} \rangle = Z_{AdS}[\phi_0] \sim e^{-S[\phi_0]}$$

Boundary conditions $\leftrightarrow$ states
Variation of boundary conditions $\leftrightarrow$ correlation functions
Radial coordinate ↔ energy scale (‘holographic renormalization’)

A realisation of holographic principle

Canonical example:
$AdS_5 \times S^5$ in Type II B string theory ↔ $\mathcal{N} = 4$ SYM in 4 dim
$SO(4, 2)_{conf} \times SO(6)_R$: isometries of the spacetime
\[ \lambda = g^2_{YM} N: \ 't \ Hooft \ coupling \]

3 versions of AdS/CFT

- weak: large \( \lambda \)

- strong: any finite \( \lambda \) but \( N \to \infty \) and \( g_s = g^2_{YM} \to 0 \)
  (exact in \( \alpha' \) but for small \( g_s \) only)

- strongest: any \( g_s \) and \( N \)
  (\( \alpha' \) and \( g_s \))
Condensed matter applications of AdS/CFT

Application to QCD relatively successful
– why not Condensed Matter Theory?

- AdS/CFT in laboratory
  - Strongly correlated systems near criticality in low dimensions
  - Rich examples
  - Good control

- A new computational tool in CMT
  - Beyond perturbative field theory and lattice

- A new arena of string theory
  - Typically, non-relativistic \( \Delta \sim (q - q_c)^{\nu z}, \xi \sim (q - q_c)^{-\nu} \)
  - \( D : t \mapsto \lambda^z t, x^i \mapsto \lambda x^i \)
  - NR AdS/CFT duality: limited technology – a new challenge
Examples of condensed matter systems

- Superconductivity
- Superfluidity
- Quantum Hall systems

**AdS/CFT involves large N SYM.**

- The AdS is realised by some other fields – CMT is assumed to be a probe on a background geometry
- Supersymmetry – broken by finite T (?)
- Phenomenological approach – be maximally optimistic and use as a computational tool

**Questions**

- Phase transition, solitonic excitation (brane excitations)
- Especially, vortices play important roles in above examples
- ABJM: M-theory on $\text{AdS}_4 \times S^7/\mathbb{Z}_k \leftrightarrow \text{CFT on M2-branes}$
Superfluidity in He

$^4\text{He}$ phase diagram (left) and $^3\text{He}$ phase diagram (right)

Source: Low Temperature Laboratory, TKK, Finland
Vortices in $^3\text{He}$ superfluidity

A closer look at the low temperature $^3\text{He}$ phase diagram (left) and two types of vortices in the superfluid B phase (right)

Source: Low Temperature Laboratory, TKK, Finland
Our focus: **vortex solutions** on the CFT side.

Solitonic solutions in ABJM

- **Abelian vortices** (Jackiw-Lee-Weinberg type) and domain walls
  
  [Arai, Montonen, Sasaki 2008]

- **Non-abelian vortices** [Kim, Kim, Kwon, Nakajima 2009] [Auzzi, Kumar 2009]

- **Abelian, non-relativistic vortices** [Kawai, Sasaki 2009]
The ABJM model

\[ S_{\text{ABJM}}^{\text{bos}} = \int d^3 x \left( \mathcal{L}_{\text{kin}}^{\text{bos}} + \mathcal{L}_{\text{CS}} - V_D^{\text{bos}} - V_F^{\text{bos}} \right), \]

\[ \mathcal{L}_{\text{kin}}^{\text{bos}} = - \text{Tr} \left[ (D_\mu Z^\hat{A})^{\dagger} (D_\mu Z^{\hat{A}}) + (D_\mu W_\hat{A})^{\dagger} (D_\mu W_{\hat{A}}) \right], \]

\[ \mathcal{L}_{\text{CS}} = \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr} \left[ A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right], \]

\[ V_D^{\text{bos}} = \frac{4\pi^2}{k^2} \text{Tr} \left[ \left| Z^\hat{B} Z_B^{\dagger} Z^{\hat{A}} - Z^\hat{A} Z_B^{\dagger} Z^{\hat{B}} - W^{\dagger} W_B Z^\hat{A} + Z^\hat{A} W_B W^{\dagger} \right|^2 \right], \]

