Kepler's laws

The explanation on Kepler's laws in the textbook is somewhat insufficient. I will show how these laws arise from the equation of motion.

1 Ellipses

We consider an ellipse of semimajor axis a, semiminor axis b, half-distance between the focuses c. Obviously, $a^2 = b^2 + c^2$. The eccentricity is $e = c/a$. If we take one of the focuses as the origin, the equation of the ellipse in the cartesian coordinates can be written as

$$\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

In polar coordinates, the ellipse is

$$r = \frac{\ell}{1 + e\cos\theta}, \tag{2}$$

where $\ell = b^2/a$. These are related, as usual, by $x = r\cos\theta$ and $y = r\sin\theta$.

2 The equation of motion

Let the mass of the heavier star ('the sun') M and that of the lighter star ('the earth') m. The gravitational force exerted on the earth is

$$\vec{F} = -G\frac{Mm}{r^2}\vec{e}_r, \tag{3}$$

where

$$\vec{e}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{4}$$

is a unit vector on the earth, pointing in the opposite direction to the sun. Also defining an orthogonal unit vector

$$\vec{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}, \tag{5}$$

it is easy to show that the differentials of these unit vectors are

$$d\vec{e}_r = \vec{e}_\theta d\theta, \qquad d\vec{e}_\theta = -\vec{e}_r d\theta. \tag{6}$$

As the position of the earth is

$$\begin{pmatrix} x \\ y \end{pmatrix} = r\vec{e}_r, \tag{7}$$

the second law of Newton, with the force (3), is

$$-G\frac{Mm}{r^2}\vec{e}_r = m\frac{d^2}{dt^2}(r\vec{e}_r). \tag{8}$$

Using (6), we have (the overdot denoting the time derivative $\frac{d}{dt}$)

$$\frac{d^2}{dt^2} r\vec{e}_r = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta. \tag{9}$$

Then the equation of motion (8) becomes two equations,

$$-G\frac{M}{r^2} = \ddot{r} - r\dot{\theta}^2, \tag{10}$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0. \tag{11}$$

3 Kepler's 2nd law

Kepler's 2nd law states that the radius vector of a planet ($r\vec{e}_r$) sweeps equal area in unit time. The equation (11) can be written as $\frac{d}{dt}(r^2\dot{\theta}) = 0$, meaning that

$$A = r^2\dot{\theta}, \tag{12}$$

is a constant. Since $A/2 = \frac{1}{2}r^2\dot{\theta}$ is the area velocity, namely the area swept by the radius vector of the planet in unit time, we have shown the 2nd law of Kepler.

4 Kepler's 1st law

The nasty-looking differential equation (10) can actually be solved easily if you introduce a new variable $q \equiv 1/r$. Using (12),

$$\frac{dq}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{A}\dot{r}, \tag{13}$$

and then

$$\frac{d^2q}{d\theta^2} = -\frac{1}{A}\frac{d\dot{r}}{d\theta} = -\frac{1}{A}\ddot{r}\frac{dt}{d\theta} = -\frac{r^2}{A^2}\ddot{r}. \tag{14}$$

Using this equation and (12), we can rewrite (10) as

$$\frac{d^2q}{d\theta^2} + q - \frac{GM}{A^2} = 0. \tag{15}$$

This is a 2nd order differential equation for a harmonic oscillator. The solution is

$$q - \frac{GM}{A^2} = B\cos(\theta + \phi), \tag{16}$$

where B is the amplitude of the oscillator and ϕ is a phase (B and ϕ are the integration constants). Choosing $\phi = 0$ by shifting the origin of time t, we find

$$r = \frac{1}{q} = \frac{\frac{A^2}{GM}}{1 + \frac{A^2B}{GM}\cos\theta}. \tag{17}$$

This is an ellipse in the polar coordinates (2), with

$$\ell = \frac{b^2}{a} = \frac{A^2}{GM}, \qquad e = \frac{c}{a} = \frac{A^2B}{GM}. \tag{18}$$

This shows Kepler's 1st law.

5 Kepler's 3rd law

The period of the orbital motion T, is equal to the area of the ellipse divided by the area swept by the radius vector in unit time, i.e.

$$T = \frac{\pi ab}{A/2}. \tag{19}$$

Using the first equation of (18), we have

$$T^2 = \frac{4\pi^2 a^2 b^2}{A^2} = \frac{4\pi^2 a^3}{GM} \propto a^3. \tag{20}$$

This shows Kepler's 3rd law.

6 The mechanical energy of a planet

The mechanical energy (the sum of the kinetic and potential energy) of the planet can be found as follows. The position of the planet is $\vec{r} = r\vec{e}_r$ and hence the velocity is

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta, \tag{21}$$

where (6) has been used. Noticing that $\vec{e}_r$ and $\vec{e}_\theta$ are unit vectors which are orthogonal to each other, we have $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$. Thus the sum of the kinetic and the potential energies is

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r}. \tag{22}$$

This expression becomes simple when the planet is at the perihelion ($r = a - c$) or at the aphelion ($r = a + c$), because there $\dot{r} = 0$. Let us consider the perihelion case below. Using $A = r^2\dot{\theta}$ (12) and $A^2 = b^2GM/a$ (18), we find

$$E = \frac{mA^2}{2r^2} - \frac{GMm}{r} = \frac{GMm}{2r^2 a}(b^2 - 2ra). \tag{23}$$

Now using the geometry of the ellipse,

$$\begin{aligned} b^2 - 2ra &= a^2 - c^2 - 2(a-c)a \\ &= -(a-c)^2 \\ &= -r^2. \end{aligned} \tag{24}$$

Hence

$$E = -\frac{GMm}{2a}. \tag{25}$$