Spherical mass distribution

In the textbook it is stated that *the gravitational force exerted by a finite-size, spherically symmetric mass distribution on a particle outside the distribution is the same as if the entire mass of the distribution were concentrated at the center.* It is not difficult to prove this and I will explain below. It is enough to consider the **gravitational potential**, defined as $V = U/m$ where U is the **gravitational potential energy** and m is the test mass. The forces, $F = -\nabla U = -m\nabla V$, in the case of the finite-size spherical distribution and in the case of the point particle source, are then shown to be equal.

Let us proceed step by step, considering a hoop, a disk and a solid sphere, with uniform mass distribution. We use the fact that the gravitational potential energy for a point source M and a test particle m, separated by distance r, is given by

$$U = -\frac{GMm}{r}. \tag{1}$$

The gravitational potential created by the point source is then

$$V = -\frac{Gm}{r}. \tag{2}$$

1 A hoop

Consider a hoop of radius a and mass M, on the x-y plane $(x^2 + y^2 = a^2, z = 0)$. The gravitational potential at $P : (0, 0, \ell)$ is given by

$$V = -\frac{GM}{\sqrt{a^2 + \ell^2}}, \tag{3}$$

since the distance from P to each point on the hoop is $\sqrt{a^2 + \ell^2}$. Note that the potential energy and hence the gravitational potential sum up.

2 A disk

Now consider the gravitational potential at point $P : (0, 0, \ell)$ created by a disk of radius r on the x-y plane $(x^2 + y^2 \le r^2, z = 0)$. Letting the surface mass density σ, the mass of the disk is $\pi r^2 \sigma$. The potential is obtained by integrating the above formula for the hoop, over the radius a varying from 0 to r. Note also that the hoop mass is now replaced by $M \to 2\pi a\sigma da$. Then we find,

$$\begin{aligned} V &= -G \int_{a=0}^{r} \frac{2\pi a\sigma da}{\sqrt{a^2 + \ell^2}} \\ &= -\pi G\sigma \int_{a^2=0}^{r^2} \frac{da^2}{\sqrt{a^2 + \ell^2}} \\ &= -2\pi G\sigma(\sqrt{r^2 + \ell^2} - \ell). \end{aligned} \tag{4}$$

3 A solid sphere

Finally, consider a solid sphere (a ball stuffed inside) of radius R, centered at the origin $(0, 0, 0)$, with mass $M = \frac{4}{3}\pi R^3 \rho$, where ρ is the mass density. This is represented in the cartesian coordinates as $x^2 + y^2 + z^2 \leq R^2$. The point $P : (0, 0, L)$, where $L > R$, is separated from a disk (a slice of the sphere) on the $z =$ const. plane by $\xi = L - z$. The gravitational potential at P is obtained by integrating the formula (4), with replacements $\ell \to \xi = L - z$, $\sigma \to \rho dz$, and $r^2 \to R^2 - z^2$. Then,

$$\begin{aligned}
V &= -2\pi G\rho \int_{-R}^{R} dz \left(\sqrt{R^2 - z^2 + (L - z)^2} - (L - z) \right) \\
&= -2\pi G\rho \left[-\frac{1}{2L}\frac{2}{3}(R^2 + L^2 - 2Lz)^{3/2} - Lz + \frac{1}{2}z^2 \right]_{z=-R}^{R} \\
&= -2\pi G\rho \left\{ \frac{1}{3L}(L + R)^3 - \frac{1}{3L}(L - R)^3 - 2LR \right\} \\
&= -\frac{4\pi R^3 G\rho}{3L} \\
&= -G\frac{M}{L}.
\end{aligned} \tag{5}$$

This is the same potential as for the point source mass M at $(0, 0, 0)$.

4 Arbitrary spherically symmetric distribution

Since the potential is additive, we may deduce from the solid sphere case above that the same statement holds for a hollow sphere (by subtracting an inner solid sphere). As any spherically symmetric distribution can be constructed by combining concentric spherical shells, we have shown that the statement holds for arbitrary spherically symmetric mass distribution.