Boundary Conformal Field Theory in Free-Field Representation

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"Nous sommes condamnés à être libres."
Jean-Paul Sartre

Abstract

This thesis presents a study on a formal aspect of two-dimensional boundary conformal field theory. We focus on a specific approach for finding boundary states and investigate several unexplored models. This method was originally introduced by Cardy to identify physical boundary conditions of conformal field theories by a purely mathematical manner. We exploit the fact that the basis of boundary states may be constructed by Fock space representations and try to reformulate the method from a Lagrangian point of view. We consider two systems in particular. One is the so-called Coulomb-gas system, developed by Dotsenko and Fateev, and the other is the symplectic fermion worked out by Kausch. The Coulomb-gas systems provide a powerful tool to calculate correlation functions and it is also advantageous because of its wide applicability. We develop a formalism to describe boundary states of Coulomb-gas and show that it reproduces conventional results in A-series Virasoro minimal models. The symplectic fermion is used to describe a model of logarithmic conformal theory, called the triplet model at $c = -2$. We investigate the boundary states of this model using free-fields and elucidate several novel features. In particular, we show the existence of boundary states with consistent modular properties, which has not been known for this class of theories.

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Publications

Chapter 1 of this thesis collects background information and motivation of this work. Chapters 2 and 3 contain original work which has been published:

Shinsuke Kawai,
Coulomb-gas Approach for Boundary Conformal Field Theory,
Nucl.Phys.B630:203-221,2002 [hep-th/0201146],

Shinsuke Kawai, John F. Wheater
Modular Transformation and Boundary States in Logarithmic Conformal Field Theory,
Phys.Lett.B508:203-210,2001 [hep-th/0103197].

Some part of the thesis has been presented as conference proceedings,

Shinsuke Kawai,
Logarithmic Conformal Field Theory with Boundary,
Lectures given at School and Workshop on Logarithmic Conformal Field Theory,
Tehran, Iran, 4-18 Sep 2001 [hep-th/0204169].

Contents

Chapter 1

Introduction

Before discussing the main subject of this thesis, namely, free-field representations of boundary conformal field theory (BCFT) in $c < 1$ Virasoro minimal models and the $c = -2$ triplet model, we review in this chapter some basic issues in conformal field theory (CFT) with and without boundary. Regarding the vastness of the subject, this chapter is by no means intended to give an overview of the whole development of the theory over the past decades. Rather, we collect material needed for the following chapters and establish notation. We start in the next section by discussing how boundary conformal field theory is used in string theory and statistical physics. The second section deals with the geometry of our work space, i.e. the two-dimensional real manifolds. In Sec.1.3 we review basic elements of CFT, where modular invariance and the Coulomb-gas formalism are treated in detail. Finally in Sec.1.4, we review basic ideas and techniques of boundary conformal field theory. Here and throughout this thesis only CFTs in two-dimensions are considered.

1 Boundary conformal field theories in string theory and statistical physics

Conformal field theory finds its physical applications in string theory and in the study of critical phenomena of statistical systems. As these applications motivate mathematical studies of CFT and they also facilitate intuitive understanding of what is happening, let us start by describing examples where BCFT is employed.

String theory was originated in the study of quark confinement and is being studied as the most promising candidate for the unified theory of all fundamental interactions including gravity. It is a quantum theory of relativistic one-dimensional objects (strings) propagating in a D-dimensional space-time. Strings are described by a field theory on the two-dimensional surface swept by the strings (world sheet), and the equivalence of different parametrisations for the same embedding amounts to the conformal invariance of the field theory. Conformal invariance at quantum level is ensured by the cancellation of the conformal anomaly, which gives the critical dimensions $D = 26$ for bosonic strings and $D = 10$ for superstrings. The anomaly cancellation also leads to the vanishing of the renormalisation group β-functions, which gives the generalised Einstein equation in the lowest order perturbation in the Regge slope parameter α'.

A boundary appears in the string theory as the end point of an open string. Apart from periodic boundary condition which leads to closed strings, the Neumann condition is the only possible boundary condition which is consistent with D-dimensional Poincaré invariance and string equations of motion. The object satisfying this boundary condition is the open string propagating freely at the speed of light. The Dirichlet boundary condition may also be imposed if one relaxes the condition for Poincaré invariance. The end points of open strings are then fixed to higher dimensional objects called D-branes, which are extremely important for the study of non-perturbative aspects of string theory. The discovery of D-branes has drastically changed the landscape of string theory. For example, the duality web relating the different perturbative string theories and the unified picture of M-theory are all fruits of this observation.

The conformal symmetry of statistical systems at criticality is attributed to the divergence of the correlation length at a second-order critical point. The absence of a characteristic length results in power-law scaling of correlation functions, and hence finding the power (scaling dimension) is one of the main objectives in the study of such systems. Solvability of two-dimensional statistical systems is closely related to conformal invariance. A landmark work in this field is by Belavin, Polyakov and Zamolozchikov [1] in 1984, where a particularly important class of CFTs called minimal models were studied, and it was shown that for such models n-point correlation functions can be found analytically.

When a critical system has a boundary, which is always the case for realistic situations since any sample of material has a finite size, the scaling laws near the boundary generally

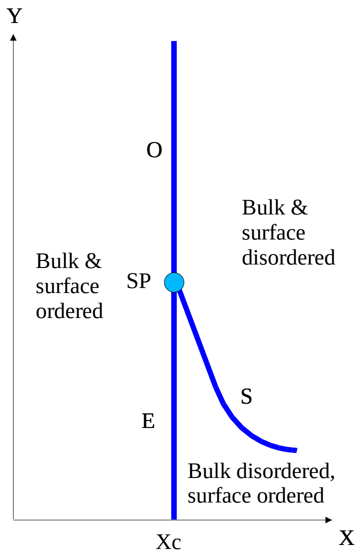


Figure 1.1: Schematic surface phase diagram of statistical systems with boundary. The symbols O, E, SP and S in the figure stand for the ordinary, extraordinary, special and surface transitions, respectively.

differ from the bulk. The purpose of BCFT is then to describe the system correctly in the presence of the boundary, in particular to find scaling laws and correlation functions. The phase diagram near criticality is schematically depicted in Fig.1.1 [2–4]. In a spin system, X is the temperature T, X_c is the bulk critical temperature $T_{c,b}$, and $Y = -1/\lambda$ where λ is called the extrapolation length which measures the decay of the order parameter near the boundary. In two dimensions, systems like the Ising model with free boundary conditions can only have the ordinary surface transition since the one-dimensional free surface cannot order independently of the bulk at a non-zero temperature without ordering fields. However, the $O(n)$ model with $n < 1$ is known to exhibit the critical behaviour as Fig.1.1 even in two dimensions. Such behaviour is believed to be generic to statistical models in more than two dimensions.

Analytical methods based on conformal invariance in the presence of boundary are quite powerful and they are exploited to solve various problems involving more than simple boundaries. For example, the two-dimensional Ising model with a defect line is studied by folding the Ising model along the defect line and mapping it to the Ashkin-Teller model [5, 6]. The crossing probability of the two-dimensional percolation is found analytically by considering the 4-point function of boundary operators [7,8]. These analytic solutions are compared with results obtained by other methods including numerical calculations, and have been shown to be in excellent agreement.

2 Two-dimensional manifolds

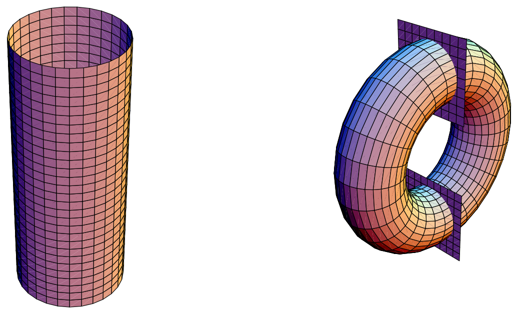
In this section we review some basic facts about manifolds of real dimension two [9–12].

2.1 Topology of two-dimensional manifolds

Compact connected real two-dimensional manifolds Σ are known to be characterised by three non-negative integers, namely the number of handles g, holes b, and crosscaps c added to a sphere. The number g is called the genus of the surface. A hole introduces a boundary to the surface (b stands for boundary). A crosscap is a hole with diametrically opposite points identified, and its insertion makes a manifold unorientable. Crosscaps are important for the construction of type I string theories. The three numbers (g, b, c) are slightly redundant to specify the topology of Σ, since three crosscaps can be traded for one handle and one crosscap. For example, a torus with a crosscap is written either as $(g, b, c) = (0, 0, 3)$ or as $(g, b, c) = (1, 0, 1)$. Hence, the number of crosscaps may be restricted to be less than 3. In this notation, a sphere is $(g, b, c) = (0, 0, 0)$, a torus is $(1, 0, 0)$, a disk is $(0, 1, 0)$, a cylinder is $(0, 2, 0)$, a Möbius strip is $(0, 1, 1)$, and a Klein bottle is $(0, 0, 2)$. The Euler characteristic is given by $\chi = 2 - 2g - b - c$.

Any compact, connected, *oriented* two-dimensional surface is topologically equivalent to a sphere with handles and holes (no crosscaps), and is specified by two non-negative integers g and b. If such an oriented manifold has no boundary, its topology is specified by the genus g only. Such a surface is called a Riemann surface and has several nice properties, as is discussed in the next subsection.

One may construct an oriented boundaryless manifold $\hat{\Sigma}$ called the *Schottky double* associated to a compact connected manifold Σ, by doubling the manifold except for the points on the boundary. This doubling process proceeds in two steps: creating a mirror image of the original manifold Σ by reflection σ, and then gluing the boundaries of Σ and its mirror image. For example, the Schottky double of a disk $(g, b, c) = (0, 1, 0)$ is a sphere $(g, b, c) = (0, 0, 0)$,



(a) Cylinder Σ. (b) Torus $\hat{\Sigma}$.

Figure 1.2: An example of Schottky double. By doubling a cylinder Σ (a) except the boundaries, one may construct a torus $\hat{\Sigma}$ (b) which is the Schottky double of Σ.

obtained by gluing the disk and its mirror image along their circumferences. Similarly, the double of a cylinder (or an annulus) $(0, 2, 0)$ is a torus $(1, 0, 0)$, as shown in Fig.1.2. The relation between the Euler characteristics of Σ and $\hat{\Sigma}$ is $\chi(\hat{\Sigma}) = 2\chi(\Sigma)$, which holds in general. The reflection σ creating a mirror image is an orientation-reversing involution ($\sigma^2 = 1$). Using this σ the original manifold Σ is written as the quotient $\Sigma = \hat{\Sigma}/\sigma$. Boundaries of Σ are fixed points of σ. If Σ is orientable and has no boundary, $\hat{\Sigma}$ is just the total space of the trivial orientation bundle, $\hat{\Sigma} = \Sigma \otimes \mathbb{Z}_2$. Note that in any case $\hat{\Sigma}$ is naturally oriented. The idea of Schottky double is important in CFT because a full (non-chiral) CFT on a conformal manifold is constructed from a chiral CFT on its double.

2.2 Riemann surfaces

A Riemann surface is a connected, analytic, orientable two-dimensional manifold without boundaries. The Schottky double $\hat{\Sigma}$ mentioned in the previous subsection is an example of a Riemann surface. Such a manifold is paracompact, and possesses a holomorphic structure (the charts take values on a complex plane and the transition functions are holomorphic). In particular, a Riemann surface allows a metric $g_{\alpha\beta}(\zeta)$ which is defined globally.

Using the metric one may define the complex structure tensor $J_\alpha{}^\beta$ as

$$J_\alpha{}^\beta = \sqrt{g}\epsilon_{\alpha\gamma}g^{\gamma\beta}, \tag{1.1}$$

where $g = \det g_{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ is an antisymmetric tensor, $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, $\epsilon_{12} = 1$. The complex structure tensor has the properties,

$$J_\alpha{}^\beta J_\beta{}^\gamma = -\delta_\alpha^\gamma, \tag{1.2}$$

$$\nabla_\gamma J_\alpha{}^\beta = 0. \tag{1.3}$$

The covariant derivative ∇_α is defined by the metric $g_{\alpha\beta}(\zeta)$. A Riemann surface is alternatively defined as a two-dimensional connected oriented manifold Σ furnished with a complex structure J. One can change the coordinate from ζ^α to $(z, \bar{z})$ in accordance with the Cauchy-Riemann equations,

$$J_\alpha{}^\beta \frac{\partial z}{\partial \zeta^\beta} = i\frac{\partial z}{\partial \zeta^\alpha}, \tag{1.4}$$

$$J_\alpha{}^\beta \frac{\partial \bar{z}}{\partial \zeta^\beta} = -i\frac{\partial \bar{z}}{\partial \zeta^\alpha}. \tag{1.5}$$

It is a special property of the two-dimensional manifolds that we can always choose a coordinate which makes the metric locally conformally flat,

$$ds^2 = g_{\alpha\beta}(\zeta)d\zeta^\alpha d\zeta^\beta = \rho(z, \bar{z})dzd\bar{z}. \tag{1.6}$$

Topological and geometrical aspects of manifolds are related by the Gauss-Bonnet theorem. On a Riemann surface it reads

$$\int d^2x\sqrt{g}R = 4\pi\chi, \tag{1.7}$$

where R is the scalar curvature of the manifold (see App. A for our conventions). The Euler characteristic is $\chi = 2 - 2g$ in our case.

For any compact Riemann surface of genus g, there are $2g$ non-contractable independent closed curves. We may choose a basis (called a canonical homology basis) of such cycles as a_i and b_i $(i = 1, 2, \cdots, g)$, satisfying

$$\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1. \tag{1.8}$$

The spin structure of a function on Σ is defined as the transformation properties around these curves a_i and b_i. A function is said to have the spin structure $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_g)$, $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_g)$ and $0 \leq \alpha_i, \beta_j < 1$, if the function is multiplied with $\exp(2\pi i \alpha_i)$ around a_i and with $\exp(2\pi i \beta_i)$ around b_i.

2.3 Teichmüller and moduli spaces

For a given Riemann surface Σ, let $\mathrm{Diff}(\Sigma)$ be the group of all diffeomorphisms of Σ, and $\mathrm{Diff}_0(\Sigma)$ consist of the elements of $\mathrm{Diff}(\Sigma)$ homotopic to the identity map. One may define the constant curvature slice $\mathcal{M}_{const}$ for the Weyl transformation group in the space of all metrics on Σ. The Teichmüller and moduli spaces are then defined by

$$\mathcal{T}_g = \frac{\mathcal{M}_{const}}{\mathrm{Diff}_0(\Sigma)}, \tag{1.9}$$

$$\mathcal{M}_g = \frac{\mathcal{M}_{const}}{\mathrm{Diff}(\Sigma)}, \tag{1.10}$$

respectively. The subscript g stands for the genus. The group

$$G_g = \frac{\mathrm{Diff}(\Sigma)}{\mathrm{Diff}_0(\Sigma)}, \tag{1.11}$$

is called the mapping class group, with which the Teichmüller and the moduli spaces are related as $\mathcal{M}_g = \mathcal{T}_g / G_g$. The dimensions of the Teichmüller and moduli spaces are

$$\dim \mathcal{T}_g = \dim \mathcal{M}_g = \begin{cases} 0, & g = 0, \\ 2, & g = 1, \\ 6g - 6, & g \geq 2. \end{cases} \tag{1.12}$$

For $g = 1$ (i.e. on the torus), the Teichmüller space is the upper half plane,

$$\mathcal{T}_1 = \{\tau \in \mathbb{C} \mid \mathrm{Im}\tau > 0\}, \tag{1.13}$$

and τ is called the modular parameter. The action of the mapping class group G_1 is

$PSL(2,\mathbb{Z})$,

$$G_1 : \tau \mapsto \frac{a\tau + b}{c\tau + d}, \tag{1.14}$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. G_1 is generated by $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$.

2.4 Theta functions

Functions defined on a closed manifold are conveniently expressed by some basis functions which have some particular periodicity with respect to the periodic directions of the manifold they inhabit. A simple example is functions defined on a circle, which, through the Fourier transformation may be expressed using trigonometric functions. The functions which play the role of trigonometric functions for the Riemann surfaces are Jacobi's theta functions. They have been studied extensively since the 19th century.

For a canonical homology basis a_i, b_i of a Riemann surface Σ, there exists a normalised basis of holomorphic 1-forms ω_i $(i = 1, \cdots, g)$ satisfying

$$\oint_{a_i} \omega_j = \delta_{ij}, \tag{1.15}$$

$$\oint_{b_i} \omega_j = \tau_{ij}, \tag{1.16}$$

where τ_{ij} is a complex symmetric $g \times g$ matrix with positive definite imaginary part, called the period matrix of the Riemann surface Σ. For $g = 1$ the period matrix is merely the modular parameter τ. The Riemann theta function is defined using the period matrix $\boldsymbol{\tau} = \tau_{ij}$, as

$$\vartheta(\boldsymbol{z}, \boldsymbol{\tau}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^g} \exp(i\pi n_i \tau_{ij} n_j + 2\pi i n_i z_i), \tag{1.17}$$

where $\boldsymbol{z} = (z_1, \cdots, z_g)$, $\boldsymbol{n} = (n_1, \cdots, n_g)$. This function has a simple transformation property on shifting $\boldsymbol{z}$ by the lattice $\mathbb{Z}^g + \boldsymbol{\tau}\mathbb{Z}^g$,

$$\vartheta(\boldsymbol{z} + \boldsymbol{\tau n} + \boldsymbol{m}, \boldsymbol{\tau}) = \exp(-i\pi \boldsymbol{n\tau n} - 2\pi i \boldsymbol{nz})\vartheta(\boldsymbol{z}, \boldsymbol{\tau}). \tag{1.18}$$

That is, on the Riemann surface Σ it is single-valued up to a phase. Theta functions are solutions of the heat equation

$$\left(4\pi i\frac{\partial}{\partial\tau_{ij}}+\frac{\partial^2}{\partial z_i\partial z_j}\right)\vartheta(\boldsymbol{z},\boldsymbol{\tau})=0. \tag{1.19}$$

Associated to the spin structure we may define the theta function with characteristics $(\boldsymbol{\alpha},\boldsymbol{\beta})$, by

$$\vartheta\begin{bmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{bmatrix}(\boldsymbol{z},\boldsymbol{\tau})=\exp(i\pi\boldsymbol{\alpha\tau\alpha}+2\pi i\boldsymbol{\alpha}(\boldsymbol{z}+\boldsymbol{\beta}))\vartheta(\boldsymbol{z}+\boldsymbol{\tau\alpha}+\boldsymbol{\beta},\boldsymbol{\tau}). \tag{1.20}$$

Its transformation laws are

$$\vartheta\begin{bmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{bmatrix}(\boldsymbol{z}+\boldsymbol{\tau n}+\boldsymbol{m},\boldsymbol{\tau})=\exp(-i\pi\boldsymbol{n\tau n}-2\pi i\boldsymbol{n}(\boldsymbol{z}+\boldsymbol{\beta})+2\pi i\boldsymbol{m\alpha})\vartheta\begin{bmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{bmatrix}(\boldsymbol{z},\boldsymbol{\tau}), \tag{1.21}$$

$$\vartheta\begin{bmatrix}\boldsymbol{\alpha}+\boldsymbol{m}\\ \boldsymbol{\beta}+\boldsymbol{n}\end{bmatrix}(\boldsymbol{z},\boldsymbol{\tau})=\exp(2\pi i\boldsymbol{\alpha n})\vartheta\begin{bmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{bmatrix}(\boldsymbol{z},\boldsymbol{\tau}). \tag{1.22}$$

As is easily shown by the definition, we also have

$$\vartheta\begin{bmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{bmatrix}(-\boldsymbol{z},\boldsymbol{\tau})=(-1)^{4\alpha\beta}\vartheta\begin{bmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{bmatrix}(\boldsymbol{z},\boldsymbol{\tau}). \tag{1.23}$$

The spin structure is called even (odd) if $4\boldsymbol{\alpha\beta}$ is even (odd).

For $g=1$ the theta functions (1.17), (1.20) reduce to

$$\vartheta(z,\tau)=\sum_{n\in\mathbb{Z}^g}\exp(i\pi n^2\tau+2\pi inz). \tag{1.24}$$

and

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z,\tau)=\exp(i\pi\alpha^2\tau+2\pi i\alpha(z+\beta))\vartheta(z+\tau\alpha+\beta,\tau), \tag{1.25}$$

respectively. They are commonly denoted as

$$\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z,\tau) = \theta_3(z,\tau), \tag{1.26}$$

$$\vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (z,\tau) = \theta_4(z,\tau), \tag{1.27}$$

$$\vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (z,\tau) = \theta_2(z,\tau), \tag{1.28}$$

$$\vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z,\tau) = -\theta_1(z,\tau). \tag{1.29}$$

These theta functions at $z = 0$ are simply denoted as $\theta_2(0,\tau) = \theta_2(\tau)$, $\theta_3(0,\tau) = \theta_3(\tau)$, $\theta_4(0,\tau) = \theta_4(\tau)$. Note that $\theta_1(0,\tau) \equiv 0$. We also use generalised theta functions at $z = 0$, defined as

$$\Theta_{\lambda,\mu}(\tau) = \sum_{k\in\mathbb{Z}} q^{(2\mu k+\lambda)^2/4\mu}. \tag{1.30}$$

In this notation,

$$\begin{aligned} \theta_2(\tau) &= 2\Theta_{1,2}(\tau), & (1.31)\\ \theta_3(\tau) &= \Theta_{0,2}(\tau) + \Theta_{2,2}(\tau), & (1.32)\\ \theta_4(\tau) &= \Theta_{0,2}(\tau) - \Theta_{2,2}(\tau). & (1.33) \end{aligned}$$

Some formulas of theta functions are collected in App.A.

3 Two-dimensional conformal field theories

In this section we review some of the basics of the two-dimensional conformal field theory without boundary [2, 9, 12–16]. We start by discussing the Virasoro algebra and primary fields in the first subsection, and then in Subsec.1.3.3 we discuss modular invariance which plays a central role throughout this thesis. In Subsec.1.3.5 we review the Coulomb-gas representation of CFT which is intended as the introduction of the discussions in Chap.2. The only non-standard material is in Subsec.1.3.7, where we introduce logarithmic conformal field theories whose behaviour in the presence of boundaries is the main topic of Chap.3.

3.1 Conformal invariance and Virasoro algebra

The central object in a two-dimensional CFT is the energy-momentum tensor $T_{\mu\nu}$. In a free-field theory it is obtained by the variation of the action,

$$T^{\mu\nu} = -2\frac{\delta \mathcal{S}}{\delta g_{\mu\nu}}. \tag{1.34}$$

However, we do not assume the existence of an action but start from $T_{\mu\nu}$ itself. The energy-momentum tensor satisfies the conservation law

$$\nabla^{\mu} T_{\mu\nu} = 0, \tag{1.35}$$

and due to local scale invariance it is traceless,

$$T^{\mu}{}_{\mu} = 0. \tag{1.36}$$

Choosing a conformal gauge $g_{\mu\nu}(z^1, z^2) = \rho(z^1, z^2)\delta_{\mu\nu}$ and introducing complex coordinates $z = z^1 + iz^2$, $\bar{z} = z^1 - iz^2$ (see App.A), the energy-momentum tensor splits into its holomorphic and antiholomorphic parts, $T \equiv -2\pi T_{zz} = \pi(T_{22} - T_{11} + 2iT_{12})/2$, $\bar{T} \equiv -2\pi T_{\bar{z}\bar{z}} = \pi(T_{22} - T_{11} - 2iT_{12})/2$. As the holomorphic and antiholomorphic parts of an operator are treated on equal footing, we often write the holomorphic part only.

The operator product expansion (OPE) of $T(z)$ with itself is

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots . \tag{1.37}$$

Here, c is called the central charge and the dots represent the terms regular in the limit $z \to w$. Such an OPE should be understood as an identity which holds when inserted into arbitrary correlation functions.

Primary fields are the fields in the theory which transform as tensors of weight $(h, \bar{h})$,

$$\Phi'_{(h,\bar{h})}(w(z), \bar{w}(\bar{z})) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \Phi_{(h,\bar{h})}(z, \bar{z}), \tag{1.38}$$

under conformal transformations $z \to w(z)$, $\bar{z} \to \bar{w}(\bar{z})$. The scaling powers $(h, \bar{h})$ are called the conformal dimensions of the primary field. The OPE of the energy-momentum tensor and a primary field takes the form

$$T(z)\phi_{(h)}(w) = \frac{h\phi_{(h)}(w)}{(z-w)^2} + \frac{\partial\phi_{(h)}(w)}{z-w} + \cdots . \tag{1.39}$$

The fields which are not primary are called secondary or descendant, and are obtained from primary fields by taking the OPE with the energy-momentum tensor.

The Laurent mode operators L_m of the energy-momentum tensor are called Virasoro operators. They are defined by

$$T(z) = \sum_{m \in \mathbb{Z}} L_m z^{-m-2}, \tag{1.40}$$

$$L_m = \frac{1}{2\pi i} \oint dz z^{m+1} T(z), \tag{1.41}$$

and the OPE (1.37) yields the Virasoro algebra,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}, \tag{1.42}$$

$$[L_n, \bar{L}_m] = 0, \tag{1.43}$$

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}. \tag{1.44}$$

The subset $\{L_{-1}, L_0, L_1\}$ generates global conformal (or Möbius) transformations, consisting of translations, dilatations, rotations and special conformal transformations.

Using the Virasoro operators the OPE (1.39) becomes

$$[L_m, \phi_{(h)}(z)] = (m+1)z^m h\phi_{(h)}(z) + z^{m+1}\partial\phi_{(h)}(z). \tag{1.45}$$

If this equation holds for $m = -1, 0, 1$, then $\phi_{(h)}(z)$ is said to be quasi-primary. Obviously, primary fields are quasi-primary but quasi-primary fields are not necessarily primary. An example of fields that are quasi-primary but not primary is the energy-momentum tensor $T(z)$. On the z-plane the Laurent mode expansion of a (quasi-) primary field is defined as

$$\phi_{(h)}(z) = \sum_{n \in \mathbb{Z}} \phi_n z^{-n-h}. \tag{1.46}$$

Using the modes ϕ_n, (1.45) is written as

$$[L_m, \phi_n] = (hm - m - n)\phi_{m+n}. \tag{1.47}$$

The OPE of two primary fields is a linear sum of primary and descendant fields,

$$\phi_i(z)\phi_j(w) = \sum_k C_{ij}{}^k (z-w)^{h_k - h_i - h_j} \phi_k(w) + (\text{descendants}). \tag{1.48}$$

Due to the conformal invariance, 2-point functions $\langle \phi_i(z)\phi_j(w) \rangle$ vanish if the conformal dimensions of the two fields differ. Then they may be normalised as

$$\langle \phi_i(z)\phi_j(w) \rangle = \frac{\delta_{ij}}{(z-w)^{2h_i}}. \tag{1.49}$$

Similarly, the conformal invariance restricts the form of 3-point functions to be

$$\langle \phi_i(z_i)\phi_j(z_j)\phi_k(z_k) \rangle = \frac{C_{ijk}}{z_{ij}^{h_i+h_j-h_k} z_{jk}^{h_j+h_k-h_i} z_{ki}^{h_k+h_i-h_j}}, \tag{1.50}$$

where $z_{ij} = z_i - z_j$, and the coupling constant C_{ijk} is the same as the OPE coefficient appearing in (1.48). The conformal invariance does not fix the forms of n-point functions with $n \geq 4$, leaving the dependence on anharmonic ratios undetermined. However, once we know the 3-point coefficients C_{ijk} any n-point function is obtained by repeated use of the OPE (1.48) within the correlators. CFT is therefore completely characterised by the central charge c, the conformal dimensions h_i of primary fields ϕ_i, and the three-point coefficients C_{ijk}[1]. One may find C_{ijk} from n-point functions, with some assumptions (existence of conformal

[1] The above discussion only applies to conventional CFTs where the OPEs are in the power-law form (1.48); for logarithmic CFTs, OPEs are modified as in (1.170).

blocks and crossing symmetry). Such a programme is called the conformal bootstrap.

