Computing moments of inertia

The moment of inertia of a rigid continuous object is given by

\[ I = \int r^2 \, dm. \]

The formulas for various homogeneous rigid objects are listed in Table 10.2 of the textbook. These are,

1. Hoop (or thin cylindrical shell) of radius \( R \)
   \[ I_{CM} = MR^2 \] (1)

2. Hollow cylinder of inner radius \( R_1 \) and outer radius \( R_2 \)
   \[ I_{CM} = \frac{1}{2}M(R_1^2 + R_2^2) \] (2)

3. Solid cylinder (or disk) of radius \( R \)
   \[ I_{CM} = \frac{1}{2}MR^2 \] (3)

4. Rectangular plate
   \[ I_{CM} = \frac{1}{12}M(a^2 + b^2) \] (4)

5. Rod of length \( L \), around its centre
   \[ I_{CM} = \frac{1}{12}ML^2 \] (5)

6. Rod of length \( L \), around one of its ends
   \[ I_{end} = \frac{1}{3}ML^2 \] (6)

7. Solid sphere of radius \( R \)
   \[ I_{CM} = \frac{2}{5}MR^2 \] (7)

8. Thin spherical shell of radius \( R \)
   \[ I_{CM} = \frac{2}{3}MR^2 \] (8)

In all cases, \( M \) is the mass of the object. In the textbook, 1, 2, 3 are derived in Example 10.5, and 5, 6 are derived in Example 10.4 and 10.6. Let us derive the formulae for the remaining cases below.
1 Rectangular plate

The moment of inertia for the rectangular plate of sides $a$ and $b$ can be found by using the formula (5) and the parallel axis theorem. The moment of inertia of a rod of mass $M$ and length $L$, with axis separated by distance $x$ from the original one (through the centre of mass), is

$$I_x = I_{CM} + Mx^2 = \frac{1}{12}ML^2 + Mx^2. \quad (9)$$

Now replacing $L \rightarrow a$, $M \rightarrow dm = \sigma adx$, and integrating over $x$ from $-b/2$ to $b/2$, one obtains

$$I = \int_{-b/2}^{b/2} \left( \frac{1}{12}a^3 \sigma + ax^2 \sigma \right) dx$$
$$= \frac{1}{12} \sigma (a^3 b + ab^3)$$
$$= \frac{1}{12} M(a^2 + b^2), \quad (10)$$

where $M = \sigma ab$ has been used.

2 Thin spherical shell

Consider a thin spherical shell of radius $R$ and mass $M$. We take spherical coordinates with azimuthal angle $\theta$ and zenith angle $\phi$ (see for example http://mathworld.wolfram.com/SphericalCoordinates.html). On the spherical shell the mass element is

$$dm = \sigma R \sin \theta d\phi R d\phi,$$  

(11)

where $\sigma = M/4\pi R^2$ is the surface mass density, and the distance from the rotational axis is $r = R \sin \phi$. Hence the moment of inertia to be calculated is

$$I = \int r^2 dm = 2\pi \sigma R^4 \int_0^\pi \sin^3 \phi d\phi. \quad (12)$$

Noting that

$$\int_0^\pi \sin^3 \phi d\phi = \int_0^\pi \sin \phi (1 - \cos^2 \phi) d\phi$$
$$= \int_0^\pi \sin \phi d\phi - \int_{-1}^{1} u^2 du$$
$$= \left[ - \cos \phi \right]_0^\pi - \left[ \frac{1}{3} u^3 \right]_{-1}^1 = \frac{4}{3} \quad (13)$$

(the variable has been changed as $u = \cos \phi$ and $du = d\cos \phi = -\sin \phi d\phi$), we now find

$$I = \frac{2}{3} MR^2. \quad (14)$$
3 Solid sphere

The moment of inertia for a solid sphere of radius $R$ and mass $M$ can be obtained by integrating the result for the disk (3) over changing distance from the axis. Choosing the $z$-axis as the axis of rotation and letting the distance from it to the mass element on the shell as $r$, we have

$$r^2 = R^2 - z^2. \quad (15)$$

Now $M \to dm = \pi r^2 \rho dz$ and $R^2 \to r^2$ in (3), we have

$$I = \int_{z=-R}^{R} \frac{1}{2} \pi r^2 \rho \cdot r^2 dz$$

$$= \frac{1}{2} \pi \rho \int_{z=-R}^{R} (R^4 - 2R^2 z^2 + z^4) dz$$

$$= \frac{2}{5} MR^2, \quad (16)$$

where the mass of the sphere is

$$M = \frac{4}{3} \pi R^3 \rho. \quad (17)$$

Alternatively, we may integrate the result for the spherical shell above, over the shell radius $r$. Rewriting $M \to dm = 4\pi r^2 \rho dr$, $R \to r$ and integrating from $r = 0$ to $r = R$, we find

$$I = \frac{2}{3} \cdot 4\pi \rho \int_{0}^{R} r^4 dr$$

$$= \frac{8\pi}{15} \rho R^5$$

$$= \frac{2}{5} MR^2. \quad (18)$$