\[ V_F^{\text{bos}} = \frac{16\pi^2}{k^2} \text{Tr} \left[ \left| \epsilon_{\hat{A}\hat{C}} \epsilon^{BD} W_B Z^\hat{C} W_D \right|^2 + \left| \epsilon_{\hat{A}\hat{C}} \epsilon_{\hat{B}\hat{D}} Z^\hat{B} Z^\hat{C} W_D \right|^2 \right]. \]

\[ A_\mu, \hat{A}_\mu, \cdots U(N) \times U(N) \text{ gauge fields, and } Z^{\hat{A}}, W^{\dagger} (\hat{A} = 1, 2, \hat{A} = 3, 4) \cdots \text{ complex scalars in } U(N) \times U(N) \text{ bi-fundamental } (\mathbf{N}, \bar{\mathbf{N}}) \text{ rep} \]
Chern-Simons-matter theory on 2+1 dimensions

\[
(Z^A, W^\dagger A) \rightarrow e^{\frac{2\pi i}{k}} (Z^A, W^\dagger A)
\]

M-theory on \( AdS_4 \times S^7 / \mathbb{Z}_k \)

\( N, k \rightarrow \infty, \lambda = \frac{N}{k} \) fixed (‘t Hooft limit): IIA on \( AdS_4 \times \mathbb{CP}^3 \)
We consider:
The ABJM model (Chern-Simons-matter theory)
  ↓ massive deformation
Mass-deformed ABJM
  ↓ non-relativistic limit
NR, mass-deformed ABJM
  ↓ solving BPS eqns
BPS vortex solutions
A bigger picture:

ABJM CSm-th $\leftrightarrow$ M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$

Massive ABJM $\rightarrow$ Fuzzy spheres

NR Massive ABJM $\rightarrow$ NR

NR BPS vortices $\leftrightarrow$ gravity dual of NR BPS vortices?
Before taking the NR limit we take massive deformation

[Hosomichi, Lee³, Park 2008], [Gomis, Rodríguez-Gómez, Van Raamsdonk, Verlinde 2008]

... introducing a scale that is necessary for the solitonic solutions

Maximally supersymmetric ($\mathcal{N} = 6$) massive deformation

- $SO(8)_R \rightarrow SU(2) \times SU(2) \times U(1) \times \mathbb{Z}_2$
- The scalars acquire equal masses

The change of (the bos. part of) the Lanrangian:

$$
\delta \mathcal{L} = \text{Tr} \left[ -m^2 Z_{A}^{\dagger} \hat{Z}^\Lambda - m^2 W^{\dagger \Lambda} \hat{W}_{\Lambda} \\
+ \frac{4\pi m}{k} \left( (Z^\Lambda Z_{A}^{\dagger})^2 - (W^{\dagger \Lambda} W_{A})^2 - (Z_{A}^{\dagger} Z^\Lambda)^2 + (W_{A} W^{\dagger \Lambda})^2 \right) \right]$

Relativistic mass-deformed ABJM
The non-relativistic limit

Now we consider the NR limit [Nakayama, Sakaguchi, Yoshida 2009], [Lee 2009]

1. Write down the action with $c$ and $\hbar$ explicitly
2. Decompose: $Z^A = \frac{\hbar}{\sqrt{2m}} \left( e^{-i \frac{mc^2}{\hbar} t} z^A + e^{i \frac{mc^2}{\hbar} t} \hat{z}^A \right)$ etc.
   
   ($z^A, \hat{z}^A$ are non-relativistic scalar fields)