An important concept related to C_{ijk} is fusion. Because of conformal invariance the descendant terms on the right hand side of the chiral OPE (1.48) factorise into conformal families associated with their primary fields. Then the OPE schematically takes the form,

$$[\phi_i] \times [\phi_j] = \sum_k N_{ij}{}^k [\phi_k]. \tag{1.51}$$

The entries of the matrix $N_{ij}{}^k$ are non-negative integers, indicating the multiplicities of $[\phi_k]$ occurring as a result of the fusion $[\phi_i] \times [\phi_j]$. The fusion rule comprises a commutative and associative algebra among the chiral components of primary operators. A fusion coefficient $N_{ij}{}^k$ is non-zero if and only if $C_{ij}{}^k$ is non-zero. This property is called naturality.

The Hilbert space of a CFT is built on the vacuum $|0\rangle$ which is a singlet under the Möbius transformation. States $|\phi\rangle$ in the Hilbert space and fields $\phi(z, \bar{z})$ are related by a one-to-one correspondence,

$$|\phi\rangle = \lim_{z,\bar{z}\to 0} \phi(z, \bar{z})|0\rangle. \tag{1.52}$$

Among these states there exist states called highest weight states $|h, c\rangle$ characterised by the properties,

$$L_0|h, c\rangle = h|h, c\rangle, \tag{1.53}$$

$$L_n|h, c\rangle = 0, \quad n > 0. \tag{1.54}$$

The highest weight states are the states associated with primary fields through the operator-state correspondence (1.52). Descendants of a highest weight states $|h, c\rangle$ are obtained from $|h, c\rangle$ as

$$L_{-k_1} L_{-k_2} \cdots L_{-k_m} |h, c\rangle, \quad k_i > 0, \tag{1.55}$$

and the set of states associated with a conformal family (h, c),

$$M(h, c) = \{L_{-k_1} L_{-k_2} \cdots L_{-k_m} |h, c\rangle \ ; \ k_1 \geq k_2 \geq \cdots \geq k_m > 0\}, \tag{1.56}$$

is called the Verma module. The Verma module splits into L_0-eigenspaces,

$$M(h,c) = \bigoplus_{n\geq 0} M(h,c)_n, \tag{1.57}$$

$$M(h,c)_n = \{v \in M(h,c)\ ;\ L_0 v = (h+n)v\}. \tag{1.58}$$

Such an eigenspace $M(h,c)_n$ is spanned by the basis states

$$L_{-k_1}L_{-k_2}\cdots L_{-k_m}|h,c\rangle,\ \ \sum_{i=1}^{m} k_i = n,\ \ k_1 \geq k_2 \geq \cdots \geq k_m > 0. \tag{1.59}$$

The determinant of the inner-product matrix $\hat{M}(h,c)_n$ of the basis vectors of $M(h,c)_n$ is called the Kac determinant, and is given by

$$\det \hat{M}(h,c)_n = \prod_{k=1}^{n}\prod_{rs=k}(h - h_{r,s}(c))^{p(n-k)}, \tag{1.60}$$

with

$$h_{r,s}(c) = \frac{1}{48}\left[(13-c)(r^2+s^2) - 24rs - 2(1-c) + (r^2-s^2)\sqrt{(1-c)(25-c)}\right]. \tag{1.61}$$

Here, r and s are positive integers and $p(n)$ is Euler's partition function, the number of ways to partition n into positive integers.

Now we are in a position to discuss reducibility and unitarity of CFT. The Verma module is called reducible if it contains invariant subspaces. The zeros of the Kac determinant correspond to the null-states $v \in M(h,c)$, which are orthogonal to every state in $M(h,c)$ and hence decouple from the theory. Such null-states form an invariant subspace $I(h,c)$. One can thus find whether the Verma module is reducible or irreducible by analysing the Kac determinant. Verma modules with $c > 1$ and $h > 0$ are easily seen to be irreducible. If a Verma module is reducible, one may construct a coset highest weight module $L(h,c) = M(h,c)/I(h,c)$ which is irreducible.

A highest weight module $L(h,c)$ is called unitary if all the states in $L(h,c)$ have positive norm. It has been shown that unitary irreducible highest weight modules occur when

$$c \geq 1, \quad h \geq 0, \tag{1.62}$$

or when

$$c = 1 - \frac{6}{m(m+1)}, \quad h_{r,s} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}, \tag{1.63}$$

where $m, r, s \in \mathbb{Z}$ and $m \geq 2$, $0 < r < m$, $0 < s < m+1$. The CFTs of the first case (1.62) are accompanied by extra symmetry other than the conformal symmetry. The series of the second case (1.63) are called the unitary Virasoro minimal models, which are associated with critical systems in statistical physics. The first few examples are, $m = 3$ ($c = 1/2$): Ising model, $m = 4$ ($c = 7/10$): tricritical Ising model, $m = 5$ ($c = 4/5$): tetracritical Ising model, and so on.

If non-unitarity is allowed, we may consider the so-called Virasoro minimal models (often called simply minimal models) $\mathcal{M}(p, p')$, characterised by two co-prime positive integers p and p' (we accept the convention $p > p' \geq 2$). In these models, the operator algebra truncates due to the existence of the singular vectors, and the fusion rule closes for a finite number of Virasoro representations. The central charge and the conformal weights of the operator content are given by the Kac formula,

$$c = 1 - 6\frac{(p-p')^2}{pp'}, \tag{1.64}$$

$$h_{r,s} = \frac{(pr - p's)^2 - (p-p')^2}{4pp'}, \tag{1.65}$$

where $1 \leq r \leq p'$ and $1 \leq s \leq p$. Because of the symmetry

$$h_{r,s} = h_{p'-r,p-s}, \tag{1.66}$$

the operators $\phi_{r,s}$ and $\phi_{p'-r,p-s}$ are identified. The explicit form of the fusion rules for these operators is

$$[\phi_{r,s}] \times [\phi_{m,n}] = \sum_{\substack{k=1+|r-m|,\\ k+r+m=1 \bmod 2.}}^{k_{\max}} \sum_{\substack{l=1+|s-n|,\\ l+s+n=1 \bmod 2.}}^{l_{\max}} [\phi_{k,l}], \tag{1.67}$$

with

$$k_{\text{max}} = \min(r + m - 1,\ 2p' - 1 - r - m), \tag{1.68}$$

$$l_{\text{max}} = \min(s + n - 1,\ 2p - 1 - s - n). \tag{1.69}$$

3.2 Free CFTs

Among CFTs other than the Virasoro minimal series, particularly simple but nevertheless significant classes of CFTs are those realised by free bosons, fermions, and ghosts. In the following we discuss them briefly in this order.

Free bosons

The CFT of a free massless boson is realised by the action

$$\mathcal{S} = \frac{1}{8\pi} \int d^2x \sqrt{g} \partial_\mu \Phi \partial^\mu \Phi. \tag{1.70}$$

The 2-point correlation function of the bosons are

$$\langle \Phi(z, \bar{z}) \Phi(w, \bar{w}) \rangle = -\ln(z - w) - \ln(\bar{z} - \bar{w}), \tag{1.71}$$

indicating that Φ itself is not primary. The derivatives of the boson,

$$\partial\varphi(z) \equiv \frac{\partial \Phi(z, \bar{z})}{\partial z}, \tag{1.72}$$

$$\bar{\partial}\bar{\varphi}(\bar{z}) = \frac{\partial \Phi(z, \bar{z})}{\partial \bar{z}}, \tag{1.73}$$

are respectively holomorphic and antiholomorphic functions due to the equation of motion $\nabla^2 \Phi = 0$. Differentiating (1.71), we find

$$\langle \partial\varphi(z) \partial\varphi(w) \rangle = -\frac{1}{(z - w)^2}, \tag{1.74}$$

$$\langle \bar{\partial}\bar{\varphi}(\bar{z}) \bar{\partial}\bar{\varphi}(\bar{w}) \rangle = -\frac{1}{(\bar{z} - \bar{w})^2}. \tag{1.75}$$

The OPE of $\partial\varphi$ with itself is now seen to be

$$\partial\varphi(z)\partial\varphi(w) = -\frac{1}{(z-w)^2}\cdots\,, \tag{1.76}$$

where the dots indicate regular terms.

The energy-momentum tensor of this system is

$$T(z) = -\frac{1}{2} : \partial\varphi(z)\partial\varphi(z) :, \tag{1.77}$$

which is obtained from the action. The OPE of $T(z)$ with $\partial\varphi(z)$ is

$$T(z)\partial\varphi(w) = \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial^2\varphi(w)}{z-w} + \cdots\,, \tag{1.78}$$

and thus $\partial\varphi(z)$ is a primary field of conformal dimension 1. The central charge of this system is $c = 1$, which is read off from the OPE of $T(z)$ with itself. The vertex operators defined by $\mathcal{V}_\alpha(z,\bar{z}) =: e^{i\sqrt{2}\alpha\Phi(z,\bar{z})} :$ are also primary. The conformal dimension of $\mathcal{V}_\alpha(z,\bar{z})$ is α^2. The OPE among themselves is

$$\mathcal{V}_\alpha(z,\bar{z})\mathcal{V}_\beta(w,\bar{w}) = |z-w|^{4\alpha\beta}\mathcal{V}_{\alpha+\beta}(w,\bar{w}) + \cdots\,. \tag{1.79}$$

Free fermions

The action of a free Majorana fermion in 2-dimensional Euclidean space is written as

$$\mathcal{S} = \frac{1}{2\pi}\int d^2x\sqrt{g}(\bar{\psi}\partial\bar{\psi} + \psi\bar{\partial}\psi), \tag{1.80}$$

where $\psi(z,\bar{z})$ and $\bar{\psi}(z,\bar{z})$ have correlators

$$\langle\psi(z,\bar{z})\psi(w,\bar{w})\rangle = \frac{1}{z-w}, \tag{1.81}$$

$$\langle\bar{\psi}(z,\bar{z})\bar{\psi}(w,\bar{w})\rangle = \frac{1}{\bar{z}-\bar{w}}, \tag{1.82}$$

$$\langle\psi(z,\bar{z})\bar{\psi}(w,\bar{w})\rangle = 0. \tag{1.83}$$

Thus one may say that the holomorphic and antiholomorphic parts decouple. The OPE of ψ with itself is

$$\psi(z)\psi(w) = \frac{1}{z-w} + \cdots . \tag{1.84}$$

The energy-momentum tensor is

$$T(z) = -\frac{1}{2} : \psi(z)\partial\psi(z) :, \tag{1.85}$$

and calculating the OPEs, the central charge of the system and the conformal dimension of ψ are shown to be $c = 1/2$ and $h = 1/2$.

Fermionic and bosonic ghosts

Generalised fermionic and bosonic ghosts are called bc and $\beta\gamma$ systems, respectively[2]. They follow from the same form of the action,

$$\mathcal{S} = \frac{1}{\pi}\int d^2x\sqrt{g}(\tilde{b}\bar{\partial}\tilde{c} + \bar{\tilde{b}}\partial\bar{\tilde{c}}), \tag{1.86}$$

where $\tilde{b} = b$ and $\tilde{c} = c$ for the anticommuting case ($\varepsilon = 1$), and $\tilde{b} = \beta$ and $\tilde{c} = \gamma$ for the commuting case ($\varepsilon = -1$). They have the OPEs,

$$\tilde{c}(z)\tilde{b}(w) = \frac{1}{z-w} + \cdots , \quad \tilde{b}(z)\tilde{c}(w) = \frac{\varepsilon}{z-w} + \cdots . \tag{1.87}$$

The energy-momentum tensor is

$$T(z) = (1-\lambda) : \partial\tilde{b}(z)\tilde{c}(z) : -\lambda : \tilde{b}(z)\partial\tilde{c}(z) :, \tag{1.88}$$

where the parameter λ comes from the freedom to add a total derivative term to the Lagrangian. The central charge then becomes

$$c = -2\varepsilon(6\lambda^2 - 6\lambda + 1), \tag{1.89}$$

[2]Needless to say, c of bc is not the central charge.

and the dimensions of $\tilde{b}(z)$ and $\tilde{c}(z)$ are found to be

$$h_{\tilde{b}} = \lambda, \quad h_{\tilde{c}} = 1 - \lambda. \tag{1.90}$$

The bc theory ($\varepsilon = 1$) with $\lambda = 1/2$ is the complex fermion with the central charge $c = 1$. The Faddeev-Popov ghosts arising in the gauge-fixing of (super) strings are realised by $(\varepsilon, \lambda) = (1, 2)$ and $(\varepsilon, \lambda) = (-1, 3/2)$. The case $(\varepsilon, \lambda) = (1, 0)$ describes the simple ghost system, which is related to the symplectic fermions discussed in Chap.3.

3.3 Modular invariance

CFTs are not restricted to the plane, but are extendible to manifolds of more general topologies. Theories defined on the torus are the simplest of such generalisations, but reveal amazingly rich structures of CFT.

As is mentioned in Subsec.1.2.3, a torus is characterised by the modular parameter $\tau = \tau_1 + i\tau_2$ with $\tau_2 > 0$. The key object which plays a central role in the study of CFT on the torus is the character, which is a function of τ. For the highest weight representation $\mathcal{V}$ of the Virasoro algebra, the Virasoro character $\chi_{\mathcal{V}}(q)$ is defined by

$$\chi_{\mathcal{V}}(q) = \mathrm{Tr}_{\mathcal{V}} q^{L_0 - c/24}, \tag{1.91}$$

where

$$q = e^{2\pi i \tau}. \tag{1.92}$$

In particular, the character of the Verma module $M(h, c)$ becomes

$$\chi_{M(h,c)}(q) = \frac{q^{h-c/24}}{\prod_{n=1}^{\infty}(1 - q^n)}. \tag{1.93}$$

The characters for representations (r, s) in the $\mathcal{M}(p, p')$ Virasoro minimal models are found by Rocha-Caridi [17], as

$$\chi_{(r,s)} = \frac{1}{\eta(\tau)}(\Theta_{pr-p's,pp'}(\tau) - \Theta_{pr+p's,pp'}(\tau)), \tag{1.94}$$

where the theta functions are defined as (1.30).

The partition function of CFT on the torus is defined as a function of τ by

$$\begin{aligned} Z(\tau) &= \mathrm{Tr} e^{-\tau_2 H} e^{-\tau_1 P} \\ &= \mathrm{Tr}(q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}), \end{aligned} \quad (1.95)$$

where $H = 2\pi(L_0 + \bar{L}_0 - c/12)$ and $P = 2\pi i(L_0 - \bar{L}_0)$ are the Hamiltonian and the momentum operators, respectively. As a consequence of conformal invariance which splits the Hilbert space of the CFT into modules associated to irreducible representations of the Virasoro algebra, the torus partition function is written as

$$Z(\tau) = \sum_{h,\bar{h}} N_{h,\bar{h}} \chi_h(q) \bar{\chi}_{\bar{h}}(\bar{q}), \quad (1.96)$$

where $\chi_h(q)$ is the character of the irreducible Virasoro representation with highest weight h, and $\bar{\chi}_{\bar{h}}(\bar{q})$ is its antiholomorphic counterpart. The entries of the multiplicity matrix $N_{h,\bar{h}}$ are non-negative integers and the uniqueness of the vacuum implies $N_{0,\bar{0}} = 1$.

As the torus partition function is a physical object (zero-point function on the torus), it must be invariant under the modular transformations S and T (defined in Subsec.1.2.3) under which the shape of the torus is unchanged. This condition imposes a stringent constraint on the CFT. Once a set of irreducible modules are specified, the classification of rational CFTs boils down to finding the matrix $N_{h,\bar{h}}$ which keeps the modular invariance and satisfies the condition $N_{0,\bar{0}} = 1$. For unitary Virasoro minimal models this classification (called ADE-classification after the associated simply-laced Lie algebra) was done in [18–21]. A modern proof of such a classification based on Galois theory is found in [22].

Another remarkable result of genus one CFT is that fusion rules are determined by the modular transformations of characters. This is highly non-trivial since fusion is a local property of operators whereas modular transformations are obviously global. The relation between fusion and modular transformations is summarised in the form of the celebrated Verlinde formula [23]:

$$N_{ij}{}^{k} = \sum_{m} \frac{S_{im} S_{jm} \bar{S}_{mk}}{S_{0m}}, \quad (1.97)$$

where ${N_{ij}}^k$ is the fusion matrix in (1.51) and S_{ij} is the modular S matrix, $\chi_i(\tilde{q}) = \sum_j S_{ij}\chi_j(q)$, where $\tilde{q} = e^{-2\pi i/\tau}$. The index 0 stands for the vacuum representation. The proof of this equation is found in [24, 25]. See also [16] for a more recent review. Using $SS^\dagger = 1$, the above relation may be written in the form

$$\sum_k {N_{ij}}^k S_{km} = \frac{S_{im}}{S_{0m}} S_{jm}, \tag{1.98}$$

meaning that the fusion matrix is diagonalised by the modular S matrix [3].

3.4 Correlation functions

From a practical point of view, the goal of a CFT is to identify its full spectrum and find all correlation functions. If this is accomplished, the CFT is said to be solved. In the case of Virasoro minimal models, the spectrum is obtained by the Kac formula. Correlation functions are found by exploiting the existence of singular vectors, with the help of conformal invariance [1].

A singular vector (also called a null state[4]) at level n is a descendant state $|\chi\rangle$ satisfying

$$\begin{aligned} L_0|\chi\rangle &= (h+n)|\chi\rangle, \\ L_k|\chi\rangle &= 0, \quad k > 0, \end{aligned} \tag{1.99}$$

where h is the conformal dimension of the ancestral primary state $|h\rangle$. Null states are by definition highest weight states as well. The singular vector at level 1 takes the form $|\chi\rangle = L_{-1}|h\rangle$. In order to satisfy the conditions (1.99), $|h\rangle$ must be the Möbius invariant vacuum $|0\rangle$ and the singular vector is $|\chi\rangle = L_{-1}|0\rangle$. At level 2, the singular vector may be written as

$$|\chi\rangle = (L_{-2} + aL_{-1}^2)|h\rangle, \tag{1.100}$$

for some a. The values of a and h satisfying the conditions (1.99) are found by using the

[3] This discussion does not hold for logarithmic CFTs

[4] There are some cases where these two concepts must be distinguished. An example is the $N = 2$ superconformal algebra, where subsingular vectors exist [26].

Virasoro algebra (1.42), to be

$$a = -\frac{3}{2(2h+1)}, \tag{1.101}$$

$$h = \frac{5 - c \pm \sqrt{(c-1)(c-25)}}{16}. \tag{1.102}$$

Hence the singular vectors at level 2 are

$$|\chi\rangle = \left[L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2\right] |h\rangle, \tag{1.103}$$

with h given by (1.102). Singular vectors at higher levels are obtained in a similar manner.

The behaviour of correlation functions under infinitesimal conformal transformtions is governed by the conformal Ward identity,

$$\delta_{\epsilon,\bar{\epsilon}}\langle X\rangle = -\frac{1}{2\pi i}\oint_C dz \epsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_C d\bar{z}\bar{\epsilon}(\bar{z})\langle \bar{T}(\bar{z})X\rangle, \tag{1.104}$$

where ϵ and $\bar{\epsilon}$ are holomorphic and antiholomorphic infinitesimal coordinate changes, X stands for an arbitrary product of primary operators, and C is a contour encircling all coordinates within X. The correlator $\langle T(z)X\rangle$ in the integrand is explicitly written as

$$\langle T(z)X\rangle = \sum_{i=1}^{n} \left\{ \frac{1}{z - w_i}\frac{\partial}{\partial w_i} + \frac{h_i}{(z-w_i)^2} \right\} \langle X\rangle, \tag{1.105}$$

which, in operator language, reads (redefining $X \to \phi(w)X$)

$$\langle (L_{-n}\phi)(w)X\rangle = \mathcal{L}_{-n}\langle \phi(w)X\rangle, \quad n \geq 1, \tag{1.106}$$

with

$$\mathcal{L}_{-n} = \sum_i \left\{ \frac{(n-1)h_i}{(w_i - w)^n} - \frac{1}{(w_i - w)^{n-1}}\frac{\partial}{\partial w_i} \right\}. \tag{1.107}$$

This indicates that the action of a Virasoro operator L_{-n} on a primary field within a correlator is described by the differential operator $\mathcal{L}_{-n}$.

Now, the decoupling of a singular vector $|\chi\rangle$ from the theory enables us to set $|\chi\rangle = 0$.

Writing the operator corresponding to the singular vector $|\chi\rangle$ as $\chi_{(h)}(w)$, this results in

$$\langle \chi_{(h)}(w) X \rangle = 0, \tag{1.108}$$

for an arbitrary product X of operators. Since $\chi_{(h)}(w)$ is obtained from $\phi_{(h)}(w)$ (primary operator of conformal dimension h) by operating with a polynomial of L_{-n}, (1.108) becomes a differential equation satisfied by the correlator $\langle \phi_{(h)}(w) X \rangle$.

As the simplest non-trivial example, let us consider the case for a level 2 singular vector. As a consequence of the projective Ward identity expressing the translational covariance of correlators, we have

$$\mathcal{L}_{-1} = -\sum_i \frac{\partial}{\partial w_i} = \frac{\partial}{\partial w}. \tag{1.109}$$

Then for a level 2 singular vector (1.103),

$$\langle \chi_{(h)}(w) X(w_i) \rangle = \left[\frac{3}{2(2h+1)} \frac{\partial^2}{\partial w^2} - \sum_i \frac{h_i}{(w-w_i)^2} - \sum_i \frac{1}{w-w_i} \frac{\partial}{\partial w_i} \right] \langle \phi_{(h)}(w) X(w_i) \rangle = 0. \tag{1.110}$$

The correlator $\langle \phi_{(h)}(w) X(w_i) \rangle$ is found by solving this second order partial differential equation. For a 4-point correlation function $\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4) \rangle$ the partial differential equation is reduced to an ordinary differential equation with respect to the anharmonic ratio

$$\eta = \frac{z_{12} z_{34}}{z_{13} z_{24}}, \tag{1.111}$$

where $z_{ij} = z_i - z_j$, and the solutions are expressed in terms of hypergeometric functions. Physical correlation functions are then obtained as particular sesquilinear combinations of two independent solutions, where the coefficients are determined by the monodromy invariance, i.e. single-valuedness of the full correlators at $\bar{z} = z^*$.

3.5 Coulomb-gas representation

It was shown by Dotsenko and Fateev [27] that all the features of the minimal models can be realised by using a single scalar field. This approach, called the Coulomb-gas formalism, has several nice features. For example, this is a free-field theory and thus all pieces of the theory are constructed from a Lagrangian. Correlation functions found by solving differential equations in the last subsection, are obtained in the Coulomb-gas formalism as contour integrals. This is not only an alternative way to obtain the same result, but is advantageous in that the method is easily extendible to CFTs on higher genus manifolds. Construction of the boundary CFT based on the Coulomb-gas formalism is the main topic of Chap.2.

The essential ingredient of the Coulomb-gas formalism is the non-minimal coupling of the free scalar field to the background curvature. This makes the $U(1)$ symmetry anomalous, modifying the central charge and the conformal dimensions of $c = 1$ theory to generate the minimal models. In this subsection we collect the basic components of the Coulomb-gas formalism without the boundary [9, 14, 27, 28]. Variation of the action,

$$\mathcal{S} = \frac{1}{8\pi} \int d^2x \sqrt{g} (\partial_\mu \Phi \partial^\mu \Phi + 2\sqrt{2}\alpha_0 i \Phi R), \tag{1.112}$$

with respect to the metric gives the energy-momentum tensor

$$T(z) = -2\pi T_{zz} = -\frac{1}{2} : \partial\varphi\partial\varphi : + i\sqrt{2}\alpha_0 \partial^2 \varphi, \tag{1.113}$$

where φ is the holomorphic part of the boson, $\Phi(z, \bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z})$. The antiholomorphic part is similar. From $T(z)$ the central charge is read off as

$$c = 1 - 24\alpha_0^2. \tag{1.114}$$

The *chiral* vertex operator defined as

$$V_\alpha(z) =: e^{i\sqrt{2}\alpha\varphi(z)} : \tag{1.115}$$

then has the conformal dimension $h_\alpha = \alpha^2 - 2\alpha_0\alpha$, which is easily verified by computing the OPE with $T(z)$. Among these vertex operators, $V_\pm(z) \equiv V_{\alpha_\pm}(z)$ with $\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1}$ play a special role. They have conformal dimensions 1 and the closed contour integral,

$$Q_\pm \equiv \oint dz V_\pm(z), \tag{1.116}$$

are the screening operators which are conformal singlets and carry charges. The condition that the fields must be screened by such screening operators leads to the quantisation of the spectrum,

$$\alpha_{r,s} = \frac{1}{2}(1-r)\alpha_+ + \frac{1}{2}(1-s)\alpha_-, \tag{1.117}$$

where r and s are positive integers. The vertex operators $V_{\alpha_{r,s}}(z)$ then have conformal dimensions

$$h_{r,s} = \frac{1}{4}(r\alpha_+ + s\alpha_-)^2 - \alpha_0^2, \tag{1.118}$$

and are identified with the operators $\phi_{r,s}$ appearing in the Kac formula. Note that $\alpha_+ = \sqrt{p/p'}$ and $\alpha_- = -\sqrt{p'/p}$ for a minimal model $\mathcal{M}(p,p')$.

The Hilbert space of the theory defined on a Riemann surface is a direct sum of charged bosonic Fock spaces (CBFSs) with BRST projection [28]. The chiral CBFS F_{α,α_0} with vacuum charge α and background charge α_0 is built on the highest-weight vector $|\alpha;\alpha_0\rangle$ as a representation of the Heisenberg algebra

$$[a_m, a_n] = m\delta_{m+n,0}, \tag{1.119}$$

where a_n are the mode operators defined by

$$\varphi(z) = \varphi_0 - ia_0 \ln z + i\sum_{n\neq 0} \frac{a_n}{n} z^{-n}. \tag{1.120}$$

The zero-mode operators satisfy the commutation relation $[\varphi_0, a_0] = i$. The highest-weight vector is constructed from the vacuum $|0;\alpha_0\rangle$ by operating with $e^{i\sqrt{2}\alpha\varphi_0}$,

$$|\alpha;\alpha_0\rangle = e^{i\sqrt{2}\alpha\varphi_0}|0;\alpha_0\rangle, \tag{1.121}$$

and is annihilated by the action of $a_{n>0}$. The charge α is related to the eigenvalue of a_0 by

$$a_0|\alpha;\alpha_0\rangle = \sqrt{2}\alpha|\alpha;\alpha_0\rangle. \tag{1.122}$$

The Virasoro generators are written in terms of the mode operators as

$$L_{n\neq 0} = \frac{1}{2}\sum_{k\in\mathbb{Z}} a_{n-k}a_k - \sqrt{2}\alpha_0(n+1)a_n, \tag{1.123}$$

$$L_0 = \sum_{k\geq 1} a_{-k}a_k + \frac{1}{2}a_0^2 - \sqrt{2}\alpha_0 a_0. \tag{1.124}$$

With these generators the CBFS F_{α,α_0} has the structure of a Virasoro module. It is easy to check that

$$L_0|\alpha;\alpha_0\rangle = (\alpha^2 - 2\alpha\alpha_0)|\alpha;\alpha_0\rangle, \tag{1.125}$$

that is, the conformal dimension of $|\alpha;\alpha_0\rangle$ is $\alpha^2 - 2\alpha\alpha_0$. Because of $[L_0, a_{-n}] = na_{-n}$ ($\forall n \geq 0$), F_{α,α_0} is graded by L_0 and written as

$$F_{\alpha,\alpha_0} = \bigoplus_{n=0}^{\infty}(F_{\alpha,\alpha_0})_n, \tag{1.126}$$

where $(F_{\alpha,\alpha_0})_n$ is the subspace with conformal dimension $\alpha^2 - 2\alpha\alpha_0 + n$. Counting the number of states the character of F_{α,α_0} is found to be

$$\chi_{\alpha,\alpha_0}(q) \equiv \operatorname*{Tr}_{F_{\alpha,\alpha_0}} q^{L_0 - c/24} = \frac{q^{(\alpha-\alpha_0)^2}}{\eta(\tau)}, \tag{1.127}$$

where $q = e^{2\pi i\tau}$, τ is the modular parameter, and $\eta(\tau) \equiv q^{1/24}\prod_{n\geq 1}(1-q^n)$ is the Dedekind eta function.

The dual space F^*_{α,α_0} of F_{α,α_0} is built on a contravariant highest-weight vector $\langle\alpha;\alpha_0|$ satisfying the condition

$$\langle\alpha;\alpha_0|\alpha;\alpha_0\rangle = \kappa, \tag{1.128}$$

where κ is a normalisation factor which is usually set to 1 in unitary models. The modules are endowed with a dual Virasoro structure

$$\langle\omega|L_{-n}\xi\rangle = \langle\omega L_n|\xi\rangle \tag{1.129}$$

for any $\langle\omega| \in F^*_{\alpha,\alpha_0}$, $|\xi\rangle \in F_{\alpha,\alpha_0}$. This dual structure naturally incorporates the transpose A^t

of an operator A through the relation

$$\langle\omega|A\xi\rangle = \langle\omega A^t|\xi\rangle. \tag{1.130}$$

In particular, $L^t_{-n} = L_n$, $a^t_{-n} = 2\sqrt{2}\alpha_0\delta_{n,0} - a_n$. With this definition of transpose, F^*_{α,α_0} is shown to be a Fock space isomorphic to $F_{2\alpha_0-\alpha,\alpha_0}$. The contravariant highest-weight vector $\langle\alpha;\alpha_0|$ is annihilated by the action of a_n for $n<0$ (or a^t_n for $n>0$),

$$\langle\alpha,\alpha_0|a_{n<0} = 0. \tag{1.131}$$

From the uniqueness of the expression $\langle\alpha;\alpha_0|a_0|\alpha;\alpha_0\rangle$ and the right operation of the zero mode (1.122) we immediately have

$$\langle\alpha;\alpha_0|a_0 = \sqrt{2}\alpha\langle\alpha;\alpha_0|. \tag{1.132}$$

Analogously to (1.121) we find

$$\langle\alpha;\alpha_0| = \langle 0;\alpha_0|e^{-i\sqrt{2}\alpha\varphi_0}, \tag{1.133}$$

where the contravariant vector $\langle 0;\alpha_0|$ is the vacuum with the normalisation $\langle 0;\alpha_0|0;\alpha_0\rangle = \kappa$. From (1.121) and (1.133), the in-state $|\alpha;\alpha_0\rangle$ and the out-state $\langle\alpha;\alpha_0|$ are interpreted as possessing charges α and $-\alpha$, respectively. The non-vanishing inner product (1.128) is consistent with the neutrality of the total charge, $-\alpha+\alpha=0$. Since the inner product must vanish when the total charge is not zero, we have in general

$$\langle\alpha;\alpha_0|\beta;\alpha_0\rangle = \kappa\delta_{\alpha,\beta}. \tag{1.134}$$

On the plane the minimal conformal theory is realized through the radial quantisation scheme, by sending the in-state to zero and the out-state to infinity. Expectation values are usually taken between $\langle 2\alpha_0;\alpha_0|$ and $|0;\alpha_0\rangle$, which is interpreted as placing a charge $-2\alpha_0$ at infinity. Correlation functions of primary operators are calculated with suitable insertion of the screening operators,

$$\langle V_{\alpha_1}V_{\alpha_2}\cdots V_{\alpha_k}Q_+^m Q_-^n\rangle, \tag{1.135}$$

where the numbers of the screening charges m and n are subject to the charge neutrality condition,

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k + m\alpha_+ + n\alpha_- = 2\alpha_0. \tag{1.136}$$

We may use the equivalence of $\alpha_{r,s}$ and $\alpha_{p'-r,p-s}$ to minimise the numbers of screening charges. The correlation functions are then expressed as contour integrals over the positions of the screening operators.

The Coulomb-gas formalism also applies to Riemann surfaces of higher genus and such theories have been studied by many authors [28–33]. On the torus it is shown that taking the trace over the BRST cohomology space is equivalent to the alternated summation [28]. For example, the zero-point function on the torus for the conformal block corresponding to the representation (r, s) of the minimal models is calculated in the Coulomb-gas method as [28]

$$\mathrm{Tr}_{(r,s)} q^{L_0 - c/24} = \frac{1}{\eta(\tau)} (\Theta_{pr-p's,pp'}(\tau) - \Theta_{pr+p's,pp'}(\tau)), \tag{1.137}$$

which is nothing but the Rocha-Caridi character formula (1.94) as it should be.

3.6 CFTs with extended symmetry

CFTs other than the Virasoro minimal models are generally accompanied by some symmetry other than the conformal symmetry. For such theories the chiral algebra is an extended algebra containing the Virasoro algebra as a subalgebra. Examples of the extra symmetries are the supersymmetry, the affine Kac-Moody symmetry, and the W-symmetry. In this subsection we review two classes of such CFTs, WZNW theory and CFTs with W-symmetry, which are relevant to later discussions.

WZNW theory

The Wess-Zumino-Novikov-Witten (WZNW) theory [34–36] is a non-linear sigma model with a Wess-Zumino (WZ) topological term, defined by the action,

$$\mathcal{S}(g) = \frac{k}{16\pi} \int_{\partial\Sigma} d^2x \mathrm{Tr}'(\partial^\mu g^{-1} \partial_\mu g) - \frac{ik}{24\pi} \int_{\Sigma} d^3y \mathrm{Tr}'(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg), \tag{1.138}$$

where Σ is the 3-dimensional ball whose boundary $\partial\Sigma$ is the 2-sphere, and $\mathrm{Tr}' = x_{\mathrm{rep}}^{-1}\mathrm{Tr}$ is a rescaled trace (x_{rep} is the Dynkin index of the representation). The scalar field g takes values in a Lie group G. The integrand of the second term (the WZ topological term) is a total derivative and thus the integration gives a surface term on $\partial\Sigma$. The WZ term is single-valued if $k \in \mathbb{Z}$.

The action (1.138) is invariant not only under the conformal transformations but also under the infinite-dimensional transformations,

$$g(z,\bar{z}) \to \Omega(z) g(z,\bar{z}) \bar{\Omega}^{-1}(\bar{z}), \tag{1.139}$$

where $\Omega(z) \in G$ and $\bar{\Omega}(\bar{z}) \in \bar{G}$. This symmetry is characterised by the currents,

$$J(z) = J^a(z) t^a = -k\partial g g^{-1} = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}, \tag{1.140}$$

$$\bar{J}(\bar{z}) = \bar{J}^a(\bar{z}) t^a = k g^{-1}\bar{\partial} g = \sum_{n\in\mathbb{Z}} \bar{J}_n \bar{z}^{-n-1}, \tag{1.141}$$

with OPEs

$$J^a(z) J^b(w) = \frac{k}{(z-w)^2}\delta^{ab} + \frac{i f_{abc}}{z-w} J^c(w) + \cdots . \tag{1.142}$$

These currents generate two commuting affine Kac-Moody algebras,

$$[J_n^a, J_m^b] = i f_{abc} J_{n+m}^c + kn\delta_{ab}\delta_{n+m,0}, \tag{1.143}$$

$$[J_n^a, \bar{J}_m^b] = 0, \tag{1.144}$$

$$[\bar{J}_n^a, \bar{J}_m^b] = i f_{abc} \bar{J}_{n+m}^c + kn\delta_{ab}\delta_{n+m,0}, \tag{1.145}$$

where k is called the level of the algebra.

The energy-momentum tensor of the WZNW model is given by the bilinear form of the current,

$$T(z) = \frac{1}{2(k+h^\vee)} \sum_{a=1}^{\dim G} : J^a(z) J^a(z) :, \tag{1.146}$$

called the Sugawara-Sommerfeld construction [37, 38]. The number $h^\vee$ is the dual Coxeter number associated of the group G. The Virasoro operators are then written as

$$L_n = \frac{1}{2(k+h^\vee)} \sum_{a=1}^{\dim G} \sum_{m\in\mathbb{Z}} : J_m^a J_{n-m}^a : . \tag{1.147}$$

The central charge of the CFT is

$$c_G = \frac{k \dim G}{k+h^\vee}. \tag{1.148}$$

Various CFTs are obtained from the WZNW model by the coset construction of Goddard, Kent and Olive (GKO) [39, 40]. For a group G and its subgroup H, the operators

$$L_n^K = L_n^G - L_n^H, \tag{1.149}$$

commute with the Virasoro operators L_m^H for the subgroup theory and define a new Virasoro algebra with central charge $c_K = c_G - c_H$. These new Virasoro operators then realise a CFT on the coset space $K = G/H$. For example, the unitary Virasoro minimal models are reproduced by taking $G = \widehat{SU(2)}_k \otimes \widehat{SU(2)}_1$ and $H = \widehat{SU(2)}_{k+1}$ (the subscripts indicate the levels). The central charge for the coset model is then

$$c_K = 1 - \frac{6}{(k+2)(k+3)}, \tag{1.150}$$

reproducing that of the $\mathcal{M}(k+2, k+3)$ minimal models.

CFT with W-algebra

Another important class of CFTs with extra symmetry is those possessing W-symmetry, generated by currents of conformal dimension (spin) greater than 2. The algebra comprising the energy-momentum tensor $T(z)$ (of spin 2), and primary fields of conformal dimensions s_2, s_3, $\cdots$, s_n ($s_i \geq 3$), is denoted by $\mathcal{W}(2, s_2, s_3, \cdots, s_n)$.

The simplest of these is $\mathcal{W}(2,3)$ (also called W_3 algebra), generated by the energy-momentum tensor and a field $W(z)$ of conformal dimension $h = 3$ [41]. The 3-state Potts model at $c = 4/5$ is known to have such an algebra. The OPEs involving $T(z)$ and $W(z)$ are

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \left[\Lambda(w) + \frac{3}{10}\partial^2 T(w)\right] + \cdots , \quad (1.151)$$

$$T(z)W(w) = \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} + \cdots , \quad (1.152)$$

$$W(z)W(w) = \frac{c}{3(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} + \frac{2\beta\Lambda(w) + 3\partial^2 T(w)/10}{(z-w)^2} + \frac{\beta\partial\Lambda(w) + \partial^3 T(w)/15}{z-w} + \cdots , \quad (1.153)$$

where $\beta = 16/(22+5c)$ and

$$\Lambda(z) =: T(z)T(z) : -\frac{3}{10}\partial^2 T(z). \quad (1.154)$$

Mode expansions of $T(z)$ and $W(z)$ define Virasoro and W-mode operators,

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad (1.155)$$

$$W(z) = \sum_{n\in\mathbb{Z}} W_n z^{-n-3}. \quad (1.156)$$

In terms of these mode operators, the above OPEs are equivalent to the commutation relations of the W_3 algebra,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \quad (1.157)$$

$$[L_n, W_m] = (2n-m)W_{n+m}, \quad (1.158)$$

$$[W_n, W_m] = \frac{c}{360}n(n^2-1)(n^2-4)\delta_{n+m,0} + (n-m)\left[\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2)\right]L_{n+m} + \beta(n-m)\Lambda_{n+m}, \quad (1.159)$$

where

$$\Lambda_n = \sum_{m\in\mathbb{Z}} (L_{n-m}L_m) - \frac{3}{10}(n+3)(n+2)L_n. \quad (1.160)$$

The W_3 algebra is quite different from the Lie-type algebra due to the appearance of the composite field $\Lambda(z)$ which is quadratic in $T(z)$.

A particularly important class of CFTs with W-algebra is the so-called unitary W-minimal series starting from the 3-state Potts model. There are also other classes, such as w_∞, W_∞, $W_{1+\infty}$, and $W_\infty(\lambda)$ algebras. The supersymmetric extension of W-algebra is also well-studied. A comprehensive review article on this subject is [15]. W-algebra of type $\mathcal{W}(2,3,3,3)$ proves to be important in the study of logarithmic CFT at $c = -2$.

3.7 Logarithmic conformal field theories

Conformal field theories with logarithmic correlation functions have been studied actively for the past few years. Such theories arise naturally as generalisations of the well-investigated Virasoro minimal theories or integral level WZNW theories, and are believed to have many applications in statistical models and string / brane physics. These logarithmic conformal field theories (LCFTs) were investigated sporadically by several authors [42–45] in the late eighties and early nineties, and systematic study started with Gurarie's work [46] in 1993. By now various models, e.g. $c = -2$ model [44, 46–50], gravitationally dressed CFTs [51, 52], WZNW models with fractional k [45, 53] and $k = 0$ [54–57] have been studied, and a number of applications, including critical polymers [44, 58, 59], percolation [7, 8], quantum Hall effect [60, 61], disordered systems [54, 55, 62–64], sandpile model [65, 66], turbulence [67–69], MHD [70], D-brane recoil [71, 72], etc. have been discussed. The state of the art of the study on LCFT is summarised in the recent lecture notes [73–76]. In this subsection we review the basic features of LCFT following the analytic approach of Gurarie [46], by examining the so-called $c = -2$ model.

Usually the operator content of the $\mathcal{M}(p, p')$ minimal model (we assume $p > p'$) is restricted to $\phi_{r,s}$ such that $0 < r < p'$, $0 < s < p$ (and also $pr > p's$ to avoid the double counting of identical operators). Although the Kac table for $(p, p') = (2, 1)$ is empty, a non-trivial theory at $c = c_{2,1} = -2$ is obtained by extending the border of the grid. Note that the Kac formulas of central charge $c_{p,p'}$ and conformal dimension $h_{r,s}$ for $\mathcal{M}(p, p')$ minimal model,

$$c_{p,p'} = 1 - 6\frac{(p-p')^2}{pp'}, \tag{1.161}$$

$$h_{r,s} = \frac{(pr-p's)^2-(p-p')^2}{4pp'}, \tag{1.162}$$

are invariant for the "rescaling" $p \to lp$ and $p' \to lp'$ for some natural number l. The table of conformal dimensions for the extended $\mathcal{M}(2,1)$ 'minimal' model is

	s											
r	1	2	3	4	5	6	7	8	9	10	11	⋯
1	0	−1/8	0	3/8	1	15/8	3	35/8	6	63/8	10	⋯
2	1	3/8	0	−1/8	0	3/8	1	15/8	3	35/8	6	
3	3	15/8	1	3/8	0	−1/8	0	3/8	1	15/8	3	
4	6	35/8	3	15/8	1	3/8	0	−1/8	0	3/8	1	
5	10	63/8	6	35/8	3	15/8	1	3/8	0	−1/8	0	
⋮	⋮											⋱

In the following we shall restrict the contents to be $0 < r < 3$, $0 < s < 6$, that is, we consider the 'next-to-minimal' model $\mathcal{M}(6,3)$. It has been shown by algebraic [47] and free-field [50] approaches that the operators with conformal dimensions $h = -1/8$, $3/8$, 0 and 1, defined with respect to the enhanced symmetry generated by a triplet of $h = 3$ fields, indeed close under the fusion rule.

We shall discuss the fusion product of the $\mu = \phi_{1,2}$ operators with conformal dimension $h = -1/8$. The necessary information is encoded in the 4-point function,

$$\langle \mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4)\rangle, \tag{1.163}$$

which is determined by the method described in Subsec.1.3.4[1]. The conformal family with

[1]In $c = -2$ theory, $\mu = \phi_{1,2}$ and $\nu = \phi_{1,4}$ are ordinary (pre-logarithmic) operators which satisfy conventional conformal Ward identities. Their correspondence with $h = -1/8$ and $h = 3/8$ operators in symplectic fermion representation (see Chap.3) is also well understood.

the primary field μ has the singular vector $(L_{-2} - 2L_{-1}^2)\mu$ at level 2, which implies,

$$\langle [(L_{-2} - 2L_{-1}^2)\mu](z_1)\mu(z_2)\mu(z_3)\mu(z_4)\rangle = 0. \tag{1.164}$$

Substituting the differential operators $\mathcal{L}_{-n}$ for L_{-n}, we obtain a second order partial differential equation satisfied by the correlation function. In order to solve the differential equation, it is convenient to use the Möbius symmetry and write

$$\langle \mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4)\rangle = (z_1 - z_3)^{1/4}(z_2 - z_4)^{1/4}[\eta(1-\eta)]^{1/4}F(\eta), \tag{1.165}$$

where

$$\eta = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}. \tag{1.166}$$

Substituting this into the partial differential equation we obtain an ordinary differential equation,

$$\eta(1-\eta)\frac{\partial^2 F(\eta)}{\partial \eta^2} + (1 - 2\eta)\frac{\partial F(\eta)}{\partial \eta} - \frac{1}{4}F(\eta) = 0. \tag{1.167}$$

This is a hypergeometric differential equation and we may choose the two independent solutions as $F(1/2, 1/2, 1; \eta)$ and $F(1/2, 1/2, 1; 1-\eta)$. The function $F(1/2, 1/2, 1; \eta)$ is a hypergeometric function of Gaussian type and reduces to the complete elliptic integral

$$\frac{\pi}{2}F(\frac{1}{2}, \frac{1}{2}, 1; k^2) = K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}. \tag{1.168}$$

The properties of $K(k)$ are well known. In particular, $F(1/2, 1/2, 1; \eta)$ is regular at $\eta = 0$ and has a logarithmic singularity at $\eta = 1$. Thus the general solution

$$F(\eta) = AF(1/2, 1/2, 1; \eta) + BF(1/2, 1/2, 1; 1-\eta), \tag{1.169}$$

has a logarithmic singularity at $\eta = 0$ unless $B = 0$, and at $\eta = 1$ unless $A = 0$. The chiral 4-point function (1.165) then necessarily has a logarithmic singularity somewhere on the manifold.

The OPE of μ with itself implied by the 4-point function is

$$\mu(z)\mu(0) = z^{1/4}[\omega(0) + \Omega(0)\log(z)] + \cdots . \tag{1.170}$$

The two operators ω and Ω, both having the conformal dimension $h = 0$, represent the two conformal blocks associated with the two independent solutions of the differential equation (1.167). The operator Ω is the Möbius invariant vacuum with respect to which vacuum expectation values are taken. Operations with L_n on the operators Ω and ω are calculated as [46, 77]

$$L_0\omega = \Omega, \tag{1.171}$$

$$L_0\Omega = 0, \tag{1.172}$$

$$L_n\omega = 0, \quad n > 0. \tag{1.173}$$

This may be regarded as a special case of the Jordan cell structure,

$$L_0 \begin{pmatrix} C \\ D_1 \\ D_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} h & 0 & 0 & \cdots \\ 1 & h & 0 & \cdots \\ 0 & 1 & h & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} C \\ D_1 \\ D_2 \\ \vdots \end{pmatrix}, \tag{1.174}$$

with $h = 0$, $C = \Omega$, and $D_1 = \omega$. Although a precise definition of logarithmic conformal field theories is absent[5], it seems to be generally accepted that they are the CFTs characterised by such Jordan cells.

4 Boundary conformal field theory

In this section we review standard techniques and concepts of boundary conformal field theory. The discussion is restricted to simple diagonal unitary minimal models, such as the Ising model. We start, in the first subsection, by discussing conformal invariance in the presence of a boundary. In Subsec.1.4.2 we review the mirroring method [78] for finding

[5] In [75], it is conjectured that LCFTs may be fully characterised by non-semisimple Zhu's algebra.

boundary correlation functions. We describe in Subsec.1.4.3 the classification of consistent boundary states based on the modular invariance, which is known as Cardy's fusion method [79]. In Subsec.1.4.4 we discuss boundary operators [79, 80] and sewing relations which lead to the concept of completeness of boundary conditions. We give in Subsec.1.4.5 an example of a statistical model [7] where the concept of boundary operators plays a central role.

4.1 Conformal transformation on half plane

Let us start by considering what is meant by conformal invariance in the presence of a boundary. Let

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu, \tag{1.175}$$

be the line element of the manifold we work on. Since the metric is a tensor, it transforms as

$$g^{\mu\nu}(x) \to \tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\lambda} \frac{\partial \tilde{x}^\nu}{\partial x^\rho} g^{\lambda\rho}(x). \tag{1.176}$$

The conformal transformation is defined as a mapping which preserves the metric $g^{\mu\nu}(x)$ up to a scale factor,

$$g^{\mu\nu}(x) \to \tilde{g}^{\mu\nu}(\tilde{x}) \propto g^{\mu\nu}(x). \tag{1.177}$$

In two dimensions this condition is written as

$$\left(\frac{\partial \tilde{x}^1}{\partial x^1}\right)^2 + \left(\frac{\partial \tilde{x}^1}{\partial x^2}\right)^2 = \left(\frac{\partial \tilde{x}^2}{\partial x^1}\right)^2 + \left(\frac{\partial \tilde{x}^2}{\partial x^2}\right)^2, \tag{1.178}$$

$$\frac{\partial \tilde{x}^1}{\partial x^1} \frac{\partial \tilde{x}^2}{\partial x^1} + \frac{\partial \tilde{x}^1}{\partial x^2} \frac{\partial \tilde{x}^2}{\partial x^2} = 0, \tag{1.179}$$

which are equivalent either to

$$\frac{\partial \tilde{x}^1}{\partial x^1} = \frac{\partial \tilde{x}^2}{\partial x^2}, \quad \frac{\partial \tilde{x}^2}{\partial x^1} = -\frac{\partial \tilde{x}^1}{\partial x^2}, \tag{1.180}$$

or to

$$\frac{\partial \tilde{x}^1}{\partial x^1} = -\frac{\partial \tilde{x}^2}{\partial x^2}, \quad \frac{\partial \tilde{x}^2}{\partial x^1} = \frac{\partial \tilde{x}^1}{\partial x^2}. \tag{1.181}$$

These are the Cauchy-Riemann equations and their antiholomorphic counterpart. Defining $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$, we conclude that the conformal transformation in two-dimensions (without considering boundary) is equivalent to analytic mapping on the complex plane [14].

On the full plane, the conformal mapping

$$z \to w(z) = \sum_n a_n z^n, \tag{1.182}$$

$$\bar{z} \to \bar{w}(\bar{z}) = \sum_n \bar{a}_n \bar{z}^n, \tag{1.183}$$

is generated by an infinite number of generators a_n and $\bar{a}_n$, which imposes strict constraints on the field theory. In a geometry with boundary, we may take the line $x^2 = 0$ as the boundary and consider a CFT on the upper half plane. As the field theory is restricted to a fixed geometry, the conformal transformation must keep the boundary $x^2 = 0$ invariant. This means

$$\text{Im } w(x)|_{x^2=0} = 0 \quad \Leftrightarrow \quad w(x^1) = \bar{w}(x^1) \quad \Leftrightarrow \quad a_n = \bar{a}_n. \tag{1.184}$$

Although the number of generators is reduced by half due to this condition, we still have an infinite dimensional conformal group and conformal invariance remains extremely powerful [81]. Note that the holomorphic and antiholomorphic generators are coupled on the boundary. This allows us to interpret the antiholomorphic part as an analytic continuation of the holomorphic part, as we shall see in the next subsection.

4.2 Boundary correlation functions

The existence of null vectors in minimal CFTs allows us to find n-point correlation functions as solutions to differential equations of hypergeometric type [1]. This method was generalised to CFTs on the half plane by Cardy [78], using the mirroring technique which is familiar in electrostatics. In this subsection we review this method, and as an example find the spin correlation functions of the Ising model on the upper half plane.

The behaviour of correlation functions under the conformal transformations is described by the conformal Ward identities (1.104). For a CFT on the upper half plane they are

$$\delta\langle\phi_1\phi_2\cdots\rangle \ = \frac{-1}{2\pi i}\oint_C dz\epsilon\langle T(z)\phi_1\phi_2\cdots\rangle + \frac{1}{2\pi i}\oint_C d\bar{z}\bar{\epsilon}\langle\bar{T}(\bar{z})\phi_1\phi_2\cdots\rangle, \tag{1.185}$$

as $z \to w = z+\epsilon$, $\bar{z} \to \bar{w} = \bar{z}+\bar{\epsilon}$, and $\bar{\epsilon} = \epsilon^*$. The contours are the semicircle C which encircles all the coordinates $(z_i, \bar{z}_i)$ of the operators (Fig.1.3a). Since there is no energy-momentum flow across the boundary, the energy-momentum tensor satisfies the condition

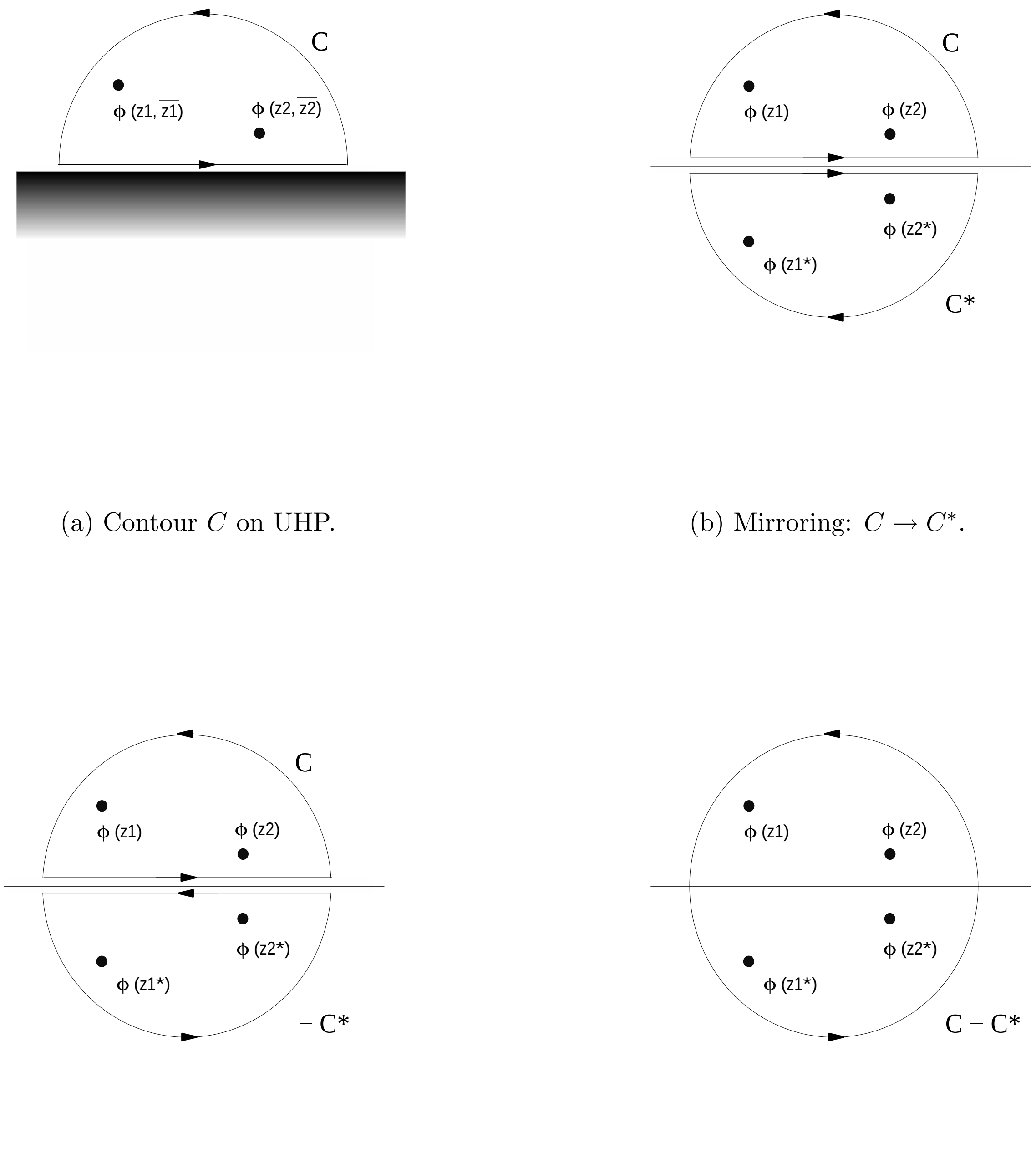
$$\left[T - \bar{T}\right]_{z=\bar{z}} = 0, \tag{1.186}$$

on the boundary $z = \bar{z}$. This condition also means the diffeomorphism invariance of the boundary as the conformal transformation is generated by the energy-momentum tensor. We can use the condition (1.186) to extend the domain of definition of $T(z)$, by mapping the antiholomorphic part on the upper half plane (UHP) to the holomorphic part on the lower half plane (LHP), as $T(z^*) = \bar{T}(\bar{z})$. The antiholomorphic dependence of the correlation function on the UHP coordinates is similarly mapped to the holomorphic dependence on the LHP coordinates. The antiholomorphic part of the Ward identities (1.185) is then mapped into the holomorphic part on the LHP, as shown in Fig.1.3b. The direction of the integration contour on the LHP is reversed (Fig.1.3c) by changing the sign of the second term in (1.185). Since the two contours along the boundary cancel each other, the contours can be concatenated to make a contour of the full circle (Fig.1.3d), leading to a much simpler conformal Ward identity,

$$\delta\langle\phi_1\phi_2\cdots\rangle = \frac{-1}{2\pi i}\oint_{C-C^*} dz\epsilon(z)\langle T(z)\phi_1(z_1)\bar{\phi}_1(z_1^*)\phi_2(z_2)\bar{\phi}_2(z_2^*)\cdots\rangle. \tag{1.187}$$

This means that the n-point function on the UHP satisfy the same differential equation as the *chiral* $2n$-point function on the full plane, with the LHP coordinates obtained through mirroring with respect to the boundary.