Keep the particle DoF ($z^A, w^\dagger_A$) & drop the antiparticles

3. Send $c, m \rightarrow \infty$ and look at the leading orders

The resulting (bos. part of the) Lagrangian:

$$\mathcal{L}^{NR, \text{bos}}_{\text{ABJM}} = \frac{k \hbar c}{4\pi} \varepsilon^{\mu\nu\lambda} \text{Tr} \left[ A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right]$$

$$+ \text{Tr} \left[ \frac{i\hbar}{2} \left( -z^A \partial_t z^A + D_t z^A \cdot z^A \right) - \frac{\hbar^2}{2m} D_i z^A \partial_i z^A \right]$$

$$+ \frac{i\hbar}{2} \left( -w_A \partial_t w_A^\dagger + D_t w_A^\dagger \cdot w_A \right) - \frac{\hbar^2}{2m} D_i w_A^\dagger \partial_i w_A$$

$$+ \frac{\pi \hbar^2}{km} \left[ (z^A z^A) - (z^A z^A)^2 - (w_A w_A^\dagger)^2 + (w_A w_A^\dagger)^2 \right].$$
The equations of motion

- The scalar part: nonlinear Schrödinger equations

\[
\begin{align*}
\imath \hbar D_t z^\hat{A} &= -\frac{\hbar^2}{2m} D_i^2 z^\hat{A} - \frac{2\pi \hbar^2}{km} (z^\hat{B} z^{\hat{A} \dagger}_B z^\hat{A} - z^\hat{A} z^{\hat{A} \dagger}_B z^\hat{B}), \\
\imath \hbar D_t w^{\dagger \hat{A}} &= -\frac{\hbar^2}{2m} D_i^2 w^{\dagger \hat{A}} + \frac{2\pi \hbar^2}{km} (w^{\dagger \hat{B}} w^{\hat{B}} w^{\dagger \hat{A}} - w^{\dagger \hat{A}} w^{\hat{B}} w^{\dagger \hat{B}}).
\end{align*}
\]

- The gauge field part: Gauss-law constraints

- The fermionic part:

\[
\begin{align*}
\imath \hbar D_t \psi^\dagger_\hat{A} - 2mc^2 \delta^\hat{A}_A \psi^\dagger_\hat{A} - \imath \hbar c D^- \psi^\dagger_+ = 0, \\
\imath \hbar D_t \psi^\dagger_+ - 2mc^2 \delta^\hat{A}_A \psi^\dagger_- \psi^\dagger_+ = 0.
\end{align*}
\]

(due to these NR Dirac eqns 1/2 of the fermionic DoF drop)
Vortex solutions on membranes

Non-relativistic mass-deformed ABJM

Non-relativistic SUSY

Non-relativistic SUSY

Apply the same procedure of NR limit to SUSY trfn rules
→ non-relativistic SUSY (super Schrödinger symmetry)
14 supercharges:

- 10 kinematical SUSY: \( (\tilde{\omega}_{+A\dot{B}}, \tilde{\omega}_{-A\dot{B}}, \tilde{\omega}_{\pm A\dot{B}}) \)
- 2 dynamical SUSY: \( (\tilde{\omega}_{-A\dot{B}}, \tilde{\omega}_{+A\dot{B}}) \)
- 2 conformal SUSY: \( (\xi_{A\dot{B}}, \xi_{\dot{A}\dot{B}}) \)

SUSY parameters defined by

\[
\omega_{AB} = \epsilon_i \Gamma_{AB}^i \quad (i = 1, 2, \ldots, 6), \quad \omega = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \tilde{\omega}_- + \tilde{\omega}_+ \\ -i\tilde{\omega}_- + i\tilde{\omega}_+ \end{array} \right), \quad \tilde{\omega}_{\pm} = (\tilde{\omega}_{\pm AB})^\dagger = \frac{1}{2} \epsilon^{ABCD} \tilde{\omega}_{\pm CD}
\]

Conformal SUSY \sim Special\ conformal\ charge \times\ Dynamical\ SUSY,

\[ S = i[K, Q_D] \]
The BPS equations

The Hamiltonian density (Noether charge of the time translation)

\[ \mathcal{H} = \text{Tr} \left[ \frac{\hbar^2}{2m} \left| D_i z^\hat{A} \right|^2 + \frac{\hbar^2}{2m} \left| D_i w^{\dagger \hat{A}} \right|^2 - \frac{\pi \hbar^2}{km} \left\{ (z^\hat{A} z^{\dagger \hat{A}})^2 - (z^{\dagger \hat{A}} z^\hat{A})^2 - (w^{\dagger \hat{A}} w_{\hat{A}})^2 + (w_{\hat{A}} w^{\dagger \hat{A}})^2 \right\} \right]. \]