Now let us see this in the example of the Ising model, and find the spin-spin correlation function on the UHP. As the boundary 2-point function on the half plane is equivalent to



(a) Contour C on UHP.

(b) Mirroring: $C \to C^*$.

(c) Reverse the direction.

(d) Merge two contours.

Figure 1.3: The antiholomorphic coordinate dependence of CFT on the UHP (a) is mapped to the holomorphic dependence on the LHP by mirroring (b). Flipping the direction of the contour on the LHP (c), and merging the two contours, the Ward identity of the n-point function on the UHP is shown to be equivalent to that of the $2n$-point function on the full plane.

the 4-point function on the full plane, one may write

$$\langle\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\rangle_{UHP} = \langle\sigma(z_1)\sigma(z_2)\sigma(z_1^*)\sigma(z_2^*)\rangle_{chiral}, \tag{1.188}$$

where σ is the spin operator. Due to the existence of a singular vector at level 2, the 4-point function of $\sigma = \phi_{1,2}$ satisfies a second order differential equation,

$$\left\{\partial_z^2 - \frac{3}{4}\sum_{i=1}^{3}\left[\frac{1}{z-z_i}\partial_{z_i} + \frac{1/16}{(z-z_i)^2}\right]\right\}\langle\sigma(z)\sigma(z_1)\sigma(z_2)\sigma(z_3)\rangle = 0. \tag{1.189}$$

Using the global conformal transformations, this partial differential equation reduces to a hypergeometric differential equation which is solved as

$$\langle\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\rangle_{UHP} = \left[\frac{(z_1-z_1^*)(z_2-z_2^*)}{|z_1-z_2|^2|z_1-z_2^*|^2}\right]^{1/8}\left\{A\sqrt{\sqrt{1-\eta}+1} + B\sqrt{\sqrt{1-\eta}-1}\right\}, \tag{1.190}$$

where $\eta = (z_1-z_2)(z_1^*-z_2^*)/(z_1-z_1^*)(z_2-z_2^*) = -|z_1-z_2|^2/4\mathrm{Im}z_1\mathrm{Im}z_2$ is the cross ratio, which takes a negative real value $-\infty < \eta < 0$ in the physical region.

The coefficients A and B are to be determined by the boundary conditions. It is convenient to introduce coordinates y_1, y_2 and ρ as in Fig.1.4. The cross ratio is then written as $\eta = -[(y_1-y_2)^2+\rho^2]/4y_1y_2$. For the free boundary condition, the correlation must vanish as we go closer to the boundary:

$$\langle\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\rangle_{UHP} \to 0, \quad \text{as} \quad \rho\to\infty. \tag{1.191}$$

Apart from the overall normalisation the coefficients are then determined as $A = 1$ and $B = -1$. The scaling law near the boundary is now found to be

$$\langle\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\rangle_{UHP} \sim \rho^{-1}. \tag{1.192}$$

In the case of the fixed boundary condition, the asymptotic behaviour near the boundary must be

$$\langle\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\rangle_{UHP} \to \langle\sigma(z_1,\bar{z}_1)\rangle_{UHP}\langle\sigma(z_2,\bar{z}_2)\rangle_{UHP}, \tag{1.193}$$

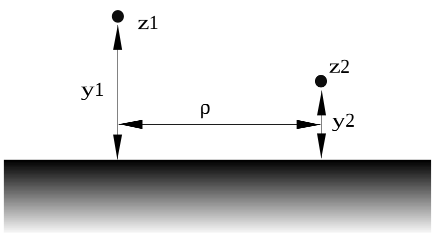


Figure 1.4: Parameters y_1, y_2 and ρ.

as $\rho \to \infty$. The coefficients may be chosen as $A = 1$ and $B = 1$ to satisfy this condition. Then, near the boundary we have

$$\langle \sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2)\rangle_{UHP} \sim (y_1 y_2)^{-1/8}. \tag{1.194}$$

In terms of conformal blocks, the free boundary condition corresponds to the process with intermediate energy operator ϵ, that is, $\langle \sigma\sigma\sigma\sigma \rangle \sim \langle \epsilon\epsilon \rangle$. The fixed boundary condition corresponds to the identity operator, $\langle \sigma\sigma\sigma\sigma \rangle \sim \langle II \rangle$.

4.3 Consistency condition and physical boundary states

Physical systems described by CFT, such as the Ising model at criticality, usually have a finite number of conformally invariant, physically realisable boundary states corresponding to various boundary conditions. For example, in the Ising model there are three physical boundary states corresponding to all spins up ($|\uparrow\rangle$), down ($|\downarrow\rangle$), and free ($|F\rangle$) along the boundary. They are not only conformally invariant but satisfy some extra conditions. Indeed, any linear combination of conformally invariant boundary states is conformally invariant, whereas the number of physical boundary states are usually finite. One of the most powerful and systematic method for finding such physical boundary states is Cardy's fusion method [79], which uses the modular invariance of partition functions as the extra information. In the following we shall review this method. In the past several years Cardy's method has attracted much attention and has been studied extensively. Generalisations to various rational CFTs, including non-diagonal minimal theories [82–84], superconformal models [85], coset models [86, 87], have been considered, and algebraic understanding of the method [88–92] has also been drastically improved. Here we shall not go into these recent developments but describe only the simplest diagonal case, following [14, 79].

The CFTs we analyse in this subsection are defined on an annulus. This geometry has a great advantage that the operators on the full plane (without boundary) may be employed without modification. This is due to the fact that in the radial quantisation, the annulus arises as a portion of the full plane bounded by two concentric circles. One may use the conformal transformation $w = (T/\pi)\ln z$ and $\zeta = \exp(-2\pi i w/L)$ to map the boundary $z = \bar{z}$ of the half plane to the two circles bordering the annulus[6]. This annulus may also be regarded as a cylinder with length T and circumference L. On the ζ-plane (annulus), the conformal invariance condition of the boundary (1.186) implies the Ishibashi condition [93] on boundary states $|B\rangle$,

$$(L_n - \bar{L}_{-n})|B\rangle = 0. \tag{1.195}$$

We shall call the boundary states $|B\rangle$ satisfying this condition as *conformally invariant* boundary states.

In ordinary rational conformal theories there is an important set of conformally invariant boundary states, called *Ishibashi* states. They are defined as

$$|j\rangle\rangle \equiv \sum_M |j;M\rangle \otimes U\overline{|j;M\rangle}, \tag{1.196}$$

where j is the label for representations (conformal towers), M is the level in the conformal tower, and U is an antiunitary operator which is the product of time reversal and complex conjugation. Ishibashi states are conformally invariant boundary states associated with conformal towers, and they form a basis spanning the space of boundary states. An important property of the Ishibashi states is that they diagonalise the closed string amplitudes and give characters for corresponding representations:

$$\langle\langle i|(q^{1/2})^{L_0+\bar{L}_0-c/12}|j\rangle\rangle = \delta_{ij}\chi_i(q). \tag{1.197}$$

[6] In Chap.2 and Sec.3.4 we shall use ζ for the half plane and z for the annulus.

These Ishibashi states are not normalisable, as the innerproducts between them (taking the limit $q \to 1$ in the expression above) are divergent.

Cardy's method uses the modular invariance of boundary partition functions as a constraint on the boundary states; partition functions calculated in open and closed string channels lead to different expressions and their equivalence imposes a condition on the boundary states (Fig.1.5). Suppose we have boundary conditions $\tilde{\alpha}$ and $\tilde{\beta}$ on the two ends of an open string. If these boundary conditions are *physical*, chiral representations labeled by i appear in the bulk with non-negative integer multiplicities $n^i_{\tilde{\alpha}\tilde{\beta}}$. The partition function is then the sum of the chiral characters with the associated multiplicities,

$$Z^{open}_{\tilde{\alpha}\tilde{\beta}}(q) = \sum_i n^i_{\tilde{\alpha}\tilde{\beta}} \chi_i(q), \tag{1.198}$$

where $q = e^{-\pi L/T}$. This is the partition function in the open-string channel. In the closed-string channel, the partition function is nothing but the amplitude between two equal-time hypersurfaces,

$$Z^{closed}_{\tilde{\alpha}\tilde{\beta}}(\tilde{q}) = \langle \tilde{\alpha} | (\tilde{q}^{1/2})^{L_0 + \bar{L}_0 - c/12} | \tilde{\beta} \rangle, \tag{1.199}$$

where $\tilde{q} = e^{-4\pi T/L}$. Note that the Hamiltonian of our system is $H = 2\pi(L_0 + \bar{L}_0 - c/12)/L$. The duality between the open and closed string channels demands $Z^{open}_{\tilde{\alpha}\tilde{\beta}}(q) = Z^{closed}_{\tilde{\alpha}\tilde{\beta}}(\tilde{q})$, or on expanding the boundary states in the closed string channel by some basis states as

$$|\tilde{\alpha}\rangle = \sum_a \langle a | \tilde{\alpha} \rangle | a \rangle, \tag{1.200}$$

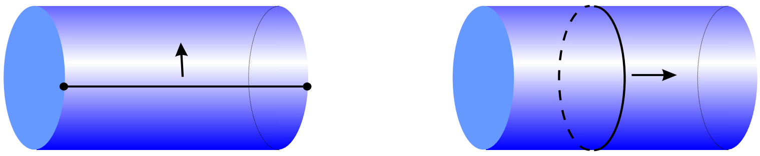
we have

$$\sum_i n^i_{\tilde{\alpha}\tilde{\beta}} \chi_i(q) = \sum_{a,b} \langle \tilde{\alpha} | a \rangle \langle a | (\tilde{q}^{1/2})^{L_0 + \bar{L}_0 - c/12} | b \rangle \langle b | \tilde{\beta} \rangle. \tag{1.201}$$

This duality constraint on the boundary states is called Cardy's consistency condition.

By solving (1.201), one may find physical boundary conditions and express the associated *consistent* boundary states[7] as particular linear combinations of basis states $|a\rangle$. Although

[7] In this thesis we call the states including coherent, Ishibashi and consistent boundary states generically as 'boundary states,' whereas some authors reserve this term for what we call 'consistent boundary states' here. Also, in some literature the term 'Ishibashi state' is used to mean any boundary state satisfying the condition (1.195), i.e. what we call 'conformally invariant boundary state' in this thesis. Our definition of Ishibashi states is in a narrower sense, meaning the particular solution (1.196) found by Ishibashi [93].



(a) Open string channel (b) Closed string channel

Figure 1.5: The open-string channel (a) and the closed-string channel (b) are related by the duality exchanging the directions of time and space. The equivalence of the partition functions calculated in each channel leads to the constraints (1.201) on the boundary states.

in principle we may use any set of basis states as long as they are complete, it is convenient to use Ishibashi states for such an expansion [79]. Using the modular transformation of the characters $\chi_i(q) \to \chi_i(\tilde{q}) = \sum_j S_{ij}\chi_j(q)$ under $\tau \to \tilde{\tau} = -1/\tau$, the left-hand side of (1.201) is written as

$$\sum_i n^i_{\tilde{\alpha}\tilde{\beta}}\chi_i(q) = \sum_{i,j} n^i_{\tilde{\alpha}\tilde{\beta}} S_{ij}\chi_j(\tilde{q}). \tag{1.202}$$

On the right-hand side, if we use the Ishibashi states as the basis, we have

$$\sum_{i,j} \langle\tilde{\alpha}|i\rangle\rangle\langle\langle i|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|j\rangle\rangle\langle\langle j|\tilde{\beta}\rangle = \sum_j \langle\tilde{\alpha}|j\rangle\rangle\langle\langle j|\tilde{\beta}\rangle\chi_j(\tilde{q}). \tag{1.203}$$

Equating the coefficients of the character functions on the both sides, we have

$$\sum_i S_{ij} n^i_{\tilde{\alpha}\tilde{\beta}} = \langle\tilde{\alpha}|j\rangle\rangle\langle\langle j|\tilde{\beta}\rangle. \tag{1.204}$$

Solutions to this equation are found by assuming the existence of a boundary state $|\tilde{0}\rangle$ satisfying $n^i_{\tilde{0}\tilde{\alpha}} = n^i_{\tilde{\alpha}\tilde{0}} = \delta^i_{\tilde{\alpha}}$ for any boundary condition $\tilde{\alpha}$. Letting $\tilde{\alpha} = \tilde{\beta} = \tilde{0}$ in (1.204), and using the positive-definiteness of S_{0j} (which is always the case for unitary models) we have

$$|\tilde{0}\rangle = \sum_j \sqrt{S_{0j}}|j\rangle\rangle. \tag{1.205}$$

Next, putting $\tilde{\alpha} = \tilde{0}$ and $\tilde{\beta} \neq \tilde{0}$ in (1.204) and using the result above, we have

$$|\tilde{\alpha}\rangle = \sum_j \frac{S_{\alpha j}}{\sqrt{S_{0j}}} |j\rangle\rangle. \tag{1.206}$$

This result (1.206) includes the $\tilde{\alpha} = \tilde{0}$ case (1.205).

Let us see this result in the case of the critical Ising model. In this model there are three operators, the identity (I), energy density (ϵ), and spin (σ) operators. In the Kac table they correspond to $I = \phi_{1,1} = \phi_{2,3}$, $\epsilon = \phi_{2,1} = \phi_{1,3}$, $\sigma = \phi_{1,2} = \phi_{2,2}$, respectively. The characters for the three representations are

$$\chi_I = \langle\langle I|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|I\rangle\rangle = \frac{1}{2}\sqrt{\frac{\theta_3(\tilde{\tau})}{\eta(\tilde{\tau})}} + \frac{1}{2}\sqrt{\frac{\theta_4(\tilde{\tau})}{\eta(\tilde{\tau})}}, \tag{1.207}$$

$$\chi_\epsilon = \langle\langle \epsilon|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|\epsilon\rangle\rangle = \frac{1}{2}\sqrt{\frac{\theta_3(\tilde{\tau})}{\eta(\tilde{\tau})}} - \frac{1}{2}\sqrt{\frac{\theta_4(\tilde{\tau})}{\eta(\tilde{\tau})}}, \tag{1.208}$$

$$\chi_\sigma = \langle\langle \sigma|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|\sigma\rangle\rangle = \frac{1}{2}\sqrt{\frac{\theta_2(\tilde{\tau})}{\eta(\tilde{\tau})}}, \tag{1.209}$$

where $|I\rangle\rangle$, $|\epsilon\rangle\rangle$, $|\sigma\rangle\rangle$ are the Ishibashi states for I, ϵ, σ. The modular S matrix for the Ising model is then calculated using the modular transformation formula in App.A, as (in the order of I, ϵ, σ),

$$S_{ij} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}. \tag{1.210}$$

Substituting the modular S matrix into (1.206) we find consistent boundary states as

$$|\tilde{0}\rangle = |\tilde{I}\rangle = \frac{1}{\sqrt{2}}|I\rangle\rangle + \frac{1}{\sqrt{2}}|\epsilon\rangle\rangle + \frac{1}{\sqrt[4]{2}}|\sigma\rangle\rangle, \tag{1.211}$$

$$|\tilde{\epsilon}\rangle = \frac{1}{\sqrt{2}}|I\rangle\rangle + \frac{1}{\sqrt{2}}|\epsilon\rangle\rangle - \frac{1}{\sqrt[4]{2}}|\sigma\rangle\rangle, \tag{1.212}$$

$$|\tilde{\sigma}\rangle = |I\rangle\rangle - |\epsilon\rangle\rangle. \tag{1.213}$$

Since $|\tilde{I}\rangle$ and $|\tilde{\epsilon}\rangle$ differ only by the sign of $|\sigma\rangle\rangle$ associated with the spin operator, they are identified as the fixed boundary conditions ($|\uparrow\rangle$, $|\downarrow\rangle$). Which is up and which is down is purely a matter of choice. The remaining $|\tilde{\sigma}\rangle$ corresponds to the free boundary condition $|F\rangle$.

Substituting (1.206) into the duality relation (1.204) we have

$$\sum_i S_{ij} n^i_{\tilde{\alpha}\tilde{\beta}} = \frac{S_{\alpha j} S_{\beta j}}{S_{0j}}. \tag{1.214}$$

Comparing this with the Verlinde formula (1.98), it is concluded that [79]

$$N_{ij}{}^k = n^k_{ij}, \tag{1.215}$$

that is, in diagonal theories the multiplicity of the representations appearing in the bulk is identical to the fusion coefficient for the operators associated with the boundary states.

4.4 Boundary operators, sewing relations, and completeness

The concept of boundary operators is introduced by Cardy and Lewellen [79, 80, 94], and has played a central role in the recent development of boundary CFT. A boundary operator, ${\psi_i}^{\tilde{\alpha}\tilde{\beta}}$, is a *chiral* operator living on a boundary. The index i refers to the representation of the chiral algebra (Virasoro or its extension), and $\tilde{\alpha}$ and $\tilde{\beta}$ are boundary conditions of the boundaries where the operator is inserted (let us define $\tilde{\alpha}$ is the left and $\tilde{\beta}$ is the right of ${\psi_i}^{\tilde{\alpha}\tilde{\beta}}$, seeing the boundary from the bulk). Thus the insertion of a boundary operator may change boundary conditions. The OPE of boundary operators takes the form,

$${\psi_i}^{\tilde{\alpha}\tilde{\beta}} {\psi_j}^{\tilde{\beta}\tilde{\gamma}} = \sum_k C^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}_{ijk} {\psi_k}^{\tilde{\alpha}\tilde{\gamma}}, \tag{1.216}$$

where $C^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}_{ijk}$ is a structure constant which determines boundary 3-point functions. The boundary 2-point functions are

$$\langle {\psi_i}^{\tilde{\alpha}\tilde{\beta}}(x_1) {\psi_i}^{\tilde{\beta}\tilde{\alpha}}(x_2) \rangle = \frac{{\alpha_i}^{\tilde{\alpha}\tilde{\beta}}}{(x_1 - x_2)^{2h_i}}, \tag{1.217}$$

where ${\alpha_i}^{\tilde{\alpha}\tilde{\beta}}$ is a normalisation constant and h_i is the conformal dimension of the boundary operators. Just as in the bulk theory, ${\alpha_i}^{\tilde{\alpha}\tilde{\beta}}$ and $C_{ijk}^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}$ determine the algebraic structure of the boundary operators completely. For a given bulk theory where the bulk operator content, the modular matrices T and S, the braid and fusion matrices B and F, the fusion and OPE coefficients ${N_{ij}}^k$ and $C_{(i\bar{\imath})(j\bar{\jmath})}^{(k\bar{k})} = C_{ij}^{k} C_{\bar{\imath}\bar{\jmath}}^{\bar{k}})$ have been found, one may ask what is the possible set of boundary operators which is consistent with the bulk theory. This is answered by solving various constraints, called the sewing relations, satisfied by ${\alpha_i}^{\tilde{\alpha}\tilde{\beta}}$ and $C_{ijk}^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}$. The resulting set of boundary operators is said to be complete [88]. The sewing constraints were solved explicitly in several models [89], including non-diagonal cases [90].

4.5 Critical percolation

The critical bond percolation problem in statistical physics is an example which is solved using boundary operators. Since this is often discussed in the context of logarithmic CFTs, whose boundary theory is the main topic of Chap.3, we shall review it here following [2,7,14].

The problem we want to solve is defined as follows. We consider a two-dimensional lattice of horizontal length $a\ell$, vertical length $b\ell$ and spacing ℓ, and set electrodes on the left and right sides of the lattice. We start placing conducting needles randomly on the grid, and observe if electric current can run between the two electrodes (horizontal percolation). Obviously, when we put no needle on the lattice there is no way the current can run through, and when all the grids are filled with conducting needles the percolation has readily been achieved. Thus there must be some occupation probability p between 0 and 1 which is barely sufficient to achieve the percolation. We may take the thermodynamic limit of this system (letting $\ell \to 0$ and $a\ell$, $b\ell$ fixed). Then there is a critical occupation probability p_c such that the horizontal percolation probability π_h is 1 for $p > p_c$ and $\pi_h = 0$ for $p < p_c$. The system at $p = p_c$ is called the critical bond percolation. At $p = p_c$, π_h still depends on the aspect ratio $r = a/b$. Our problem is to find π_h as a function of r.

This percolation problem is translated into $Q \to 1$ limit of the Q-state Potts model in two dimension [95, 96]. The interaction energy of the Q-state Potts model is $J \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j}$, where the sum is over nearest-neighbours and the indices i and j label one of the Q states.

The partition function of the Q-state Potts model is then

$$Z = \sum_{\text{config}} \prod_{\langle ij \rangle} \exp\left\{-\beta J \delta_{\sigma_i \sigma_j}\right\}. \tag{1.218}$$

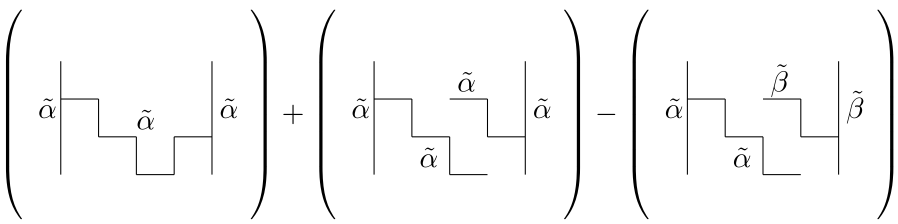
Defining $1/(1-p) = e^{-\beta J}$, one may rewrite this partition function as

$$Z = \sum_{\text{activation}} p^B (1-p)^{b-B} Q^{N_c}, \tag{1.219}$$

where b is the number of bonds, B is the number of activated bonds (needles), and N_c is the number of disjoint clusters. The Q-state Potts model has been mapped into a system with active bonds of Q possible colours (probability $= p$) and inert bonds (probability$= 1-p$). It is now obvious that the critical percolation is realised by taking $Q \to 1$ limit of this system.

The relation between the Q-state Potts models and the $\mathcal{M}(m+1, m)$ unitary Virasoro minimal series ($m = 3$ is the Ising model) is well established for $Q = 2, 3, 4$. The correspondence is given by $Q = 4\cos^2(\pi/(m+1))$ for these models. Extrapolating this formula for arbitrary value of m, the bond percolation problem then correspond to the minimal model of $\mathcal{M}(3, 2)$, whose central charge is $c = 0$. Using this correspondence, the percolation problem is described in a boundary CFT language as follows. Let us consider a rectangle with a pair of opposing sides having free boundary condition $\tilde{f}$ (Fig.1.6). The remaining two sides have fixed boundary conditions $\tilde{\alpha}$ and $\tilde{\beta}$, with fixed 'colours' out of the Q possible colours of the activated bonds. Such a configuration is realised by inserting four boundary (changing) operators at the four corners of the rectangle. The boundary operator $\psi^{\tilde{\alpha}\tilde{f}}$, which changes a fixed boundary condition $\tilde{\alpha}$ to the free boundary condition $\tilde{f}$, is identified as $\phi_{1,2}$ by analogy with the Q-state Potts models of $Q = 2, 3, 4$.

The crossing probability π_h is obtained by calculating the partition functions for the configurations



(1.220)

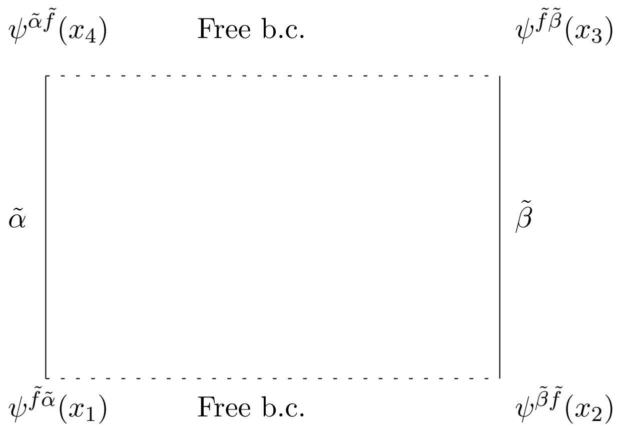


Figure 1.6: Horizontal bond percolation is modelled by a rectangle with the free boundary condition on the top and bottom sides, and fixed boundary conditions on the left and right sides.

and taking the limit $Q \to 1$ (that is, $\tilde{\alpha} = \tilde{\beta}$) afterwards. In this graphical notation, the first term indicates configurations with the left and right boundaries linked by bonds of the same colour $\tilde{\alpha}$. The second term means configurations with no percolation, with left and right boundaries in the same colour $\tilde{\alpha}$, and the third one means no percolation with left and right boundaries in different colours ($\tilde{\alpha}$ and $\tilde{\beta}$). Note that after taking the limit the last two terms cancel and only the first one (realising the percolation) survives. If we write the partition function for the configurations with the boundary condition (or colour) $\tilde{\alpha}$ on the left and $\tilde{\beta}$ on the right as $Z_{\tilde{\alpha}\tilde{\beta}}$, the crossing probability is given by

$$\pi_h = \lim_{Q\to 1} (Z_{\tilde{\alpha}\tilde{\alpha}} - Z_{\tilde{\alpha}\tilde{\beta}}), \tag{1.221}$$

since the first two terms of the graphical representation (1.220) are $Z_{\tilde{\alpha}\tilde{\alpha}}$ and the last one is $Z_{\tilde{\alpha}\tilde{\beta}}$. These partition functions for particular boundary conditions are given by the four-point functions of the boundary operators. Then up to a multiplicative constant we have

$$\begin{aligned} \pi_h &\sim \langle \psi^{\tilde{f}\tilde{\alpha}}(x_1)\psi^{\tilde{\alpha}\tilde{f}}(x_2)\psi^{\tilde{f}\tilde{\beta}}(x_3)\psi^{\tilde{\beta}\tilde{f}}(x_4)\rangle_{Q=1} \\ &= \langle \phi_{1,2}(x_1)\phi_{1,2}(x_2)\phi_{1,2}(x_3)\phi_{1,2}(x_4)\rangle_{Q=1}. \end{aligned} \tag{1.222}$$

The four-point function $\langle \phi_{1,2}(x_1)\phi_{1,2}(x_2)\phi_{1,2}(x_3)\phi_{1,2}(x_4)\rangle$ is found by solving a second order ordinary differential equation, as in Subsec.1.3.4. Introducing the cross ratio $\eta = [(z_1 -$

$z_2)(z_3 - z_4)]/[(z_1 - z_3)(z_2 - z_4)]$, where z_i are the coordinates after the Schwartz-Christoffel transformation $x_i \to z_i$ mapping the interior of the rectangle to the upper half plane, the differential equation for $\pi_h(\eta) = g(\eta)$ is

$$\eta(1-\eta)\frac{d^2g(\eta)}{d\eta^2} + \frac{2(1-2\eta)}{3}\frac{dg(\eta)}{d\eta} = 0. \tag{1.223}$$

This differential equation has two independent solutions. One is $g(\eta) = const$, and the other is

$$g(\eta) = \eta^{1/3}F(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; \eta). \tag{1.224}$$

The crossing probability π_h is a linear combination of the two solutions. The coefficients are determined by demanding $\pi_h \to 1$ when the rectangle is infinitely narrow and $\pi_h \to 0$ when it is infinitely wide. The percolation probability is then found to be

$$\pi_h(\eta) = \frac{3\Gamma(2/3)}{\Gamma(1/3)^2}\eta^{1/3}F(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; \eta). \tag{1.225}$$

This analytic result is compared with numerical calculations and exhibits excellent agreement [7, 97, 98]. Although the extrapolation of the $Q \neq 1$ results to $Q \to 1$ may seem somewhat speculative, this agreement justifies the method of the analysis as well as the underlying concepts such as conformal invariance and boundary operators.

Finally we emphasise that this CFT at $c = 0$ is *not* the minimal model of $\mathcal{M}(3,2)$, which consists only of the identity operator. Recall that the differential equation (1.223) has two independent solutions, one corresponds to the conformal block $\phi_{1,1}$ and the other to $\phi_{1,3}$. Obviously, the former solution is the constant and the latter is (1.224). If we were dealing with $\mathcal{M}(3,2)$ minimal model, the solution (1.224) should have been discarded since it is associated with the operator outside the Kac table. Hence the percolation problem must be considered in the framework of a CFT with extended conformal grid, possibly to $\mathcal{M}(9,6)$ [2, 74]. From this example we may expect the existence of bona-fide CFTs which are not minimal models but something that should be called 'next-to-minimal' models, which may well include logarithmic operators [99]. The existence of such statistical models motivates the study of boundary logarithmic CFT.

Chapter 2

Coulomb-gas approach for boundary conformal field theory

In this chapter we discuss a construction of boundary states based on the Coulomb-gas formalism of Dotsenko and Fateev. After addressing the motivation and advertising the merits of this formalism, we start in Sec.2.2 by defining the charged bosonic Fock space (CBFS) for the theory on an annulus. We then construct boundary coherent states on CBFS and find conditions for the conformal invariance of such states. In Sec.2.3 the charge-neutrality conditions for the boundary Coulomb-gas are considered and the closed-string channel amplitudes are calculated. We illustrate the method in Sec.2.4 using the Ising model as an example, and in Sec.2.5 we conclude by discussing possible applications to other models. The result of this section has been published in [100].

1 Why Coulomb-gas?

Among the series of seminal papers on BCFT written in the eighties, two results are of particular importance: one is the method to calculate boundary correlation functions [78], and the other is the observation that bulk operator content is restricted by boundary conditions [101]. The former result is essential from a phenomenological point of view since correlation functions are the only observable quantities which may be compared with measurements, while the latter result is more conceptual and leads to the study of algebraic aspects of BCFT, namely, the systematic classification of boundary states based on the modular transformations [79, 88, 91, 102, 103].

These two aspects of BCFT seem to be, in the author's opinion, somewhat remote. Although the link between them may be made through boundary operators [80, 94, 101][1], a more direct connection is desirable. In string theory, boundary states are defined on the Fock space and correlation functions are obtained by inserting vertex operators on string world sheets. If such a free-field construction of boundary states is available for any CFT, it should be in principle possible to calculate correlation functions directly from algebraically classified boundary states. The formalism we present in this chapter is intended to be a first step in this direction.

So far the boundary theory of Coulomb-gas picture was only considered by Schulze [104], where the CFT is defined on the half plane and the results for the Ising model are reproduced in a contour integration form using the mirroring technique of [78]. In the following we formulate the Coulomb-gas picture on an annulus and discuss modular properties of boundary states built on a Fock space on the boundaries. The Coulomb-gas formalism is quite attractive in many respects. It is a Lagrangian theory and hence the whole theory may be constructed from an action (which is comfortable for those who were brought up in the physics community). Extension to the non-critical (massive) regime is also possible by perturbation. Thus once the boundary theory of the Coulomb-gas picture is formulated, we may expect development of the theory in many directions. In the following, however, we shall not discuss such topics but shall restrict ourselves to presenting the formalism.

2 CBFS with boundary

In this section we discuss the Fock space representation of BCFT where the interplay between holomorphic and antiholomorphic sectors is important. Let us start with the geometry of the upper half-plane. We define $\zeta = x + iy$, $x,\ y \in \mathbb{R}$ and consider a CFT defined on the region $\mathrm{Im}\zeta \geq 0$. The boundary is $y = 0$, or $\zeta = \bar{\zeta}$. As is discussed in Subsec.1.4.2, the antiholomorphic dependence of the correlators on the upper half plane may be mapped into the holomorphic dependence on the lower half plane [78]. This introduces a mirror image on the lower half plane, and the boundary condition tells how the images on the upper and lower

[1] If we can find a set of boundary operators which reside on a algebraically classified boundary state, the leading terms of a correlation function near the boundary should be obtained by the OPE expanded with such boundary operators.

half-planes are glued on the mirror, $\zeta = \bar{\zeta}$. The energy-momentum tensor on the lower half plane is obtained by the mapping from the upper half plane, $T(\zeta^*) = \bar{T}(\bar{\zeta})$. The condition on the boundary

$$\left[T(\zeta) - \bar{T}(\bar{\zeta})\right]_{\zeta=\bar{\zeta}} = 0, \tag{2.1}$$

indicates the absence of the energy-momentum flow across the boundary. Since the energy-momentum tensor is the generator of conformal transformations, (2.1) also means the conformal invariance of the boundary. Going from the upper half plane (or holomorphic part) to the lower half plane (antiholomorphic part) is generally accompanied by a *parity* transformation $\mathcal{P}$. The free boson transforms under $\mathcal{P}$ as $\varphi(\zeta) \to \Omega\bar{\varphi}(\bar{\zeta})$, $\Omega = \pm 1$. This leads to the condition on the boundary

$$\left[\varphi(\zeta) - \Omega\bar{\varphi}(\bar{\zeta})\right]_{\zeta=\bar{\zeta}} = 0. \tag{2.2}$$

When $\Omega = 1$, the non-chiral free boson $\Phi(\zeta, \bar{\zeta}) = \varphi(\zeta) + \bar{\varphi}(\bar{\zeta})$ is a scalar and the boundary condition is called Neumann, whereas when $\Omega = -1$, $\Phi(\zeta, \bar{\zeta})$ is a pseudo-scalar and such a boundary condition is called Dirichlet. Under the parity transformation the chiral vertex operators $V_\alpha(\zeta) =: e^{i\sqrt{2}\alpha\varphi(\zeta)}$: are mapped into $\bar{V}_\alpha(\bar{\zeta}) =: e^{i\sqrt{2}\alpha\Omega\bar{\varphi}(\bar{\zeta})}$:. When $\Omega = -1$ (Dirichlet) the mirror image has a charge $\Omega\alpha = -\alpha$ which has the opposite sign from the original one. In the Neumann case ($\Omega = 1$), the mirror and the original vertex operators have the same charge α. The Coulomb-gas system on the half plane was studied in [104], where the boundary correlation functions of the Ising model are calculated using the mirroring technique [78].

In this chapter we mainly study BCFT defined on a finite cylinder, or an annulus. We consider a finite cylinder of length T and circumference L, or an annulus on the z-plane with $1 \leq |z| \leq \exp(2\pi T/L)$. We also introduce a modular parameter as $\tilde{q} = e^{2\pi i\tilde{\tau}}$, $\tilde{\tau} = 2iT/L$. With this the annulus is $1 \leq |z| \leq \tilde{q}^{-1/2}$. We regard this cylinder as a propagating closed string, and call the direction along it as *time*. A merit of considering such a geometry is that the familiar energy-momentum tensor for the full-plane may be used without modification. We conformally map a semi-annular domain in the upper-half ζ-plane onto a full-annulus in the z-plane by $z = \exp(-2\pi iw/L)$ and $w = (T/\pi)\ln\zeta$. The boundary $\zeta = \bar{\zeta}$ is then mapped on the z-plane to $|z| = 1$, $\exp(2\pi T/L)$. Since the z-plane allows radial quantization, the conformal invariance (2.1) on the $|z| = 1$ boundary becomes the conditions on the quantum states $|B\rangle$ [79, 93],

$$(L_k - \bar{L}_{-k})|B\rangle = 0. \tag{2.3}$$

As $\varphi(\zeta)$ and $\bar{\varphi}(\bar{\zeta})$ are not primary, the condition (2.2) cannot be mapped to the annulus. However, the derivatives of *uncharged* bosons are primary and

$$\left[\partial\varphi(\zeta) - \Omega\bar{\partial}\bar{\varphi}(\bar{\zeta})\right]_{\zeta=\bar{\zeta}} = 0 \tag{2.4}$$

on the ζ-plane is mapped on the z-plane as

$$(a_n + \Omega\bar{a}_{-n})|B\rangle = 0. \tag{2.5}$$

This expression no longer makes sense for the *charged* bosons since $\partial\varphi$ and $\bar{\partial}\bar{\varphi}$ cease to be primary when they are couple to the background curvature. However, (2.3) is still valid and is indeed a necessary condition for the conformally invariant boundary states. The vertex operators are safely mapped to z-plane since they remain primary. In the rest of this section we construct a Fock space representation of boundary states which satisfy the conformal invariance condition (2.3).

Our starting point is recalling that a BCFT consists of a pair of chiral CFTs whose holomorphic and antiholomorphic sectors are glued together on the boundary. The construction of the boundary states then requires a Fock space which is common to both holomorphic and antiholomorphic sectors. As we have the same central charge c for both holomorphic and antiholomorphic sectors, α_0, which is related to c by (1.114), is common to both sectors[2], although we are free to choose different vacuum charges for each sector. Hence let us define the highest-weight vectors at the two boundaries of the annulus as $|\alpha, \bar{\alpha}; \alpha_0\rangle$ and $\langle\alpha, \bar{\alpha}; \alpha_0|$,

[2]Even if one relaxes this condition and starts by allocating different background charges α_0 and $\bar{\alpha}_0$ to holomorphic and antiholomorphic sectors, the condition (2.3) restricts either $\alpha_0 = \pm\bar{\alpha}_0$. For $\alpha_0 = -\bar{\alpha}_0$ we have $\Omega = -1$ (Dirichlet) and $\alpha - \bar{\alpha} - 2\alpha_0 = 0$ instead of (2.20) and (2.21), respectively, but this merely flips the sign of all antiholomorphic charges and thus does not give any new results.

satisfying

$$a_0|\alpha,\bar{\alpha};\alpha_0\rangle = \sqrt{2}\alpha|\alpha,\bar{\alpha};\alpha_0\rangle, \tag{2.6}$$

$$\bar{a}_0|\alpha,\bar{\alpha};\alpha_0\rangle = \sqrt{2}\bar{\alpha}|\alpha,\bar{\alpha};\alpha_0\rangle, \tag{2.7}$$

$$\langle\alpha,\bar{\alpha};\alpha_0|a_0 = \langle\alpha,\bar{\alpha};\alpha_0|\sqrt{2}\alpha, \tag{2.8}$$

$$\langle\alpha,\bar{\alpha};\alpha_0|\bar{a}_0 = \langle\alpha,\bar{\alpha};\alpha_0|\sqrt{2}\bar{\alpha}, \tag{2.9}$$

which are essentially the direct products of holomorphic and antiholomorphic parts of (1.122), (1.132). The state $|\alpha,\bar{\alpha};\alpha_0\rangle$ has *holomorphic* charge α and *antiholomorphic* charge $\bar{\alpha}$, and $\langle\alpha,\bar{\alpha};\alpha_0|$ has holomorphic charge $-\alpha$ and antiholomorphic charge $-\bar{\alpha}$. The mode operators of the antiholomorphic sector are defined, similarly to the holomorphic part (1.120), by the mode expansion of $\bar{\varphi}(\bar{z})$ as

$$\bar{\varphi}(\bar{z}) = \bar{\varphi}_0 - i\bar{a}_0 \ln \bar{z} + i\sum_{n\neq 0}\frac{\bar{a}_n}{n}\bar{z}^{-n}. \tag{2.10}$$

The antiholomorphic mode operators satisfy the same Heisenberg algebra as their holomorphic counterpart:

$$[\bar{a}_m, \bar{a}_n] = m\delta_{m+n,0}, \tag{2.11}$$

$$[\bar{\varphi}_0, \bar{a}_0] = i. \tag{2.12}$$

There is a subtlety in the treatment of $\bar{\varphi}_0$ and $\bar{a}_0$ since the zero mode of the boson $\Phi(z,\bar{z})$ does not naturally decouple into left and right. We split them into two identical and independent copies such that $[\varphi_0, \bar{a}_0] = [\bar{\varphi}_0, a_0] = 0$. In such decomposition the existence of the dual field is implicit [105]. The highest-weight vector $|\alpha,\bar{\alpha};\alpha_0\rangle$ is annihilated by the action of $a_{n>0}$ and $\bar{a}_{n>0}$, and the contravariant highest-weight vector $\langle\alpha,\bar{\alpha};\alpha_0|$ is annihilated by $a_{n<0}$ and $\bar{a}_{n<0}$. Following (1.134) we assume the highest-weight vectors are normalised as

$$\langle\alpha,\bar{\alpha};\alpha_0|\beta,\bar{\beta};\alpha_0\rangle = \kappa'\delta_{\alpha,\beta}\delta_{\bar{\alpha},\bar{\beta}}, \tag{2.13}$$

where κ' is a normalisation factor, which may be set to 1 if the sector is unitary. If κ' is negative we set it to -1.

We are looking for conformally invariant boundary states built on the highest-weight vectors $|\alpha, \bar{\alpha}; \alpha_0\rangle$ and $\langle\alpha, \bar{\alpha}; \alpha_0|$. Since we know that such states for (uncharged) bosonic strings are found in the form of coherent states in string theory, let us start with an ansatz

$$|B_{\alpha,\bar{\alpha};\alpha_0}\rangle_\Omega = \prod_{k>0} \exp\left(-\frac{\Omega}{k} a_{-k}\bar{a}_{-k}\right) |\alpha, \bar{\alpha}; \alpha_0\rangle, \tag{2.14}$$

$$_\Omega\langle B_{\alpha,\bar{\alpha};\alpha_0}| = \langle\alpha, \bar{\alpha}; \alpha_0| \prod_{k>0} \exp\left(-\frac{1}{k\Omega} a_k\bar{a}_k\right). \tag{2.15}$$

These states satisfy

$$(a_n + \Omega\bar{a}_{-n})|B_{\alpha,\bar{\alpha};\alpha_0}\rangle_\Omega = 0 \ \ (n \neq 0), \tag{2.16}$$

$$_\Omega\langle B_{\alpha,\bar{\alpha};\alpha_0}|(a_n + \Omega\bar{a}_{-n}) = 0 \ \ (n \neq 0). \tag{2.17}$$

Using the expression of Virasoro operators (1.123) (1.124) we see that $|B_{\alpha,\bar{\alpha};\alpha_0}\rangle_\Omega$ does *not* satisfy the condition (2.3) straightaway. For example, we have

$$\begin{aligned}
&(L_n - \bar{L}_{-n})|B_{\alpha,\bar{\alpha};\alpha_0}\rangle_\Omega \\
&= \prod_{k>0} \exp\left(-\frac{\Omega}{k} a_{-k}\bar{a}_{-k}\right) \\
&\times \left\{\sqrt{2}\bar{a}_{-n}[(\Omega - 1)n\alpha_0 + (\Omega + 1)\alpha_0 - \Omega\alpha - \bar{\alpha}]\right. \\
&\left. + \frac{1}{2} \sum_{0<j<n} \bar{a}_{-j}\bar{a}_{j-n}(\Omega^2 - 1)\right\} |\alpha, \bar{\alpha}; \alpha_0\rangle
\end{aligned} \tag{2.18}$$

for $n > 0$, and

$$\begin{aligned}
&(L_0 - \bar{L}_0)|B_{\alpha,\bar{\alpha};\alpha_0}\rangle_\Omega \\
&= \prod_{k>0} \exp\left(-\frac{\Omega}{k} a_{-k}\bar{a}_{-k}\right) \\
&\times \{(\alpha - \bar{\alpha})(\alpha + \bar{\alpha} - 2\alpha_0)\} |\alpha, \bar{\alpha}; \alpha_0\rangle,
\end{aligned} \tag{2.19}$$

which are in general not zero. However, it can be easily seen that the expressions (2.18) and

(2.19) do vanish when

$$\Omega = 1, \tag{2.20}$$

and

$$\alpha + \bar{\alpha} - 2\alpha_0 = 0, \tag{2.21}$$

even for $\alpha_0 \neq 0$. It is easily verified that these conditions also lead to $(L_n - \bar{L}_{-n})|B_{\alpha,\bar{\alpha};\alpha_0}\rangle_\Omega = 0$ for $n < 0$ and are indeed a sufficient condition for the conformal invariance. Similarly it can be checked that ${}_\Omega\langle B_{\alpha,\bar{\alpha};\alpha_0}|(L_n - \bar{L}_{-n}) = 0$ as long as (2.20) and (2.21) hold. Note that the "Dirichlet" condition $\Omega = -1$ is not compatible with the conformal invariance for non-zero α_0 $(= \bar{\alpha}_0)$ because of the term proportional to n in (2.18). In the rest of this chapter we shall consider the conformally invariant boundary states satisfying the conditions (2.20) and (2.21). Since the antiholomorphic charge is determined by the condition (2.21), such boundary states are characterised by only one parameter α, apart from the value of the background charge α_0 which is fixed by the central charge. For simplicity we shall denote these boundary states as

$$|B(\alpha)\rangle = |B_{\alpha,2\alpha_0-\alpha;\alpha_0}\rangle_{\Omega=1}, \tag{2.22}$$

and

$$\langle B(\alpha)| = {}_{\Omega=1}\langle B_{\alpha,2\alpha_0-\alpha;\alpha_0}|. \tag{2.23}$$

The background charge α_0 is suppressed since no confusion arises.

3 Coherent and consistent boundary states

Identifying boundary states which may be realised in a physical system is one of the main goals in BCFT. In order to study the modular properties of the coherent states we defined in the last section and discuss their physical relevance, we need to calculate the closed string amplitudes between $\langle B(\alpha)|$ and $|B(\beta)\rangle$. Such amplitudes generally involve screening operators, or floating charges in the bulk. Let us consider the situation where m positive (α_+) and n negative (α_-) floating charges are present. The closed-string amplitude for such a process is

$$\begin{aligned} \mathcal{A}_{\alpha,\beta} &= \langle B(\alpha)|e^{-TH}Q_+^m Q_-^n \bar{Q}_+^m \bar{Q}_-^n|B(\beta)\rangle \\ &= \langle B(\alpha)|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}Q_+^m Q_-^n \bar{Q}_+^m \bar{Q}_-^n|B(\beta)\rangle, \end{aligned} \tag{2.24}$$

where $Q_\pm$ is defined in (1.116) and

$$\bar{Q}_\pm \equiv \oint d\bar{z}\bar{V}_\pm(\bar{z}), \tag{2.25}$$

$$\bar{V}_\pm(\bar{z}) =: e^{i\sqrt{2}\alpha_\pm\bar{\varphi}(\bar{z})} : . \tag{2.26}$$

The integration contours must be non-self-intersecting closed curves with non-trivial homotopy. In our geometry such contours are the ones which simply go around the cylinder just once. A comment on the uniqueness of the amplitude (2.24) is in order. It is easy to show that $[Q_+, Q_-] = 0$, $[\bar{Q}_+, \bar{Q}_-] = 0$. Also, $[Q_\pm, \bar{Q}_\pm] = 0$, $[Q_\pm, \bar{Q}_\mp] = 0$ because the holomorphic and antiholomorphic mode operators commute. As the screening operators have trivial conformal dimension, they commute with the Virasoro operators: $[L_n, Q_\pm] = 0$, $[\bar{L}_n, \bar{Q}_\pm] = 0$. In particular, $[L_0, Q_\pm] = 0$ and $[\bar{L}_0, \bar{Q}_\pm] = 0$. Hence the order and the position of the screening operators do not matter and the amplitude with m positive and n negative floating charges may be always written in the form (2.24).

The numbers of the screening charges m and n are not arbitrary but they must satisfy the charge neutrality condition (otherwise the amplitude vanishes). Note that our formalism (see the normalisation (2.13)) demands charge neutrality in both holomorphic and antiholomorphic sectors. In the holomorphic sector, we have charges $-\alpha$ and β on the boundaries, and m positive and n negative screening charges in the bulk. The total charge in the holomorphic part is then

$$-\alpha + \beta + m\alpha_+ + n\alpha_-, \tag{2.27}$$

which must be zero. Similarly, the total charge in the antiholomorphic part is $-\bar{\alpha} + \bar{\beta} + m\alpha_+ + n\alpha_-$, or, using the condition (2.21),

$$\alpha - \beta + m\alpha_+ + n\alpha_-, \tag{2.28}$$

which is also zero. Since the sum of the holomorphic and antiholomorphic charges must also vanish, summing the above two expressions we have $m\alpha_+ + n\alpha_- = 0$. Now let us recall that the screening charges of the minimal models are characterised by two co-prime integers p and p' ($p > p'$) as $\alpha_+ = \sqrt{p/p'}$, $\alpha_- = -\sqrt{p'/p}$. Then we have

$$pm - p'n = 0. \tag{2.29}$$

Since p and p' are co-prime, m and n are written using an integer l as $m = lp'$, $n = lp$. This means the net floating charges must vanish in both holomorphic and antiholomorphic sectors. The simplest charge configuration obeying this condition is $m = n = 0$, or *no* screening operators. In this case the amplitude (2.24) is particularly easily evaluated. The oscillating part is calculated with the Heisenberg algebras (1.119) (2.11) and repeated use of Hausdorff formula, as

$$\prod_{k=1}^{\infty} \frac{1}{1-q^k} = \frac{q^{1/24}}{\eta(\tau)}. \tag{2.30}$$

The zero-mode part,

$$\langle \alpha, \bar{\alpha}; \alpha_0 | (q^{1/2})^{(a_0^2+\bar{a}_0^2)/2 - \sqrt{2}\alpha_0(a_0+\bar{a}_0) - c/12} | \beta, \bar{\beta}; \alpha_0 \rangle, \tag{2.31}$$

is simplified with the central charge (1.114), the condition on boundary charges for conformal invariance (2.21) and the operation of zero-modes on the highest-weight vectors (2.6)-(2.9), as

$$\langle \alpha, 2\alpha_0 - \alpha; \alpha_0 | q^{\alpha^2 - 2\alpha_0\alpha + \alpha_0^2 - 1/24} | \beta, 2\alpha_0 - \beta; \alpha_0 \rangle. \tag{2.32}$$

Using the normalisation of the highest weight vectors (2.13) we have

$$\begin{aligned} \mathcal{A}_{\alpha,\beta} &= \langle B(\alpha) | (\tilde{q}^{1/2})^{L_0 + \bar{L}_0 - c/12} | B(\beta) \rangle \\ &= \frac{\tilde{q}^{(\alpha-\alpha_0)^2}}{\eta(\tilde{\tau})} \kappa' \delta_{\alpha,\beta}. \end{aligned} \tag{2.33}$$

Note the similarity of these amplitudes to the characters (1.127) of CBFS. This is a consequence of the fact that the Schottky double of the annulus is the torus.

In order to describe the minimal models, it is convenient to introduce boundary states

$|a_{r,s}\rangle$ and $|a_{r,-s}\rangle$ defined as

$$|a_{r,s}\rangle = \sum_{k\in\mathbb{Z}} |B(k\sqrt{pp'} + \alpha_{r,s})\rangle, \tag{2.34}$$

$$|a_{r,-s}\rangle = \sum_{k\in\mathbb{Z}} |B(k\sqrt{pp'} + \alpha_{r,-s})\rangle. \tag{2.35}$$

Similarly we define

$$\langle a_{r,s}| = \sum_{k\in\mathbb{Z}} \langle B(k\sqrt{pp'} + \alpha_{r,s})|, \tag{2.36}$$

$$\langle a_{r,-s}| = \sum_{k\in\mathbb{Z}} \langle B(k\sqrt{pp'} + \alpha_{r,-s})|. \tag{2.37}$$

These are linear sums of countably many coherent states (2.22) and (2.23) defined in the previous section. Using (2.33) it is shown that

$$\begin{aligned} \langle a_{r,s}|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|a_{r',s'}\rangle &= \frac{\Theta_{pr-p's,pp'}(\tilde{\tau})}{\eta(\tilde{\tau})}\kappa'\delta_{r,r'}\delta_{s,s'} \\ &= \frac{\Theta_{pr-p's,pp'}(\tilde{\tau})}{\eta(\tilde{\tau})}\delta_{r,r'}\delta_{s,s'}, \end{aligned} \tag{2.38}$$

$$\begin{aligned} \langle a_{r,-s}|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|a_{r',-s'}\rangle &= \frac{\Theta_{pr+p's,pp'}(\tilde{\tau})}{\eta(\tilde{\tau})}\kappa'\delta_{r,r'}\delta_{s,s'} \\ &= -\frac{\Theta_{pr+p's,pp'}(\tilde{\tau})}{\eta(\tilde{\tau})}\delta_{r,r'}\delta_{s,s'}, \end{aligned} \tag{2.39}$$

and

$$\langle a_{r,\pm s}|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|a_{r',\mp s'}\rangle = 0. \tag{2.40}$$

Here, we have assumed $1 \leq r, r' < p'$ and $1 \leq s, s' < p$. See App. A for our convention of Jacobi theta functions. We have set $\kappa' = 1$ in (2.38) and $\kappa' = -1$ in (2.39). This means the states $|a_{r,s}\rangle$, $\langle a_{r,s}|$ belong to an unitary sector whereas $|a_{r,-s}\rangle$, $\langle a_{r,-s}|$ belong to a non-unitary sector. The amplitudes include all the theta functions appearing in the characters of minimal models (1.94) and thus we have reproduced the necessary set of boundary states covering the right hand side of Cardy's consistency condition (1.201). We shall see this in detail for the Ising model in the next section. It can be easily checked by using (2.38) - (2.40) and the character formula (1.94) that the states defined as sums of the coherent states,

$$|(r,s)\rangle\rangle = |a_{r,s}\rangle + |a_{r,-s}\rangle, \tag{2.41}$$

$$\langle\langle(r,s)| = \langle a_{r,s}| + \langle a_{r,-s}|, \tag{2.42}$$

diagonalise the amplitude and reproduce the minimal characters. These states $|(r,s)\rangle\rangle$ may then be regarded as the Ishibashi states.