Using \( D_\pm \equiv D_1 \pm iD_2 \) and Bogomol’nyi completion \( \implies \) BPS bound:

\[ E = \int d^2x \mathcal{H} = \int d^2x \text{Tr} \left[ \frac{\hbar^2}{2m} \left| D_- z^\hat{A} \right|^2 + \frac{\hbar^2}{2m} \left| D_+ w^{\dagger \hat{A}} \right|^2 \right] \geq 0 \]

saturated by BPS equations

\[ D_- z^\hat{A} = 0, \quad D_+ w^{\dagger \hat{A}} = 0, \]

Recall: \( z^\hat{A} \) and \( w^{\dagger \hat{A}} \) are matrix-valued \((N \times N)\)
The fuzzy 3-sphere ansatz [Gomis, Rodríguez-Gómez, Van Raamsdonk, Verlinde 2008]

\[ z^A(x) = \psi_z(x)S^I, \quad w^\dagger \tilde{A}(x) = \psi_w(x)S^I, \quad A_i(x) = a_i(x)S^I S_i^\dagger, \quad \hat{A}_i(x) = a_i(x)S_i^\dagger S^I. \]

Here, \( \psi_z, \psi_w, a_i \in \mathbb{C} \), \( (S_1^\dagger)_{mn} = \sqrt{m - 1}\delta_{mn} \), \( (S_2^\dagger)_{mn} = \sqrt{N - m}\delta_{m+1,n} \)

\[ s^I = s^I s_j^\dagger s^I - s^I s_j^\dagger s^J, \]
\[ s_i^\dagger = s_i^\dagger s^J s_j^\dagger - s_i^\dagger s^J s_i^\dagger, \]
\[ \text{Tr} \ s^I s_i^\dagger = \text{Tr} \ s_i^\dagger s^I = N(N - 1). \]

Then the BPS eqns \( \implies \) the Jackiw-Pi vortex eqns [Jackiw and Pi 1990]

\[ (\mathcal{D}_1 - i\mathcal{D}_2)\psi_z(x) = 0, \quad (\mathcal{D}_1 + i\mathcal{D}_2)\psi_w(x) = 0 \quad (\mathcal{D}_i \equiv \partial_i + ia_i) \]

J-P eqns allow exact solutions
Finding solutions:
Set $w^{\dagger \hat{A}} = 0$ & solve for $z^{\hat{A}}$, $A_i$ and $\hat{A}_i$ (BPS I), or set $z^{\hat{A}} = 0$ & solve for $w^{\dagger \hat{A}}$, $A_i$ and $\hat{A}_i$ (BPS II)

The radius of the fuzzy $S^3$ (in the case of BPS I):

$$R^2 = \frac{2}{NT_{M2}} \text{Tr} \left[ Z^{\hat{A}} Z^{\dagger \hat{A}} \right] = \frac{N - 1}{T_{M2}} \frac{|\psi_z|^2}{m},$$

$T_{M2}$: the tension of an M2-brane
The BPS I solutions

1. Changing the variables $\psi_z(x) = e^{i\theta(x)} \rho_{\frac{1}{2}}(x)$, $(\theta, \rho \in \mathbb{R})$, the BPS eqns become

$$a_i(x) = -\partial_i \theta + \frac{1}{2} \epsilon_{ij} \partial^j \ln \rho.$$ 

2. Using the Gauss law constraint

$\Rightarrow$ Liouville equation $\nabla^2 \ln \rho = -\frac{4\pi}{k} \rho$, solved by

$$\rho(x) = \frac{k}{2\pi} \nabla^2 \ln (1 + |f(z)|^2)$$

($f(z)$: a holomorphic function of $z = x_1 + ix_2$)

3. $\theta$ is fixed by regularity of $\psi_z$ at $z = 0$
Examples of BPS I solutions
Choose a profile: \( f(z) = \left( \frac{z_0}{z} \right)^n, \ n \in \mathbb{Z}, \ z_0: \text{a complex const} \); yielding

\[
\rho(x) = \frac{k}{2\pi} \frac{4n^2}{r_0^2} \frac{\left( \frac{r}{r_0} \right)^{2(n-1)}}{1 + \left( \frac{r}{r_0} \right)^{2n}}^2, \quad \theta = -(n-1) \arg z = -(n-1) \arctan(x_2/x_1)
\]