Before discussing the Ising model, we have three points to make about the boundary states $\{|a_{r,s}\rangle, |a_{r,-s}\rangle\}$. Firstly, the amplitudes (2.38), (2.39), (2.40) are diagonal, i.e. the boundary states are all orthogonal to each other. This is a consequence of the diagonal amplitude (2.33). Indeed, since the boundary charges $k\sqrt{pp'} + \alpha_{r,\pm s}$ are all different for each set of $(r, \pm s, k)$ and the boundary states $\{|a_{r,s}\rangle, |a_{r,-s}\rangle\}$ contain no charges in common, the amplitudes (2.38), (2.39) must vanish unless $(r,s) = (r',s')$. The second point is that these boundary states are unique (besides the degeneracy $(r,s) \leftrightarrow (p'-r, p-s)$) as long as we want to reproduce the theta functions as amplitudes between such boundaries. The infinite sum expressions (A.15) for the theta functions are power series of q, and the power is related to the boundary charge through the expression (2.33). By superimposing the boundary charges appearing in the expression of theta functions, the boundary states are constructed without ambiguity. Third, the negative-norm states $|a_{r,-s}\rangle$ seem to be unavoidable even for the unitary minimal models. The highest-weight vector $|\alpha, \bar{\alpha}; \alpha_0\rangle$ is built on the vacuum $|0,0;\alpha_0\rangle$ by operating with $e^{i\sqrt{2}\alpha\varphi_0}$ and $e^{i\sqrt{2}\bar{\alpha}\bar{\varphi}_0}$, and its norm κ' is due to the normalisation of the vacuum $\langle 0,0;\alpha_0|0,0;\alpha_0\rangle = \kappa'$. This κ' may be rescaled to an arbitrary real number as long as it is either positive or negative definite, but the sign cannot be changed by the rescaling. The states with $\kappa' = 1$ and $\kappa' = -1$ (that is, $|a_{r,s}\rangle$ and $|a_{r,-s}\rangle$ above) therefore belong to different sectors with no intersection.

4 Ising model boundary states

In this section we demonstrate that the boundary states constructed in the previous section are enough to reproduce the known physical boundary states of the Ising model, by a parallel discussion with Cardy's original paper [79]. We shall follow the procedure of Subsec.1.4.3 using the coherent states on CBFS instead of the Ishibashi states.

From the Rocha-Caridi formula (1.94) the characters for the three representations I, ϵ, σ of the Ising model are expressed as

$$\chi_I(q) = \chi_{1,1}(q) = \frac{1}{\eta(\tau)}\left[\Theta_{1,12}(\tau) - \Theta_{7,12}(\tau)\right], \tag{2.43}$$

$$\chi_\epsilon(q) = \chi_{2,1}(q) = \frac{1}{\eta(\tau)}\left[\Theta_{5,12}(\tau) - \Theta_{11,12}(\tau)\right], \tag{2.44}$$

$$\chi_\sigma(q) = \chi_{2,2}(q) = \frac{1}{\eta(\tau)}\left[\Theta_{2,12}(\tau) - \Theta_{10,12}(\tau)\right]. \tag{2.45}$$

These expressions are equivalent to (1.207)-(1.209) in Subsec.1.4.3 due to the relations (A.20)-(A.22). Using the modular transformation formula of the theta and Dedekind functions (A.24) they are written as

$$\chi_I(q) = \frac{\Theta_{1,12}(\tilde{\tau}) + \Theta_{5,12}(\tilde{\tau}) - \Theta_{7,12}(\tilde{\tau}) - \Theta_{11,12}(\tilde{\tau})}{2\eta(\tilde{\tau})} + \frac{\Theta_{2,12}(\tilde{\tau}) - \Theta_{10,12}(\tilde{\tau})}{\sqrt{2}\eta(\tilde{\tau})}, \tag{2.46}$$

$$\chi_\epsilon(q) = \frac{\Theta_{1,12}(\tilde{\tau}) + \Theta_{5,12}(\tilde{\tau}) - \Theta_{7,12}(\tilde{\tau}) - \Theta_{11,12}(\tilde{\tau})}{2\eta(\tilde{\tau})} - \frac{\Theta_{2,12}(\tilde{\tau}) - \Theta_{10,12}(\tilde{\tau})}{\sqrt{2}\eta(\tilde{\tau})}, \tag{2.47}$$

$$\chi_\sigma(q) = \frac{\Theta_{1,12}(\tilde{\tau}) - \Theta_{5,12}(\tilde{\tau}) - \Theta_{7,12}(\tilde{\tau}) + \Theta_{11,12}(\tilde{\tau})}{\sqrt{2}\eta(\tilde{\tau})}. \tag{2.48}$$

These are the character functions appearing in the open-string channel (left hand side) of the consistency equation (1.201). In the closed string channel (right hand side) of (1.201) we expand the boundary states in terms of $|a_{r,\pm s}\rangle$ and $\langle a_{r,\pm s}|$ defined in (2.34) - (2.37), with $1 \leq r \leq 2$, $1 \leq s \leq 3$, and $3s < 4r$. In the Ising model the non-trivial amplitudes (2.38), (2.39) are

$$\begin{aligned}
\langle a_{1,1}|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|a_{1,1}\rangle &= \Theta_{1,12}(\tilde{\tau})/\eta(\tilde{\tau}), && (2.49)\\
\langle a_{2,2}|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|a_{2,2}\rangle &= \Theta_{2,12}(\tilde{\tau})/\eta(\tilde{\tau}), && (2.50)\\
\langle a_{2,1}|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|a_{2,1}\rangle &= \Theta_{5,12}(\tilde{\tau})/\eta(\tilde{\tau}), && (2.51)\\
\langle a_{1,-1}|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|a_{1,-1}\rangle &= -\Theta_{7,12}(\tilde{\tau})/\eta(\tilde{\tau}), && (2.52)\\
\langle a_{2,-2}|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|a_{2,-2}\rangle &= -\Theta_{10,12}(\tilde{\tau})/\eta(\tilde{\tau}), && (2.53)\\
\langle a_{2,-1}|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|a_{2,-1}\rangle &= -\Theta_{11,12}(\tilde{\tau})/\eta(\tilde{\tau}). && (2.54)
\end{aligned}$$

Using these amplitudes and equating the coefficients of $\Theta_{1,12}(\tilde{\tau})/\eta(\tilde{\tau})$, $\Theta_{2,12}(\tilde{\tau})/\eta(\tilde{\tau})$, $\Theta_{5,12}(\tilde{\tau})/\eta(\tilde{\tau})$, $\Theta_{7,12}(\tilde{\tau})/\eta(\tilde{\tau})$, $\Theta_{10,12}(\tilde{\tau})/\eta(\tilde{\tau})$ and $\Theta_{11,12}(\tilde{\tau})/\eta(\tilde{\tau})$ on both sides of (1.201), we have

$$\frac{1}{2}n^I_{\tilde{\alpha}\tilde{\beta}} + \frac{1}{2}n^\epsilon_{\tilde{\alpha}\tilde{\beta}} + \frac{1}{\sqrt{2}}n^\sigma_{\tilde{\alpha}\tilde{\beta}} = \langle\tilde{\alpha}|a_{1,1}\rangle\langle a_{1,1}|\tilde{\beta}\rangle, \tag{2.55}$$

$$\frac{1}{\sqrt{2}}n^I_{\tilde{\alpha}\tilde{\beta}} - \frac{1}{\sqrt{2}}n^\epsilon_{\tilde{\alpha}\tilde{\beta}} = \langle\tilde{\alpha}|a_{1,2}\rangle\langle a_{1,2}|\tilde{\beta}\rangle, \tag{2.56}$$

$$\frac{1}{2}n^I_{\tilde{\alpha}\tilde{\beta}} + \frac{1}{2}n^\epsilon_{\tilde{\alpha}\tilde{\beta}} - \frac{1}{\sqrt{2}}n^\sigma_{\tilde{\alpha}\tilde{\beta}} = \langle\tilde{\alpha}|a_{1,3}\rangle\langle a_{1,3}|\tilde{\beta}\rangle, \tag{2.57}$$

$$\frac{1}{2}n^I_{\tilde{\alpha}\tilde{\beta}} + \frac{1}{2}n^\epsilon_{\tilde{\alpha}\tilde{\beta}} + \frac{1}{\sqrt{2}}n^\sigma_{\tilde{\alpha}\tilde{\beta}} = \langle\tilde{\alpha}|a_{1,-1}\rangle\langle a_{1,-1}|\tilde{\beta}\rangle, \tag{2.58}$$

$$\frac{1}{\sqrt{2}}n^I_{\tilde{\alpha}\tilde{\beta}} - \frac{1}{\sqrt{2}}n^\epsilon_{\tilde{\alpha}\tilde{\beta}} = \langle\tilde{\alpha}|a_{1,-2}\rangle\langle a_{1,-2}|\tilde{\beta}\rangle, \tag{2.59}$$

$$\frac{1}{2}n^I_{\tilde{\alpha}\tilde{\beta}} + \frac{1}{2}n^\epsilon_{\tilde{\alpha}\tilde{\beta}} - \frac{1}{\sqrt{2}}n^\sigma_{\tilde{\alpha}\tilde{\beta}} = \langle\tilde{\alpha}|a_{1,-3}\rangle\langle a_{1,-3}|\tilde{\beta}\rangle. \tag{2.60}$$

Let us find the coefficients assuming that they are real and $\langle\tilde{\alpha}|a_{r,\pm s}\rangle = \langle a_{r,\pm s}|\tilde{\alpha}\rangle$. We start by letting $\tilde{\alpha} = \tilde{\beta} = \tilde{0}$. The first equation (2.55) gives $|\langle\tilde{0}|a_{1,1}\rangle|^2 = 1/2$ and we can choose $\langle\tilde{0}|a_{1,1}\rangle = 1/\sqrt{2}$. Likewise, from (2.56) - (2.60) we find $\langle\tilde{0}|a_{2,2}\rangle = \langle\tilde{0}|a_{2,-2}\rangle = 2^{-1/4}$, $\langle\tilde{0}|a_{2,1}\rangle = \langle\tilde{0}|a_{1,-1}\rangle = \langle\tilde{0}|a_{2,-1}\rangle = 1/\sqrt{2}$. Next, letting $\tilde{\alpha} = \tilde{\epsilon}$ and $\tilde{\beta} = \tilde{0}$ we find $\langle\tilde{\epsilon}|a_{1,1}\rangle = \langle\tilde{\epsilon}|a_{2,1}\rangle = \langle\tilde{\epsilon}|a_{1,-1}\rangle = \langle\tilde{\epsilon}|a_{2,-1}\rangle = 1/\sqrt{2}$, and $\langle\tilde{\epsilon}|a_{2,2}\rangle = \langle\tilde{\epsilon}|a_{2,-2}\rangle = -2^{-1/4}$. Lastly, putting $\tilde{\alpha} = \tilde{\sigma}$ and $\tilde{\beta} = \tilde{0}$ we find $\langle\tilde{\sigma}|a_{1,1}\rangle = \langle\tilde{\sigma}|a_{1,-1}\rangle = 1$, $\langle\tilde{\sigma}|a_{2,2}\rangle = \langle\tilde{\sigma}|a_{2,-2}\rangle = 0$ and $\langle\tilde{\sigma}|a_{2,1}\rangle = \langle\tilde{\sigma}|a_{2,-1}\rangle = -1$. Then the consistent boundary states are expressed in terms of the coherent states as

$$|\tilde{I}\rangle = |\tilde{0}\rangle = 2^{-1/2}(|a_{1,1}\rangle + |a_{1,-1}\rangle + |a_{2,1}\rangle + |a_{2,-1}\rangle) + 2^{-1/4}(|a_{2,2}\rangle + |a_{2,-2}\rangle), \tag{2.61}$$

$$|\tilde{\epsilon}\rangle = 2^{-1/2}(|a_{1,1}\rangle + |a_{1,-1}\rangle + |a_{2,1}\rangle + |a_{2,-1}\rangle) - 2^{-1/4}(|a_{2,2}\rangle + |a_{2,-2}\rangle), \tag{2.62}$$

$$|\tilde{\sigma}\rangle = |a_{1,1}\rangle + |a_{1,-1}\rangle - |a_{2,1}\rangle - |a_{2,-1}\rangle. \tag{2.63}$$

These are exactly the same result as (1.211) - (1.213), as the relation between the Ishibashi states and the coherent states are given in (2.41) and (2.42). We have thus shown for the Ising model that the coherent states constructed on CBFS are not merely a subspace of the boundary states but they cover the space spanned by Cardy's consistent boundary states.

In the case of the Ising model, a similar construction of the boundary states from coherent states has been done using free Majorana fermions [85, 106, 107]. This is summarised in App.B. In a sense the present analysis is a generalisation of such a construction to general minimal theories.

5 Beyond minimal models

In this chapter we have constructed a set of coherent states on CBFS which preserve the conformal invariance, and argued that Cardy's consistent boundary states for minimal models are expressed as linear combinations of such states. We have demonstrated this explicitly in the example of Ising model. Our approach provides a new intuitive picture of boundary states in CFT, in terms of the boundary charges which obey the charge neutrality conditions with bulk screening operators.

Once consistent boundary states are expressed in terms of the coherent states, it is in principle possible to compute boundary n-point functions on an annulus directly without resorting to extra information on the boundary. The n-point function on the upper half plane involving an operator $\phi_{r,s}$ is found in the conventional method of Subsec.1.4.2 by solving the $(r \times s)$-th order differential equations satisfied by the $2n$-point function on the full plane [78]. Solutions to such a differential equation are in the form $A_1F_1 + A_2F_2 + \cdots$

where F_i represent the conformal blocks, and the coefficients A_i reflect boundary conditions and are determined by considering e.g. the asymptotic behaviour of the n-point function. In our Coulomb-gas approach, n-point functions on an annulus are obtained by inserting vertex operators between the boundary-to-boundary amplitudes, with appropriate inclusion of screening operators, leading to an integral representation of the correlation functions. In practice, however, such expressions involving multiple integrals of theta-functions are not always easy to evaluate.

We would like to conclude this chapter by mentioning applications of this formalism to two classes of CFTs with extended symmetry, WZNW model and CFTs with W-symmetry. In WZNW theories, the Sugawara-Sommerfeld (SS) energy-momentum tensor is expressed using free-fields by employing the Wakimoto free-field representation [108] of current operators. It is shown in [109] that bosonisation of this SS tensor leads to a sum of Coulomb-gas systems. For example, SS energy-momentum tensor for the $\widehat{sl(2)_k}$ WZNW theory is expressed as

$$T(z) = \sum_{j=1}^{3} T_j(z), \tag{2.64}$$

with

$$T_j(z) = -\frac{1}{2}(\partial\varphi_j)^2 + i\sqrt{2}\alpha_0^{(j)}\partial^2\varphi_j, \tag{2.65}$$

$$\alpha_0^{(1)} = \frac{1}{2\sqrt{2}}, \quad \alpha_0^{(2)} = \frac{1}{2\sqrt{2}}, \quad \alpha_0^{(3)} = -\frac{1}{2\sqrt{k+2}}. \tag{2.66}$$

The central charge of this model is $c = 3-6/(k+2)$, which is the sum of the three Coulomb-gas systems. It is also suggested in [110, 111] that CFTs involving W-algebra are realised by multiple scalar fields. For the system with W-operators T, $W^{(3)}$, $W^{(4)}$, $\cdots$, $W^{(n)}$, the energy-momentum tensor takes the form

$$T(z) = -\frac{1}{2}\partial\vec{\varphi}\cdot\partial\vec{\varphi} + 2i\alpha_0\vec{\rho}\cdot\partial^2\vec{\varphi}, \tag{2.67}$$

where $\vec{\varphi}$ is a $(n-1)$-component scalar field and $\vec{\rho}$ is a $(n-1)$-component constant vector. The central charge is $c = n-1-24\alpha_0^2\vec{\rho}\cdot\vec{\rho}$. As these systems have been reduced to Coulomb-gas formalism, such systems in the presence of boundaries should be able to analysed with the method presented in this chapter. An obvious merit for this is that the action of current operators on boundary states is explicitly seen. We postpone these topics to publications elsewhere.

Chapter 3

Logarithmic CFT with boundary

Happy conformal families are all alike; every unhappy family is unhappy in its own way.

– Leo Tolstoy, Anna Karenina (adapted).

In this chapter we discuss several features of logarithmic CFTs in the presence of boundary. Our main result is the existence of consistent boundary states which are found by using a free-field construction of the $c = -2$ triplet theory.

1 Boundary logarithmic CFT – Overview

For the past years the so-called logarithmic conformal field theories (LCFTs) have attracted much attention, and there are good reasons for this. After the landmarking paper by Belavin, Polakov and Zamolodchikov [1], the idea of conformal invariance in two-dimensional field theory has been investigated with great interest and has achieved a remarkable success. The interest in CFT is not restricted in physics but also spread in mathematics, as almost all areas of mathematics, ranging from algebra, analysis, geometry, to number theory, are proved to be related to CFT. After the successful establishment of the concept of rational CFTs, it is natural to ask how far the standard mathematical results applies to less well-behaved 'left-over' models, that is, LCFT.

By now many problems in physics have been claimed to be solved using LCFTs. In string theory context, it is argued that logarithmic operators may play important roles in several situations, such as D-brane recoil [71, 72] and dimensional up-grading [112]. Although these theories are rather speculative due to the non-unitary nature of LCFT, they certainly deserve more serious investigation because they may lead to new physics. In statistical physics, non-unitary theories are not exotic; a subsystem interacting with the outer environment may in general be non-unitary. Indeed, there are a number of statistical systems which are conjectured to be modelled by non-unitary, possibly logarithmic CFTs.

The current research on LCFT may be said to be driven by two motivations: to understand the mathematical structure of LCFT, and to model a certain class of physical systems using LCFT. From either point of view, it now seems to be both natural and rewarding to consider boundary theory of LCFT. Regarding the enormous recent progress in the algebraic study of boundary rational CFTs, it is quite interesting from the mathematical side to see what is happening in the algebraic structure of boundaries in LCFT. In the field of statistical physics, the study of polymers has become a big area and interesting boundary properties have been found [113–115]. As certain universality class of critical polymers are conjectured to be modelled by LCFT, we expect such boundary properties of polymers are somehow related to boundary LCFT. Furthermore, boundaries is potentially essential for LCFT itself because LCFTs are in general non-unitary and hence being far from a boundary does not guarantee the irrelevance of the boundary effect; once we have a boundary, its effect may change the system globally. Despite all this interest in the subject, the study of boundary LCFT started only recently. This is probably because the bulk theory itself is fairly complicated for LCFT and much of the attention had been focused on it. Now that the structure of modules and fusion algebra are understood [47, 48, 77] and that a free-field representation is available [49, 50] at least for the simplest model, we are in a good position to start discussing the theory with boundary.

Apart from the study of the surface critical behaviour of the $O(n)$ model [114, 115], the first paper that mentioned boundary in LCFT is by Moghimi-Araghi and Rouhani [116]. Soon after this, boundary theories are more extensively considered by Kogan and Wheater [117], who obtained explicit boundary correlation functions in $c = -2$ and $c = 0$ models. We shall review the $c = -2$ case in the next section. They also proposed a possible form of boundary operators and addressed a problem in the standard Cardy's construction of boundary states applied to these models, which arises from the unconventional modular transformation properties of characters. The main topic of this chapter is on the resolution of this problem [118], which is discussed in Sec.3.4. Using a free-field representation, we show that boundary states with consistent modular properties are found by Cardy's method. Before discussing the boundary states, we review in Sec.3.3 algebraic [47, 48, 75, 77] and free-field [49, 50] constructions of the so-called triplet model at $c = -2$. We summarise the results and discuss physical implications in Sec.3.5.

2 Boundary correlation functions in LCFT

An important goal of boundary CFT is to find correlation functions near the boundary. In the following we see this for the boundary 2-point function of $h = -1/8$ operators in $c = -2$ model, following [117].

The CFT at $c = c_{2,1} = -2$ is one of the logarithmic CFTs that has been studied most intensively and are so far best understood. This was used by Gurarie [46] to discuss the importance of logarithmic operators (see Subsec.1.3.7), and it is also claimed that certain universality classes of two-dimensional statistical models (such as critical polymers in the dense phase [44] and the Abelian sandpile model [65, 66]) are described by this theory. The operator content of this theory is as given in the extended Kac table of Subsec.13.7, where we assume $0 < r < 3$ and $0 < s < 6$. Although the Kac determinant for operators with non-standard transformation properties (such as ω) must be suitably modified [74, 119], the original Kac formula still applies to operators like Ω and μ. For these operators, the structure of singular vectors are the same as the Virasoro minimal models. Therefore the procedure in Subsec.1.3.7 to find the bulk 4-point function of μ operators is indeed the correct calculation.

For the boundary theory defined on the half plane, boundary n-point functions of the Virasoro minimal models are found by solving differential equations for the chiral $2n$-point functions on the plane, with the mirroring technique reviewed in Subsec.1.4.2. This method safely applies to operators like μ in the $c = -2$ model [116, 117]. The boundary 2-point function of the spin $-1/8$ operators $\langle\mu\mu\rangle_B$ is then equivalent to the 4-point function,

$$\langle\mu(z_1, \bar{z}_1)\mu(z_2, \bar{z}_2)\rangle_B = \langle\mu(z_1)\mu(z_2)\mu(z_1^*)\mu(z_2^*)\rangle_{chiral}. \tag{3.1}$$

As is discussed in Subsec.3.7, this chiral 4-point function satisfies a second order differential equation, which reduces to a hypergeometric differential equation due to the Möbius symmetry. The general form of the solution is

$$
\begin{aligned}
&\langle\mu(z_1,\bar{z}_1)\mu(z_2,\bar{z}_2)\rangle_B \\
&= |z_1 - z_2^*|^{1/2} z^{1/4}(1-z)^{1/4}\left\{AF(\frac{1}{2},\frac{1}{2},1;z) + BF(\frac{1}{2},\frac{1}{2},1;1-z)\right\},
\end{aligned}
\tag{3.2}
$$

where z is the cross ratio,

$$
z = \frac{(z_1 - z_2)(z_1^* - z_2^*)}{(z_1 - z_2^*)(z_1^* - z_2)} = \frac{|z_1 - z_2|^2}{|z_1 - z_2^*|^2}, \tag{3.3}
$$

and A and B are constants to be determined by boundary conditions. This is a single-valued function since z is always real and $0 < z < 1$.

If the points z_1 and z_2 are away from the boundary but the separation is kept fixed, we have $z \to 0$ and then

$$
\langle\mu(z_1,\bar{z}_1)\mu(z_2,\bar{z}_2)\rangle_B \to A|z_1 - z_2|^{1/2} + 2B|z_1 - z_2|^{1/2}\ln|z_1 - z_2|. \tag{3.4}
$$

The first term is the same as the bulk 2-point function. Hence, we may let $B = 0$ if we want to recover the bulk result by letting z_1 and z_2 away from the boundary. However, there is no physical motivation to do so because our theory is not unitary and the 2-point function grows with the separation, that is, being away from the boundary does not guarantee the negligibility of the boundary effect.

If the two operators are close to the boundary, we have $z \to 1$ and then $F(\frac{1}{2},\frac{1}{2},1;1-z) \to 1$. The correlation function is dominated by the first term,

$$
\langle\mu(z_1,\bar{z}_1)\mu(z_2,\bar{z}_2)\rangle_B \sim 2A|z_1 - z_1^*|^{1/2}\ln\frac{|z_1 - z_1^*|}{|z_1 - z_2|}, \tag{3.5}
$$

displaying a logarithmic behaviour. Therefore the 2-point function of μ operators exhibit logarithmic divergences at least either away from or close to the boundary. This is, of course, in a stark contrast with unitary cases.

3 Triplet $c = -2$ model and symplectic fermions

Extension of the Kac table discussed in Subsec.1.3.7 allows us to find correlation functions with or without boundary by solving differential equations. This method is quite powerful and is apparently a correct approach at least for some cases. For example, in the percolation problem reviewed in Subsec.1.4.5, the result obtained by solving such a differential equation is supported by a remarkable agreement with numerical calculations.

It is however obviously necessary to stand on more solid ground, in order to discuss the content and algebra of LCFT in detail. As logarithmic operators are not included in the (extended) Kac table, one cannot treat logarithmic theories in a full analogy with the minimal models. One desirable direction of research is to consider the structure of the singular vectors in logarithmic theories and formulate logarithmic Kac tables. Such an attempt has been recently made by Flohr [119], but at the moment this task has yet to be completed.

Although it is not easy to discuss LCFTs in general, at $c = -2$ there exists a well-understood model of LCFT, called the triplet model. This was found by Kausch [120] as one of the series of CFTs characterised by a triplet of W-algebra, and has been studied in detail for many years [47, 48, 75, 77]. The triplet model has many nice properties similar to the minimal models, and hence it is speculated to be a proper realisation of a next-to-minimal CFT, if such a theory does exist. There is also a free-field representation of a $c = -2$ CFT, realised by the so-called symplectic fermions [49, 50, 121], which is basically a fermionic ghost system with a modified zero-mode. It has been shown that the triplet model is realised by the bosonic sector of the symplectic fermions.

In the following we review the algebraic and free-field (symplectic fermion) realisations of the triplet model, and discuss the representations of the triplet model and collect the results needed for later discussions.

3.1 The triplet model in the algebraic approach

The triplet algebra $\mathcal{W}(2, 3, 3, 3)$ is an extension of the Virasoro algebra by a triplet of spin 3 fields $W^a(z)$, which is identified with $\phi_{3,1}(z)$ in the Kac table. The commutation relations for this algebra are

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \tag{3.6}$$

$$[L_n, W^a_m] = (2n-m)W^a_{n+m}, \tag{3.7}$$

$$\begin{aligned}[W^a_n, W^b_m] &= g^{ab}\Big[2(n-m)\Lambda_{n+m} + \frac{1}{20}(n-m)(2n^2+2m^2-mn-8)L_{n+m} \\ &- \frac{1}{120}n(n^2-1)(n^2-4)\delta_{n+m}\Big] \\ &+ f^{ab}{}_c\left[\frac{5}{14}(2n^2+2m^2-3mn-4)W^c_{n+m} + \frac{12}{5}V^c_{n+m}\right],\end{aligned} \tag{3.8}$$

where

$$\Lambda =: L^2 : -\frac{3}{10}\partial^2 L, \quad V^a =: LW^a : -\frac{3}{14}\partial^2 W^a, \tag{3.9}$$

and g^{ab} and $f^{ab}{}_c$ are the metric and the structure constants of $su(2)$, which are in an orthonormal basis $g^{ab} = \delta^{ab}$ and $f^{ab}{}_c = i\epsilon^{abc}$.

A crucial ingredient of the triplet model is the existence of vacuum null vectors at level 6,

$$N^a = \left[2L_{-3}W^a_{-3} - \frac{4}{3}L_{-2}W^a_{-4} + W^a_{-6}\right]\Omega, \tag{3.10}$$

$$\begin{aligned}N^{ab} = &\Big[W^a_{-3}W^b_{-3} - g^{ab}\left(\frac{8}{9}L^3_{-2} + \frac{19}{36}L^2_{-3} + \frac{14}{9}L_{-4}L_{-2} - \frac{16}{9}L_{-6}\right) \\ &- f^{ab}{}_c\left(\frac{5}{4}W^c_{-6} - 2L_{-2}W^c_{-4}\right)\Big]\Omega,\end{aligned} \tag{3.11}$$

where Ω is the Möbius invariant vacuum. It can be shown that any highest weight state ψ compatible[1] with these null vectors must satisfy the following two conditions:

$$\left[W^a_0 W^b_0 - \frac{1}{9}g^{ab}L^2_0(8L_0+1) - \frac{1}{5}f^{ab}{}_c(6L_0-1)W^c_0\right]\psi = 0, \tag{3.12}$$

$$W^a_0(8L_0-3)(L_0-1)\psi = 0. \tag{3.13}$$

[1]This implies that any correlation functions of ψ's with a null vector must vanish.

Multiplying (3.13) with W_0^a and using (3.12), we have

$$L_0^2(8L_0 + 1)(8L_0 - 3)(L_0 - 1)\psi = 0. \tag{3.14}$$

It is important to notice that the L_0 factor appears quadratically. This implies that irreducible representations must have highest weights $h = 0$, $-1/8$, $3/8$ or 1, but Jordan-cell representations

$$L_0\omega = \Omega, \quad L_0\Omega = 0, \tag{3.15}$$

are also allowed since $L_0^2 = 0$ does not necessarily require $L_0 = 0$.

These irreducible representations are classified by using the (rescaled) $su(2)$ algebra

$$[W_0^a, W_0^b] = \frac{2}{5}(6h - 1)f^{ab}{}_c W_0^c, \tag{3.16}$$

which follows from (3.12). Letting m and $j(j+1)$ be the eigenvalues of W_0^3 and the quadratic Casimir $\sum_a (W_0^a)^2$ after rescaling, we have $j(j+1) = 3m^2$ because of $W_0^a W_0^a = W_0^b W_0^b$ (no sum) which is obtained from (3.12). The only possible values of j are either 0 or 1/2, and this restricts the allowed irreducible representations as [47, 75, 122]

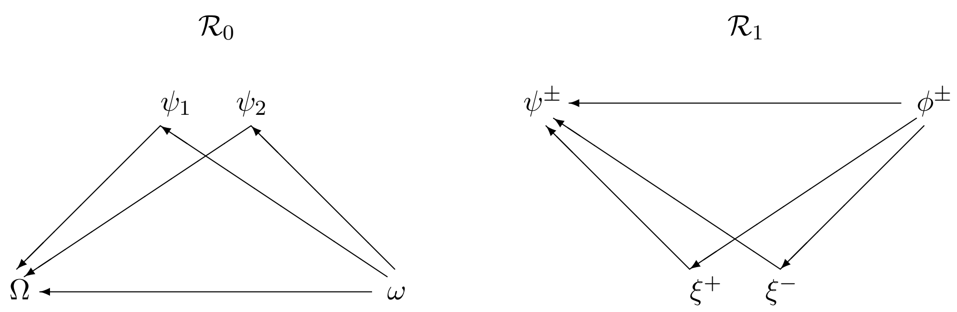
$\mathcal{V}_0$: the singlet $(j = 0)$ representation at $h = 0$.
$\mathcal{V}_{-1/8}$: the singlet $(j = 0)$ representation at $h = -1/8$.
$\mathcal{V}_1$: the doublet $(j = 1/2)$ representation at $h = 1$.
$\mathcal{V}_{3/8}$: the doublet $(j = 1/2)$ representation at $h = 3/8$.

The above discussion is made mathematically more rigorous by resorting to Zhu's algebra [47, 75, 123].

The fusion rules of these irreducible representations are calculated using a comultiplication formula [47, 75]. It is shown that the fusions do not close for the irreducible representations but they involve 'reducible but indecomposable' representations $\mathcal{R}_0$ and $\mathcal{R}_1$. The explicit fusion results are summarised as follows:

$$
\begin{aligned}
\mathcal{R}_i \times \mathcal{R}_j &= 2\mathcal{R}_0 + 2\mathcal{R}_1 && i, j = 0, 1, \\
\mathcal{R}_i \times \mathcal{V}_j &= \mathcal{R}_0 && (i, j) = (0, 0), (1, 1), \\
&= \mathcal{R}_1 && (i, j) = (0, 1), (1, 0), \\
&= 2\mathcal{V}_{-1/8} + 2\mathcal{V}_{3/8} && i = 0, 1; j = -\tfrac{1}{8}, \tfrac{3}{8}, \\
\mathcal{V}_i \times \mathcal{V}_j &= \mathcal{V}_0 && (i, j) = (0, 0), (1, 1), \\
&= \mathcal{V}_1 && (i, j) = (0, 1), (1, 0), \\
&= \mathcal{V}_{-1/8} && (i, j) = (0, -\tfrac{1}{8}), (1, \tfrac{3}{8}), \\
&= \mathcal{V}_{3/8} && (i, j) = (1, -\tfrac{1}{8}), (0, \tfrac{3}{8}), \\
&= \mathcal{R}_0 && (i, j) = (-\tfrac{1}{8}, -\tfrac{1}{8}), (\tfrac{3}{8}, \tfrac{3}{8}), \\
&= \mathcal{R}_1 && (i, j) = (-\tfrac{1}{8}, \tfrac{3}{8}), (\tfrac{3}{8}, -\tfrac{1}{8}).
\end{aligned}
\tag{3.17}
$$

The structure of the representations $\mathcal{R}_0$ and $\mathcal{R}_1$ are schematically depicted as (after [47, 48, 75])



where the arrows indicate the action of the triplet algebra, L_n and W_n^a. Each element represents an irreducible representation. In both diagrams, the uppers rows are $h = 1$ and the bottom are $h = 0$. In $\mathcal{R}_0$, the singlet Ω is the ground state of $\mathcal{V}_0$. The states ψ_1 and ψ_2 are actually four states $L_{-1}\omega$ and $W_{-1}^a\omega$, forming two doublets under $su(2)$. The elements in $\mathcal{R}_1$ are generated from the doublet states $\phi^{\pm}$ of weight $h = 1$, and $\psi^{\pm}$ are the highest weight states of the $su(2)$ doublet irreducible representation $\mathcal{V}_1$. The ground states of $\mathcal{R}_1$ are the doublet $\xi^{\pm}$ at $h = 0$. The two doublets $\psi^{\pm}$ and $\phi^{\pm}$ form an L_0 Jordan block at $h = 1$. More precisely, the subrepresentations in $\mathcal{R}_0$ and $\mathcal{R}_1$ are characterised by the following defining relations [47, 48, 75]:

$$L_0\omega = \Omega, \quad L_0\Omega = 0,$$
$$W_0^a\omega = 0, \quad W_0^a\Omega = 0, \tag{3.18}$$

and

$$\begin{aligned}
L_1\phi^\alpha &= -\xi^\alpha, & W_1^a\phi^\alpha &= t^{a\alpha}_\beta\xi^\beta,\\
L_0\phi^\alpha &= \phi^\alpha + \psi^\alpha, & W_0^a\phi^\alpha &= 2t^{a\alpha}_\beta\phi^\beta,\\
L_0\xi^\alpha &= 0, & W_0^a\xi^\alpha &= 0,\\
L_{-1}\xi^\alpha &= \psi^\alpha, & W_{-1}^a\xi^\alpha &= t^{a\alpha}_\beta\psi^\beta,\\
L_0\psi^\alpha &= \psi^\alpha, & W_0^a\psi^\alpha &= 2t^{a\alpha}_\beta\psi^\beta,
\end{aligned} \tag{3.19}$$

where α and β are the $su(2)$ doublet indices. In Cartan-Weyl basis, $t^{0\pm}_\pm = \pm 1/2$, $t^{\pm\mp}_\pm = 1$. Note that, for an instance, $L_{-1}\omega$ is not null since $L_1L_{-1}\omega = [L_1, L_{-1}]\omega = 2\Omega \neq 0$.

The fusion rules (3.17) indicate that they close for the four representations $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{V}_{-1/8}$ and $\mathcal{V}_{3/8}$. Thus the triplet model is regarded as a rational conformal field theory, with a weakened definition of rationality [47]. The local and non-chiral representations of the triplet model are discussed in [48], where the theory with finite multiplicity consists of three non-chiral representations, namely, $\mathcal{V}_{-1/8}\otimes\bar{\mathcal{V}}_{-1/8}$, $\mathcal{V}_{3/8}\otimes\bar{\mathcal{V}}_{3/8}$ and $\mathcal{R}$ which is a combination of $(\mathcal{R}_0\otimes\bar{\mathcal{R}}_0)/\mathcal{N}_{0\bar{0}}$ and $(\mathcal{R}_1\otimes\bar{\mathcal{R}}_1)/\mathcal{N}_{1\bar{1}}$, where $\mathcal{N}_{0\bar{0}}$ and $\mathcal{N}_{1\bar{1}}$ are subspaces to be quotiented out.

3.2 Symplectic fermions

It is well known that the Ising model at $c = 1/2$ is related to the free Majorana fermion of the same central charge, through the Jordan-Wigner transformation. The triplet $c = -2$ model discussed so far is similarly related to a two-component free fermion, called the symplectic fermion. The correspondence is rather direct, and it has been shown that its bosonic sector reproduces the triplet model [49, 50].

The symplectic fermion is closely related to the simple ghost system of the same central charge $c = -2$, whose action is

$$S = \frac{1}{\pi}\int d^2z(\eta\bar{\partial}\xi + \bar{\eta}\partial\bar{\xi}). \tag{3.20}$$

Here, η and ξ are fermionic ghosts with conformal dimensions $h_\eta = 1$ and $h_\xi = 0$. The operator products are

$$\eta(z)\xi(w) \sim \xi(z)\eta(w) \sim 1/(z-w), \tag{3.21}$$

reflecting the Grassmannian nature of the operators. Symplectic fermions are introduced as operators of conformal dimensions $h_{\chi^\pm} = 1$, defined as

$$\chi^+ \equiv \eta, \ \ \chi^- \equiv \partial\xi. \tag{3.22}$$

The mode expansions,

$$\chi^\pm(z) = \sum_{k\in\mathbb{Z}} \chi_k^\pm z^{-k-1}, \tag{3.23}$$

define the mode operators $\chi_k^\pm$ with anti-commutation relations,

$$\{\chi_m^\alpha, \chi_n^\beta\} = m d^{\alpha\beta}\delta_{m+n}, \tag{3.24}$$

where $d^{\alpha\beta}$ is antisymmetric and $d^{\pm\mp} = \pm 1$. These symplectic fermions differ from the η-ξ simple ghost system by the zero-mode of ξ. The absence of ξ_0 enhances the symmetry and realises the triplet model at $c = -2$.

The orbifold structure is obtained by considering twisted sectors as well as the untwisted sector [44, 49, 50]. The twisted sectors are built on the vacuum by operating with a twisting field $\sigma_{k/N}$, and the resulting theory becomes $\mathbb{Z}_N$ invariant. It is argued that the $\mathbb{Z}_N$ orbifold model constructed like this has a W-algebra of type $\mathcal{W}(2, 3, N(N+1)/2, N(N+1)/2)$ [121]. In the case of $N = 2$ the model possesses $\mathcal{W}(2, 3, 3, 3)$ symmetry which is generated by the stress tensor

$$T(z) =: \chi^-(z)\chi^+(z) + \lambda(\lambda-1)/2, \tag{3.25}$$

where $\lambda = 0$ for the untwisted and $\lambda = 1/2$ for the twisted sector, and a triplet of W-fields with conformal dimension 3,

$$\begin{aligned} W^0 &= -\frac{1}{2}(:\partial\chi^+(z)\chi^-(z): + :\partial\chi^-(z)\chi^+(z):), \\ W^\pm &=: \partial\chi^\pm(z)\chi^\pm(z):. \end{aligned} \tag{3.26}$$

In the following we only consider the $N = 2$ case which reproduces the triplet model. Virasoro operators and W-mode operators are found from these as

$$L_n = \frac{1}{2} d_{\alpha\beta} \sum_{m\in\mathbb{Z}+\lambda} :\chi^\alpha_m \chi^\beta_{n-m}: + \frac{\lambda(\lambda-1)}{2}\delta_{n0}, \tag{3.27}$$

and

$$\begin{aligned} W^0_n &= -\frac{1}{2}\sum_{j\in\mathbb{Z}+\lambda} j\left\{:\chi^+_{n-j}\chi^-_j: + :\chi^-_{n-j}\chi^+_j:\right\}, \\ W^\pm_n &= \sum_{j\in\mathbb{Z}+\lambda} j\chi^\pm_{n-j}\chi^\pm_j. \end{aligned} \tag{3.28}$$

Derivation of these mode operators based on the twisted Borcherds identity is detailed in App.C.

The irreducible representations with conformal weights $-1/8$, 0, 3/8 and 1 are reproduced in the Fock space representations of the symplectic fermions as follows [48–50, 75]. The ground state of the twisted sector which is obtained by operating with $\sigma_{1/2}$ on the vacuum is denoted by μ. This μ has conformal weight $-1/8$, and then the singlet representation $\mathcal{V}_{-1/8}$ is defined as states built on μ. The doublet of the states $\nu^\pm \equiv \chi^\pm_{-1/2}\mu$ has conformal dimension 3/8 and this is the highest-weight state of the doublet representation $\mathcal{V}_{3/8}$. The representations belonging to the untwisted sector are more complicated. Let ω be a state annihilated by operations with $\chi^\pm_{n>0}$. Then there are four ground states, ω, $\theta^\pm = -\chi^\pm_0\omega$, and $\Omega = \chi^-_0\chi^+_0\omega = L_0\omega$. As Ω is annihilated by further operations with zero modes, it is identified as the Möbius invariant vacuum. The irreducible vacuum representation $\mathcal{V}_0$ is built on the ground state Ω. Similarly, the irreducible doublet representation $\mathcal{V}_1$ at $h = 1$ is built on the doublet $\psi^\pm = \chi^\pm_{-1}\Omega$.

As is discussed in the previous subsection, these four irreducible representations $\mathcal{V}_{-1/8}$, $\mathcal{V}_{3/8}$, $\mathcal{V}_0$ and $\mathcal{V}_1$ do not close under the fusion; extra representations $\mathcal{R}_0$ and $\mathcal{R}_1$ are necessary. The 'reducible but indecomposable' representation $\mathcal{R}_0$ is obtained by extending the vacuum $\mathcal{V}_0$ to include ω, $L_{-1}\omega$ and $W^a_{-1}\omega$ as well as Ω. The two bosonic ground states Ω and ω span a two-dimensional Jordan cell on the action of L_0, forming a 'logarithmic pair.' The representation $\mathcal{R}_1$ is obtained likewise, by extending $\mathcal{V}_1$ to include $\phi^\pm = \chi^\pm_{-1}\omega$ and $\xi^\pm = -L_1\phi^\pm$ in addition to $\psi^\pm = \chi^\pm_{-1}\Omega$. The doublet states $\psi^\pm$ and $\phi^\pm$ form a logarithmic pair at $h = 1$.

The four representations $\mathcal{V}_{-1/8}$, $\mathcal{V}_{3/8}$, $\mathcal{R}_0$ and $\mathcal{R}_1$ then close under the fusion. Now it is obvious that neglecting the fermionic states (such as $\theta^{\pm}$), the symplectic fermion reproduces the triplet model. The correspondence between the triplet model and the symplectic fermion is summarised as follows:

Twisted sector

$$\begin{array}{rcl} \mu & \leftrightarrow & \mathcal{V}_{-1/8} \\ \nu^{\pm} \equiv \chi^{\pm}_{-1/2}\mu & \leftrightarrow & \mathcal{V}_{3/8} \end{array}$$

Untwisted sector

$$\begin{array}{rcl} \Omega = L_0\omega & \leftrightarrow & \mathcal{V}_0 \\ \psi^{\pm} \equiv \chi^{\pm}_{-1}\Omega & \leftrightarrow & \mathcal{V}_1 \end{array}$$

$$\begin{array}{rcl} \omega,\ \ \Omega,\ \ L_{-1}\omega,\ \ W^a_{-1}\omega & \leftrightarrow & \mathcal{R}_0 \\ \phi^{\pm} \equiv \chi^{\pm}_{-1}\omega,\ \ \psi^{\pm},\ \ \xi^{\pm} \equiv -L_1\phi^{\pm} & \leftrightarrow & \mathcal{R}_1 \end{array}$$

3.3 Modular invariants

The fusion rule of the triplet model (3.17) is quite unusual in that it is not diagonalisable. This is in a sharp contrast with ordinary rational theories, where fusion matrices are always diagonalised with modular S matrices through the Verlinde formula. As is expected from the failure of the Verlinde formula, the characters of the triplet model are quite unusual as well. They are calculated in [47, 50, 124, 125] as

$$\begin{aligned} \chi_{\mathcal{V}_0}(\tau) &= \frac{\Theta_{1,2}(\tau)}{2\eta(\tau)} + \frac{1}{2}\eta(\tau)^2, \\ \chi_{\mathcal{V}_1}(\tau) &= \frac{\Theta_{1,2}(\tau)}{2\eta(\tau)} - \frac{1}{2}\eta(\tau)^2, \\ \chi_{\mathcal{V}_{-1/8}}(\tau) &= \frac{\Theta_{0,2}(\tau)}{\eta(\tau)}, \\ \chi_{\mathcal{V}_{3/8}}(\tau) &= \frac{\Theta_{2,2}(\tau)}{\eta(\tau)}, \\ \chi_{\mathcal{R}_0}(\tau) = \chi_{\mathcal{R}_1} &= \frac{2\Theta_{1,2}(\tau)}{\eta(\tau)}, \end{aligned} \tag{3.29}$$

where $\Theta_{k,l}(\tau)$ and $\eta(\tau)$ are Jacobi theta functions and Dedekind eta function, respectively (see App.A for definitions). Note that these character functions are not independent, $\chi_{\mathcal{R}_0} = \chi_{\mathcal{R}_1} = 2\chi_{\mathcal{V}_0} + 2\chi_{\mathcal{V}_1}$. The mutual dependence of the character functions is reminiscent of the minimal Virasoro theories with extra symmetry (such as the three-state Potts model). What is pathological about these characters is their modular transformation property. Since $\eta(\tau)^2 \to \eta(\tilde{\tau})^2 = -i\tau\eta(\tau)^2$ as $\tau \to \tilde{\tau} = -1/\tau$, the character functions do not transform into each other linearly under the modular transformation.

Nevertheless, we may construct a modular-invariant partition function from these character functions [48, 75]. Indeed,

$$\begin{aligned} Z(\tau,\bar{\tau}) &= \chi_{\mathcal{V}_{-1/8}}\bar{\chi}_{\mathcal{V}_{-1/8}} + \chi_{\mathcal{V}_{3/8}}\bar{\chi}_{\mathcal{V}_{3/8}} + 2\chi_{\mathcal{V}_0}\bar{\chi}_{\mathcal{V}_0} + 2\chi_{\mathcal{V}_0}\bar{\chi}_{\mathcal{V}_1} + 2\chi_{\mathcal{V}_1}\bar{\chi}_{\mathcal{V}_0} + 2\chi_{\mathcal{V}_1}\bar{\chi}_{\mathcal{V}_1} \\ &= \chi_{\mathcal{V}_{-1/8}}\bar{\chi}_{\mathcal{V}_{-1/8}} + \chi_{\mathcal{V}_{3/8}}\bar{\chi}_{\mathcal{V}_{3/8}} + 2\chi_{\mathcal{R}_0}\bar{\chi}_{\mathcal{R}_0} + 2\chi_{\mathcal{R}_0}\bar{\chi}_{\mathcal{R}_1} + 2\chi_{\mathcal{R}_1}\bar{\chi}_{\mathcal{R}_0} + 2\chi_{\mathcal{R}_1}\bar{\chi}_{\mathcal{R}_1} \\ &= \frac{1}{|\eta(\tau)|^2}\sum_{k=0}^{3}|\Theta_{k,2}(\tau)|^2, \end{aligned} \tag{3.30}$$

is easily verified to be invariant under both $\tau \to -1/\tau$ and $\tau \to \tau + 1$.

4 Boundary states of $c = -2$ triplet model

As $c = -2$ theory is expected to model statistical systems such as polymers, it is not conceivable that this theory has no consistent boundary states. On the other hand, as is expected from the failure of the Verlinde formula, we have a difficulty applying Cardy's method illustrated in Subsec.1.4.3 to the triplet model at $c = -2$. When we try to find consistent boundary states as linear combinations of Ishibashi states, we immediately notice that the modular S-matrices for the triplet model cannot be defined because the characters (3.29) do not close under the modular S transformation. One way to circumvent this difficulty is to add another character involving $-i\tau\eta^2(\tau)$ and enlarge the set of character functions [74, 124, 125]. This however introduces a mysterious extra representation which appears neither in algebraic nor free-field construction of the triplet model.

In this section we analyse this problem carefully by using the symplectic fermion representation of the triplet model. Our starting point is noticing that the closed string boundary states in (1.201) may be expanded with any basis of boundary states, not necessarily Ishibashi states. Since the construction of Ishibashi states rely on the existence of well-defined conformal towers which are unavailable in the $c = -2$ triplet model, we shall construct more 'sound' boundary states in the Fock space using the symplectic fermions. As boundary states in CFT must satisfy the conditions for diffeomorphism invariance (1.195), and it is well known in string theory that such boundary states for boson, fermion and ghost fields are found in the form of coherent states, we shall construct coherent states of symplectic fermion and use them as a basis to span the boundary states of the triplet model. As we shall see below in detail, it turns out that boundary states with consistent modular properties are obtained as linear sums of such coherent states. In particular, the modular function $-i\tau\eta^2(\tau)$ naturally appears in the amplitudes between such coherent states, without introducing any extra representations of the theory. The results of this section have been published in [118].

4.1 Boundary conditions for symplectic fermion

Let us consider a situation where CFT is defined on the upper ζ-plane and the action of a general chiral field J on the boundary $\zeta = \bar{\zeta}$ is given by

$$\left[J(\zeta) - \Gamma \bar{J}(\bar{\zeta})\right]_{\zeta=\bar{\zeta}} = 0. \tag{3.31}$$

Here, Γ is an element of a gluing automorphism group, which tells how the holomorphic and antiholomorphic J-operators are related on the boundary. When $\Gamma = 1$ (identity element), this simply means the trivial continuity of J across the boundary. As we have done in Subsec.1.4.3 and in Sec.2.2, we conformally map a semi-annular domain in the upper half ζ-plane onto a full annulus in the z-plane, by $z = \exp(-2\pi i w/L)$ and $w = (T/\pi)\ln\zeta$. Under this mapping the boundary $\zeta = \bar{\zeta}$ is mapped to $|z| = 1$, $\exp(2\pi T/L)$, and the condition (3.31) reads

$$z^{s_J} J(z) = (-\bar{z})^{s_J} \Gamma \bar{J}(\bar{z}), \tag{3.32}$$

on the boundary. Here, s_J is the spin of J. Now that the z-plane allows radial quantization, the continuity of J may be translated into conditions on the boundary states at $|z| = 1$ as [79, 93]

$$(J_m - (-1)^{s_J} \Gamma \bar{J}_{-m})|B\rangle = 0. \tag{3.33}$$

When $J(z)$ is the energy-momentum tensor $T(z)$, (3.33) becomes

$$(L_m - \Gamma \bar{L}_{-m})|B\rangle. \tag{3.34}$$

In this case, Γ must be the identity in order that the boundary be diffeomorphism invariant.

Now consider the action (3.20), defined on a cylinder of circumference L and length T. Modular parameters are defined as $\tilde{q} = e^{-4\pi T/L}$ and $\tilde{\tau} = 2iT/L$. Following the standard construction of Dirichlet and Neumann boundary states in open superstring theory [126–128], we assume that the boundary term in the action vanishes, $\eta\xi + \bar{\eta}\bar{\xi} = 0$. Since η and ξ have different scaling dimensions, this condition is decomposed into the linear conditions that $\eta = e^{i\phi}\bar{\eta}$ and $\xi = e^{i(\pi-\phi)}\bar{\xi}$ on the boundary. Here, ϕ is a phase factor reflecting the $U(1)$ symmetry of the system. These conditions are trivially consistent with the conformal invariance. Although it might be possible to include non-trivial interactions between bulk and boundary by introducing conformally invariant boundary terms, we do not consider such terms here. These conditions are translated through radial quantisation into conditions on the boundary states $|B\rangle$ as in (3.33). The difference of the η-ξ ghost and the symplectic fermion is taken into account by neglecting the conditions involving ξ_0 and $\bar{\xi}_0$. One may now look for the conformally invariant symplectic fermion boundary states $|B\rangle$ and $\langle B|$ as states satisfying the conditions,

$$\begin{aligned} \left(\chi_m^{\pm} - e^{\pm i\phi}\bar{\chi}_{-m}^{\pm}\right)|B\rangle &= 0, \\ \langle B|\left(\chi_m^{\pm} - e^{\pm i\phi}\bar{\chi}_{-m}^{\pm}\right) &= 0, \end{aligned} \tag{3.35}$$

where $\chi_m^{\pm}$ and $\bar{\chi}_m^{\pm}$ are the symplectic fermion mode operators.

4.2 Conformally invariant boundary states

Again by analogy with open string theory, let us consider the coherent states,

$$\begin{aligned} |B_{0\phi}\rangle &= N \exp\left(\sum_{k>0} \frac{e^{i\phi}}{k}\chi_{-k}^{-}\bar{\chi}_{-k}^{+} + \frac{e^{-i\phi}}{k}\bar{\chi}_{-k}^{-}\chi_{-k}^{+}\right)|0_\phi\rangle, \\ \langle B_{0\phi}| &= \langle 0_\phi|N^* \exp\left(\sum_{k>0} \frac{e^{i\phi}}{k}\chi_k^{-}\bar{\chi}_k^{+} + \frac{e^{-i\phi}}{k}\bar{\chi}_k^{-}\chi_k^{+}\right), \end{aligned} \tag{3.36}$$

where N and N^* are normalization constants, and k runs over integers in the untwisted sector and half-integers in the twisted sector. The non-chiral ground states $|0_\phi\rangle$ are annihilated by the positive modes,

$$\chi^{\pm}_{k>0}|0_\phi\rangle = 0 = \bar{\chi}^{\pm}_{k>0}|0_\phi\rangle, \tag{3.37}$$

and $\langle 0_\phi|$ by the negative modes,

$$\langle 0_\phi|\chi^{\pm}_{n<0} = 0 = \langle 0_\phi|\chi^{\pm}_{n<0}. \tag{3.38}$$

It is easily shown that the states (3.36) indeed satisfy the conditions (3.35) as long as the untwisted sector ground states $|0_\phi\rangle$ and $\langle 0_\phi|$ satisfy the conditions

$$\left(\chi^{\pm}_0 - e^{\pm i\phi}\bar{\chi}^{\pm}_0\right)|0_\phi\rangle = 0, \quad \langle 0_\phi|\left(\chi^{\pm}_0 - e^{\pm i\phi}\bar{\chi}^{\pm}_0\right) = 0. \tag{3.39}$$

Then the states (3.36) satisfy the condition (1.195) and its bra counterpart.

In the untwisted (or NS) sector of the triplet model, the ground states are doubly degenerate (Ω and ω) and thus we have $|\Omega\rangle$ and $|\omega\rangle$. The ground state of the twisted (R) sector is unique (μ) so we have $|\mu\rangle$. We normalize these ground states to be

$$\begin{aligned}
&\langle\omega|\omega\rangle = \kappa,\\
&\langle\Omega|\omega\rangle = \langle\omega|\Omega\rangle = \rho \equiv -1,\\
&\langle\Omega|\Omega\rangle = 0,\\
&\langle\mu|\mu\rangle = 1.
\end{aligned} \tag{3.40}$$

For the convenience of later discussions we have chosen $\rho = -1$. Although the negative sign may look strange, this is the choice of the sign which simplifies the following results enormously. We leave the value of κ unfixed[2]. The boundary states (3.36) are indexed by the phase factor ϕ and the choice of ground state 0_ϕ which may also depend on ϕ.