These are non-topological vortices since \(|\psi_z| \rightarrow 0\) as \(|z| \rightarrow \infty\)

Figure: \( |\psi_z|^2 \) shown for \( f(z) = \frac{1}{z}, \frac{1}{z^2} \) and \( \frac{1}{z(z-1)} \), with \( k = 1 \)
Examples of BPS II solutions

Setting $z^\hat{A} = 0$ we find similar solutions for $w^\dagger\hat{A}$:

$$
\psi_w(x) = e^{i\theta(x)} \rho^\frac{1}{2}(x),
$$

$$
\rho(x) = -\frac{k}{2\pi} \nabla^2 \ln (1 + |f(z)|^2),
$$

$$
\theta = (n - 1) \arctan(x_2/x_1).
$$
Check the SUSY trfns against the BPS eqns (in the BPS-I case)

\[ w^\dagger \hat{A} = 0, \quad D_- z^{\hat{A}} = 0. \]

The fermion transformation rules:

\[
\begin{align*}
\delta_K \psi_{+\hat{A}} &= -\omega_{+\hat{A}\hat{B}} z^{\hat{B}}, \\
\delta_D \psi_{+\hat{A}} &= \frac{i}{2m} \omega_{-\hat{A}\hat{B}} D_+ z^{\hat{B}}, \\
\delta_S \psi_{+\hat{A}} &\sim \xi_{\hat{A}\hat{B}} z^{\hat{B}}, \\
\delta_K \psi_{-\hat{A}} &= +\omega_{-\hat{A}\hat{B}} z^{\hat{B}}, \\
\delta_D \psi_{-\hat{A}} &= 0, \\
\delta_S \psi_{-\hat{A}} &\sim 0.
\end{align*}
\]

Hence \( \delta \psi = 0 \implies \omega_{+\hat{A}\hat{B}} = \omega_{-\hat{A}\hat{B}} = \omega_{-\hat{A}\hat{B}} = \xi_{\hat{A}\hat{B}} = 0 \)

This means that the BPS-I solutions break 5 kinematical, 1 dynamical and 1 conformal SUSYs (i.e. exactly 1/2).
The BPS II case is similar.
Summarising the results,

<table>
<thead>
<tr>
<th>Type of SUSY</th>
<th>Kinematical</th>
<th>Dynamical</th>
<th>Conformal</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPS I</td>
<td>$\omega_{+AB}$</td>
<td>$\omega_{+\dot{A}\dot{B}}$</td>
<td>$\omega_{-\dot{A}\dot{B}}$</td>
</tr>
<tr>
<td>BPS II</td>
<td>$\times$</td>
<td>$\circ$</td>
<td>$\times$</td>
</tr>
</tbody>
</table>

Table: $\circ$: preserved, $\times$: broken
Summary of our solutions

- We find exact solutions of abelian vortices by solving BPS equations in the non-relativistic ABJM model: Jackiw-Pi combined with fuzzy 3-sphere.
- These solutions preserve half of the super Schrödinger symmetry.
- Any relevance in real physics? – more realistic, parity broken models with external fields desirable.
Vortex solutions in AdS

- Vortex line
  - Pure AdS [Dehghani Ghezelbash Mann, 2001]
  - AdS-Sch [Dehghani Ghezelbash Mann, 2001]

- with boundary magnetic field
  - [Albash Johnson 2009]
  - [Montull Pomarol Silva 2009]
  - [Maeda Natuume Okamura 2009]

- with vanishing magnetic field on the boundary
  - [Keränen Keski-Vakkuri Nowling Yogendran 2009]
Unsorted list of problems

- Non-relativistic AdS/CFT exists at all?
- In 2 + 1 dim, spontaneous breaking of continuous symmetry is not possible at finite temperature (Mermin-Wagner). However, such a phase transition is found in holographic superconductor. How do we interpret? Large N artefact?
- Berezinskii-Kosterlitz-Thouless transition in AdS?