The phase factor ϕ is related to the gluing automorphism Γ of the triplet W-current W^a. The action of the triplet W-operators on the boundary,

$$\left[W^a - \Gamma\bar{W}^a\right]_{\zeta=\bar{\zeta}} = 0, \tag{3.41}$$

[2]This κ may depend on the cutoff scale or geometry.

leads to the conditions

$$(W_m^a + \Gamma \bar{W}_{-m}^a)|B\rangle = 0, \tag{3.42}$$

on the boundary states. Clearly, the simplest case is when the automorphism is trivial, $\Gamma = 1$. This restricts the value of the phase ϕ to be either 0 or π. We will see that this choice is sufficient to construct boundary states with consistent modular properties in the triplet model, allowing the appearance of bulk representations $\mathcal{V}_0$, $\mathcal{V}_1$, $\mathcal{V}_{-1/8}$, $\mathcal{V}_{3/8}$, $\mathcal{R}_0$ and $\mathcal{R}_1$. When we deal with e.g. $\mathbb{Z}_4$-orbifold symplectic fermion model, we need to consider non-trivial automorphism as well as $\Gamma = 1$. The distinction of $\phi = 0$ and π is analogous to that of Dirichlet and Neumann boundary conditions for bosonic string. In the following discussions of the triplet model which is realised by the $\mathbb{Z}_2$ orbifold symplectic fermion, we restrict the values of ϕ to be either $\phi = 0$ or $\phi = \pi$. For simplicity we write $\phi = 0$ as $+$ and $\phi = \pi$ as $-$, and hence we shall consider the boundary states, $|B_{0+}\rangle = |B_{0,\phi=0}\rangle$, $|B_{0-}\rangle = |B_{0,\phi=\pi}\rangle$, and $\langle B_{0+}| = \langle B_{0,\phi=0}|$, $\langle B_{0-}| = \langle B_{0,\phi=\pi}|$. Then we have six distinct boundary states, $|B_{\omega+}\rangle$, $|B_{\omega-}\rangle$, $|B_{\Omega+}\rangle$, $|B_{\Omega-}\rangle$, $|B_{\mu+}\rangle$ and $|B_{\mu-}\rangle$, which we collectively write $|a\rangle = |B_{0\phi}\rangle$. Now that we have explicit expressions of the boundary states, we may calculate the amplitudes $\langle a|(\tilde{q}^{1/2})^{L_0+\bar{L}_0+1/6}|b\rangle$ between them, which appear on the right hand side of the consistency condition (1.201). Setting $|N|^2 = 1$, they are summarized as follows:

Untwisted Sector

$\langle a\|$	$\|b\rangle$ $B_{\omega+}$	$B_{\omega-}$	$B_{\Omega+}$	$B_{\Omega-}$
$B_{\omega+}$	$(\kappa - \ln\tilde{q})\eta(\tilde{\tau})^2$	0	$-\eta(\tilde{\tau})^2$	$-\Lambda_{1,2}(\tilde{\tau})$
$B_{\omega-}$	0	$(\kappa - \ln\tilde{q})\eta(\tilde{\tau})^2$	$-\Lambda_{1,2}(\tilde{\tau})$	$-\eta(\tilde{\tau})^2$
$B_{\Omega+}$	$-\eta(\tilde{\tau})^2$	$-\Lambda_{1,2}(\tilde{\tau})$	0	0
$B_{\Omega-}$	$-\Lambda_{1,2}(\tilde{\tau})$	$-\eta(\tilde{\tau})^2$	0	0

Twisted Sector

$\langle a\vert$	$\vert b\rangle$	
	$B_{\mu+}$	$B_{\mu-}$
$B_{\mu+}$	$\Lambda_{0,2}(\tilde{\tau}) - \Lambda_{2,2}(\tilde{\tau})$	$\Lambda_{0,2}(\tilde{\tau}) + \Lambda_{2,2}(\tilde{\tau})$
$B_{\mu-}$	$\Lambda_{0,2}(\tilde{\tau}) + \Lambda_{2,2}(\tilde{\tau})$	$\Lambda_{0,2}(\tilde{\tau}) - \Lambda_{2,2}(\tilde{\tau})$

We have denoted $\Lambda_{k,l}(\tau) = \Theta_{k,l}(\tau)/\eta(\tau)$. Strictly speaking, the boundary states (3.36) are not normalisable. Their non-trivial inner products are

$$\begin{aligned}
\langle B_{\omega\pm}|B_{\omega\mp}\rangle &= \kappa \lim_{\tilde{\tau}\to 0} \Lambda_{1,2}(\tilde{\tau}), \\
\langle B_{\Omega\pm}|B_{\omega\mp}\rangle &= \langle B_{\omega\pm}|B_{\Omega\mp}\rangle = -\lim_{\tilde{\tau}\to 0} \Lambda_{1,2}(\tilde{\tau}), \\
\langle B_{\mu\pm}|B_{\mu\mp}\rangle &= \lim_{\tilde{\tau}\to 0}(\Lambda_{0,2}(\tilde{\tau}) + \Lambda_{2,2}(\tilde{\tau})).
\end{aligned} \tag{3.43}$$

The right hand sides of (3.43) are all divergent, which is a well-known feature of such boundary states. Because of the conditions (3.39), the amplitudes between states built on ω are non-vanishing only when the bra and ket have the same value of ϕ (for example, $\langle\omega_{\phi'}|\chi_0^{\pm}|\omega_\phi\rangle = e^{\pm i\phi}\langle\omega_{\phi'}|\bar{\chi}_0^{\pm}|\omega_\phi\rangle = e^{\pm i\phi'}\langle\omega_{\phi'}|\bar{\chi}_0^{\pm}|\omega_\phi\rangle$, leading to $\langle\omega_{\phi'}|\chi_0^{\pm}|\omega_\phi\rangle = 0$ unless $\phi = \phi'$). An important point we would like to emphasize is that we cannot have all of the four boundary states of the untwisted sector simultaneously; a GSO-type projection is necessary to construct the triplet model from the symplectic fermions. In our case, either $|B_{\omega+}\rangle$ or $|B_{\omega-}\rangle$ must be excluded to make a consistent theory. Which one should be discarded is purely a matter of choice, so for definiteness let us discard $\omega+$ in the following calculation. The same thing happens when we construct the Ising model using Majorana fermions, where a suitable GSO projection is necessary.

4.3 Modular properties of the boundary states

Now that we have cylinder amplitudes between boundaries, we may discuss modular properties of boundary states and look for consistent boundary states as in Subsec.1.4.3. In the following we show that, using the modular transformations of Jacobi and Dedekind functions (see App. A), relations between $|a\rangle$ and $|\tilde{\alpha}\rangle$ are found by comparing the coefficients of functions on both sides of the duality relation (1.201). Some care is needed: in (1.201), $\langle\tilde{\alpha}|a\rangle$

and $\langle b|\tilde{\beta}\rangle$ should be understood merely as coefficients in the expansions $\langle\tilde{\alpha}| = \sum_a \langle\tilde{\alpha}|a\rangle\langle a|$ and $|\tilde{\beta}\rangle = \sum_b \langle b|\tilde{\beta}\rangle|b\rangle$, as ordinary orthonormal bra-ket operations are not possible in our non-unitary case. Also, all non-diagonal contributions have to be considered since on the right hand side amplitudes are not diagonalised into characters.

Equating the coefficients of $\Lambda_{1,2}(\tilde{\tau})\ln\tilde{q}$, $\eta(\tilde{\tau})^2\ln\tilde{q}$, $\Lambda_{1,2}(\tilde{\tau})$, $\eta(\tilde{\tau})^2$, $\Lambda_{0,2}(\tilde{\tau}) + \Lambda_{2,2}(\tilde{\tau})$ and $\Lambda_{0,2}(\tilde{\tau}) - \Lambda_{2,2}(\tilde{\tau})$, we have

$$\langle\tilde{\alpha}|B_{\omega+}\rangle\langle B_{\omega-}|\tilde{\beta}\rangle + \langle\tilde{\alpha}|B_{\omega-}\rangle\langle B_{\omega+}|\tilde{\beta}\rangle = 0, \tag{3.44}$$

$$\langle\tilde{\alpha}|B_{\omega+}\rangle\langle B_{\omega+}|\tilde{\beta}\rangle + \langle\tilde{\alpha}|B_{\omega-}\rangle\langle B_{\omega-}|\tilde{\beta}\rangle = \frac{n^{\mathcal{V}_0}_{\tilde{\alpha}\tilde{\beta}} - n^{\mathcal{V}_1}_{\tilde{\alpha}\tilde{\beta}}}{4\pi}, \tag{3.45}$$

$$\begin{aligned}&\langle\tilde{\alpha}|B_{\omega+}\rangle\langle B_{\Omega-}|\tilde{\beta}\rangle + \langle\tilde{\alpha}|B_{\omega-}\rangle\langle B_{\Omega+}|\tilde{\beta}\rangle + \langle\tilde{\alpha}|B_{\Omega+}\rangle\langle B_{\omega-}|\tilde{\beta}\rangle \\ &\qquad + \langle\tilde{\alpha}|B_{\Omega-}\rangle\langle B_{\omega+}|\tilde{\beta}\rangle = n^{\mathcal{V}_{3/8}}_{\tilde{\alpha}\tilde{\beta}} - n^{\mathcal{V}_{-1/8}}_{\tilde{\alpha}\tilde{\beta}},\end{aligned} \tag{3.46}$$

$$\begin{aligned}&\langle\tilde{\alpha}|B_{\omega+}\rangle\langle B_{\Omega+}|\tilde{\beta}\rangle + \langle\tilde{\alpha}|B_{\omega-}\rangle\langle B_{\Omega-}|\tilde{\beta}\rangle \\ &\quad + \langle\tilde{\alpha}|B_{\Omega+}\rangle\langle B_{\omega+}|\tilde{\beta}\rangle + \langle\tilde{\alpha}|B_{\Omega-}\rangle\langle B_{\omega-}|\tilde{\beta}\rangle \\ &\quad - \kappa(\langle\tilde{\alpha}|B_{\omega-}\rangle\langle B_{\omega-}|\tilde{\beta}\rangle + \langle\tilde{\alpha}|B_{\omega+}\rangle\langle B_{\omega+}|\tilde{\beta}\rangle) = 0,\end{aligned} \tag{3.47}$$

$$\langle\tilde{\alpha}|B_{\mu+}\rangle\langle B_{\mu-}|\tilde{\beta}\rangle + \langle\tilde{\alpha}|B_{\mu-}\rangle\langle B_{\mu+}|\tilde{\beta}\rangle = \frac{n^{\mathcal{V}_{-1/8}}_{\tilde{\alpha}\tilde{\beta}} + n^{\mathcal{V}_{3/8}}_{\tilde{\alpha}\tilde{\beta}}}{2}, \tag{3.48}$$

$$\begin{aligned}&\langle\tilde{\alpha}|B_{\mu+}\rangle\langle B_{\mu+}|\tilde{\beta}\rangle + \langle\tilde{\alpha}|B_{\mu-}\rangle\langle B_{\mu-}|\tilde{\beta}\rangle \\ &\qquad = n^{\mathcal{R}_0}_{\tilde{\alpha}\tilde{\beta}} + n^{\mathcal{R}_1}_{\tilde{\alpha}\tilde{\beta}} + \frac{1}{4}(n^{\mathcal{V}_0}_{\tilde{\alpha}\tilde{\beta}} + n^{\mathcal{V}_1}_{\tilde{\alpha}\tilde{\beta}}).\end{aligned} \tag{3.49}$$

Now following Cardy [79], let us find the consistent physical boundary states one by one. We assume that bra and ket boundary states have the same real coefficients $\langle a|\tilde{\alpha}\rangle = \langle\tilde{\alpha}|a\rangle$. We start by looking for a reference state $|\tilde{\mathcal{V}}_0\rangle$ such that $n^i_{\tilde{\mathcal{V}}_0\tilde{\alpha}} = n^i_{\tilde{\alpha}\tilde{\mathcal{V}}_0} = \delta^i_{\tilde{\alpha}}$. Let $\tilde{\alpha} = \tilde{\beta} = \tilde{\mathcal{V}}_0$ in (3.44)-(3.49). In the untwisted sector, from (3.44) we have $\langle\tilde{\mathcal{V}}_0|B_{\omega+}\rangle\langle B_{\omega-}|\tilde{\mathcal{V}}_0\rangle = 0$. Since we can exchange $\phi = 0$ and $\phi = \pi$ as a consequence of $\mathbb{Z}_2$ symmetry, we may put $\langle\tilde{\mathcal{V}}_0|B_{\omega+}\rangle = \langle B_{\omega+}|\tilde{\mathcal{V}}_0\rangle = 0$ without loss of generality. Then from (3.45) we have $|\langle\tilde{\mathcal{V}}_0|B_{\omega-}\rangle|^2 = 1/(4\pi)$, so $\langle\tilde{\mathcal{V}}_0|B_{\omega-}\rangle = \langle B_{\omega-}|\tilde{\mathcal{V}}_0\rangle = 1/(2\sqrt{\pi})$. Substituting these values, (3.46) gives $\langle\tilde{\mathcal{V}}_0|B_{\Omega+}\rangle = \langle B_{\Omega+}|\tilde{\mathcal{V}}_0\rangle = 0$. From (3.47) we find $\langle\tilde{\mathcal{V}}_0|B_{\Omega-}\rangle = \langle B_{\Omega-}|\tilde{\mathcal{V}}_0\rangle = \kappa/(4\sqrt{\pi})$. In the twisted sector, (3.48) becomes $\langle\tilde{\mathcal{V}}_0|B_{\mu+}\rangle\langle B_{\mu-}|\tilde{\mathcal{V}}_0\rangle = 0$, and again without losing generality we can choose $\langle\tilde{\mathcal{V}}_0|B_{\mu+}\rangle = \langle B_{\mu+}|\tilde{\mathcal{V}}_0\rangle = 0$. Then from (3.49) we find $\langle\tilde{\mathcal{V}}_0|B_{\mu-}\rangle = \langle B_{\mu-}|\tilde{\mathcal{V}}_0\rangle = 1/2$. Thus we found $|\tilde{\mathcal{V}}_0\rangle = (1/2\sqrt{\pi})|B_{\omega-}\rangle + (\kappa/4\sqrt{\pi})|B_{\Omega-}\rangle + (1/2)|B_{\mu-}\rangle$.

Next, we put $\tilde{\alpha} = \tilde{\mathcal{V}}_1$ and $\tilde{\beta} = \tilde{\mathcal{V}}_0$. We find $\langle\tilde{\mathcal{V}}_1|B_{\omega-}\rangle = \langle B_{\omega-}|\tilde{\mathcal{V}}_1\rangle = -1/(2\sqrt{\pi})$, $\langle\tilde{\mathcal{V}}_1|B_{\Omega-}\rangle = \langle B_{\Omega-}|\tilde{\mathcal{V}}_1\rangle = -\kappa/(4\sqrt{\pi})$, $\langle\tilde{\mathcal{V}}_1|B_{\mu-}\rangle = \langle B_{\mu-}|\tilde{\mathcal{V}}_1\rangle = 1/2$, and $\langle\tilde{\mathcal{V}}_1|B_{\omega+}\rangle = \langle B_{\omega+}|\tilde{\mathcal{V}}_1\rangle = \langle\tilde{\mathcal{V}}_1|B_{\Omega+}\rangle = \langle B_{\Omega+}|\tilde{\mathcal{V}}_1\rangle = \langle\tilde{\mathcal{V}}_1|B_{\mu+}\rangle = \langle B_{\mu+}|\tilde{\mathcal{V}}_1\rangle = 0$. The rest of the states are found similarly by putting $\tilde{\alpha} = \tilde{\mathcal{V}}_{-1/8}$, $\tilde{\mathcal{V}}_{3/8}$, $\tilde{\mathcal{R}}_0$, $\tilde{\mathcal{R}}_1$ one by one, all with $\tilde{\beta} = \tilde{\mathcal{V}}_0$. Then we find

$$\begin{aligned}
|\tilde{\mathcal{V}}_0\rangle &= \frac{1}{2\sqrt{\pi}}|B_{\omega-}\rangle + \frac{\kappa}{4\sqrt{\pi}}|B_{\Omega-}\rangle + \frac{1}{2}|B_{\mu-}\rangle,\\
|\tilde{\mathcal{V}}_1\rangle &= \frac{-1}{2\sqrt{\pi}}|B_{\omega-}\rangle - \frac{\kappa}{4\sqrt{\pi}}|B_{\Omega-}\rangle + \frac{1}{2}|B_{\mu-}\rangle,\\
|\tilde{\mathcal{V}}_{-1/8}\rangle &= |B_{\mu+}\rangle - 2\sqrt{\pi}|B_{\Omega+}\rangle,\\
|\tilde{\mathcal{V}}_{3/8}\rangle &= |B_{\mu+}\rangle + 2\sqrt{\pi}|B_{\Omega+}\rangle,\\
|\tilde{\mathcal{R}}\rangle &\equiv |\tilde{\mathcal{R}}_0\rangle = |\tilde{\mathcal{R}}_1\rangle = 2|B_{\mu-}\rangle.
\end{aligned} \tag{3.50}$$

Since $\tilde{\mathcal{R}}_0$ and $\tilde{\mathcal{R}}_1$ are the same state[3], we shall denote it as $\tilde{\mathcal{R}}$. There are other solutions obtained from the above by exchanging $\omega+$ and $\omega-$, $\Omega+$ and $\Omega-$, $\mu+$ and $\mu-$ (first and second pairs have to be exchanged simultaneously), as a consequence of the $\mathbb{Z}_2$ symmetry. Apart from this, the solutions are unique. Therefore, the duality of open and closed string channels provides strong enough constraints for the physical boundary states to be determined without ambiguity. Substituting these states back into the Cardy's constraint (1.201), possible $n^i_{\tilde{\alpha}\tilde{\beta}}$ on the left hand side are found. Note that $n^i_{\tilde{\alpha}\tilde{\beta}}$ cannot be determined uniquely by this procedure, since the characters are not independent but $\chi_{\mathcal{R}_0} = \chi_{\mathcal{R}_1} = 2(\chi_{\mathcal{V}_0} + \chi_{\mathcal{V}_1})$. Up to this ambiguity $n^i_{\tilde{\alpha}\tilde{\beta}}$ is identical to the fusion matrix ${N_{jk}}^i$ of (1.51) for the fusion rule of the triplet model (3.17).

4.4 Discussion

The essential point in our analysis is the appearance of the term $\eta(\tilde{\tau})^2 \ln \tilde{q}$ in the closed string amplitude through the proper treatment of the zero-mode. Note that the *five* modular functions $\eta(\tilde{\tau})^2$, $\eta(\tilde{\tau})^2 \ln \tilde{q}$, $\Lambda_{0,2}(\tilde{\tau})$, $\Lambda_{1,2}(\tilde{\tau})$, $\Lambda_{2,2}(\tilde{\tau})$ close under the modular transformation $\tilde{\tau} \to -1/\tilde{\tau}$. Discarding either $B_{\omega+}$ or $B_{\omega-}$ by a GSO-type projection, we obtained a set

[3] This simply comes from the fact that $\mathcal{R}_0$ and $\mathcal{R}_1$ have the same character functions. It is not clear if this degeneracy may be resolved, as in the 3-state Potts model

of boundary states including the reference state $\tilde{\mathcal{V}}_0$ which is necessary for the Cardy fusion procedure. This situation is quite similar to what happens in the Ising model case [85, 106, 107], where one of the two R sector states has to be discarded to give three boundary states, namely spin up, down, and free, which behave appropriately under modular transformations. See App.B for detail.

However, our model differs from the Ising model in one important respect. Neglecting the row and column involving the discarded state $B_{\omega+}$, the closed string amplitude of the untwisted sector gives a matrix

$$\begin{pmatrix} (\kappa - \ln\tilde{q})\eta(\tilde{\tau})^2 & -\Lambda_{1,2}(\tilde{\tau}) & -\eta(\tilde{\tau})^2 \\ -\Lambda_{1,2}(\tilde{\tau}) & 0 & 0 \\ -\eta(\tilde{\tau})^2 & 0 & 0 \end{pmatrix}, \tag{3.51}$$

which is not regular. Since one of the three eigenvalues is zero, the untwisted sector has only two non-trivial partition functions on diagonalisation. This means that the net content of the space spanned by $|B_{\omega-}\rangle$, $|B_{\Omega\pm}\rangle$, $|B_{\mu\pm}\rangle$ consists of only four states, not five. Therefore it is *not* possible to allocate five boundary states to the five modular functions.

This is related to the difficulty in expressing the physical boundary states in terms of the Ishibashi states. In ordinary CFTs, the solutions to (3.34) are found in the form of the Ishibashi states (1.196) which diagonalise the cylinder amplitudes to give characters. In our model, we can find candidates for the Ishibashi states such as

$$\begin{aligned}
|\mathcal{V}_0\rangle\rangle &= \frac{1}{2}|B_{\Omega+}\rangle + \frac{1}{2}|B_{\Omega-}\rangle, \\
|\mathcal{V}_1\rangle\rangle &= \frac{1}{2}|B_{\Omega+}\rangle - \frac{1}{2}|B_{\Omega-}\rangle, \\
|\mathcal{V}_{-1/8}\rangle\rangle &= \frac{1}{2}|B_{\mu+}\rangle + \frac{1}{2}|B_{\mu-}\rangle, \\
|\mathcal{V}_{3/8}\rangle\rangle &= \frac{1}{2}|B_{\mu+}\rangle - \frac{1}{2}|B_{\mu-}\rangle, \\
|\mathcal{R}\rangle\rangle \equiv |\mathcal{R}_0\rangle\rangle &= |\mathcal{R}_1\rangle\rangle = \sqrt{2}|B_{\Omega+}\rangle,
\end{aligned} \tag{3.52}$$

and

$$
\begin{aligned}
\langle\langle \mathcal{V}_0| &= \frac{-1}{2}\langle B_{\omega-}| - \frac{1}{2}\langle B_{\omega+}|, \\
\langle\langle \mathcal{V}_1| &= \frac{1}{2}\langle B_{\omega+}| - \frac{1}{2}\langle B_{\omega-}|, \\
\langle\langle \mathcal{V}_{-1/8}| &= \frac{1}{2}\langle B_{\mu+}| + \frac{1}{2}\langle B_{\mu-}|, \\
\langle\langle \mathcal{V}_{3/8}| &= \frac{1}{2}\langle B_{\mu-}| - \frac{1}{2}\langle B_{\mu+}|, \\
\langle\langle \mathcal{R}| \equiv \langle\langle \mathcal{R}_0| &= \langle\langle \mathcal{R}_1| = -\sqrt{2}\langle B_{\omega-}|,
\end{aligned}
\tag{3.53}
$$

whereby the characters (3.29) are reproduced in the form (1.197), and the orthogonality holds for $\mathcal{V}_0$, $\mathcal{V}_1$, $\mathcal{V}_{-1/8}$, and $\mathcal{V}_{3/8}$. Note that it is not possible to find such states with the same bra and ket coefficients. It can be easily checked that the physical boundary states $\tilde{\mathcal{V}}_0$ and $\tilde{\mathcal{V}}_1$ cannot be expressed as linear combinations of the states (3.52), (3.53). As a consequence, it is not possible to derive the Verlinde-type expression as in [101], since $\tilde{\mathcal{V}}_0$ plays an essential role in such discussions. This result is consistent with the non-diagonalisable fusion rule (3.17) which indicates the failure of the Verlinde formula.

Alternatively, the *four* representations $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{V}_{-1/8}$ and $\mathcal{V}_{3/8}$ can be regarded as fundamental constituents of the theory, since they themselves close under the fusion. As is mentioned in Subsec.3.3.1, it is argued by Kausch and Gaberdiel [48] that local and non-chiral bulk theory with finite multiplicity is given by three non-chiral representations, namely, $\mathcal{V}_{-1/8}\otimes\bar{\mathcal{V}}_{-1/8}$, $\mathcal{V}_{3/8}\otimes\bar{\mathcal{V}}_{3/8}$, and $\mathcal{R}$. This is analogous to our result that the physical boundary states for $\mathcal{R}_0$ and $\mathcal{R}_1$ are identical. Considering the four representations $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{V}_{-1/8}$ and $\mathcal{V}_{3/8}$, we see from (3.50) and (3.52) that the physical ket-states and Ishibashi ket-states are related as

$$
\begin{aligned}
|\tilde{\mathcal{R}}\rangle &= 2|\mathcal{V}_{-1/8}\rangle\rangle - 2|\mathcal{V}_{3/8}\rangle\rangle, \\
|\tilde{\mathcal{V}}_{-1/8}\rangle &= |\mathcal{V}_{-1/8}\rangle\rangle + |\mathcal{V}_{3/8}\rangle\rangle - \sqrt{2\pi}|\mathcal{R}\rangle\rangle, \\
|\tilde{\mathcal{V}}_{3/8}\rangle &= |\mathcal{V}_{-1/8}\rangle\rangle + |\mathcal{V}_{3/8}\rangle\rangle + \sqrt{2\pi}|\mathcal{R}\rangle\rangle.
\end{aligned}
\tag{3.54}
$$

These are the combinations of $|B_{\mu\pm}\rangle$ and $|B_{\Omega+}\rangle$. However, the boundary bra-states for these representations cannot be expressed in terms of the corresponding Ishibashi bra-states

(3.53), since the former are the combinations of $\langle B_{\mu\pm}|$ and $\langle B_{\Omega+}|$, whereas the latter are of $\langle B_{\mu\pm}|$ and $\langle B_{\omega-}|$. The candidate of the Ishibashi states (3.52), (3.53) are not unique, and alternatively, we can define such states so that the bra-states are linearly related to the consistent boundary states, but then the ket-states cannot be. That is, it is possible to express the consistent boundary states in terms of such Ishibashi states on either of the two boundaries, but not on both.

We started from the free-field representation of the $c = -2$ LCFT model and presented a possible solution for physical boundary states. Modular invariance imposes tight enough constraints on the partition function to identify the boundary states which allow the appearance of bulk representations. Although we could find five consistent boundary states $\tilde{\mathcal{R}}$, $\tilde{\mathcal{V}}_0$, $\tilde{\mathcal{V}}_1$, $\tilde{\mathcal{V}}_{-1/8}$ and $\tilde{\mathcal{V}}_{3/8}$, their implication is still not evident. Although the three states $\tilde{\mathcal{R}}$, $\tilde{\mathcal{V}}_{-1/8}$, $\tilde{\mathcal{V}}_{3/8}$ may be considered as *genuinely* physical as they correspond to non-chiral bulk representations, this speculation is not necessarily persuasive. In a recent study of the Abelian sandpile model (ASM), it was suggested in [66] that open and closed boundary conditions correspond to $\tilde{\mathcal{V}}_0$ and $\tilde{\mathcal{V}}_{-1/8}$, respectively. As the open boundary condition is dissipative, 'logarithmic' boundary states corresponding to irreducible subrepresentations of 'reducible but indecomposable' representations might be associated with interfaces to external systems. In order to investigate this conjecture and to see how general it can be, more study on concrete statistical models is obviously needed.

Another issue which is important in Cardy's construction is the completeness of the boundary states. Among well-studied unitary minimal models, the 3-state Potts model is known to possess a W-algebra, and its complete boundary states were found quite recently [82–84]. In that model, only the fixed and mixed boundary states are obtained by Cardy's method from the W-invariant conformal towers; in order to obtain the complete set including "free" and "new" boundary states, all chiral representations from the Kac table not constrained by the W-symmetry had to be considered. Looking at our boundary states of the triplet model in an analogy with the 3-state Potts model, the boundary states (3.50) we have obtained may be considered as the W-invariant (diagonal) set, whose 3-state Potts model counterpart is the fixed and mixed boundary states. Indeed, for all the states (3.50), the gluing automorphism Γ associated with the triplet W-algebra is trivial. The compatibility of the multiplicity matrix $n^i_{\tilde{\alpha}\tilde{\beta}}$ and the fusion matrix $N_{jk}{}^i$ also seems to suggest the similarity of our boundary states to the diagonal boundary states in Virasoro minimal models.

5 Summary

In this chapter we have discussed boundary theories of the simplest and so far the best understood model of LCFT, the $c = -2$ triplet model. After seeing the behaviour of a boundary correlation function calculated by the standard mirroring method, we reviewed the algebraic and free-field constructions of the triplet model, and then discussed boundary states in this model. As we do not know well-defined Ishibashi states in the $c = -2$ triplet model, we used coherent states constructed from the symplectic fermion as the basis of the states, and found boundary states which satisfy Cardy's consistency condition.

Since boundary LCFT is a very young subject and its study has just started, it should be appropriate to conclude this chapter by addressing some problems that have to be tackled for the next step of its progress.

Now that we know the existence of consistent boundary states in the $c = -2$ triplet model, the question of completeness, as is mentioned at the end of the last section, is obviously one of the things to be considered next. The completeness of boundary conditions comes from the complete solutions of the sewing relations, which can be spelled out when the bulk theory is fully solved. Although our knowledge on the bulk theory for $c = -2$ triplet model has increased in recent years, it is far from being fully solved since our algebraic understanding of the model (corresponding to the quantum group structure in conventional CFTs) is still limited. For example, the sewing relations rely on the existence of a bona-fide unity operator in the theory, which is obviously absent in the $c = -2$ triplet model. In order to discuss the completeness of boundary states in LCFT, we need to know the algebraic structure of the theory which may be much more complicated than conventional CFTs.

Studying boundary LCFTs other than $c = -2$ is of course an important issue, but at the moment this seems to be far beyond our reach. For studying boundary states of general LCFT models where a free-field representation is not available, we need to construct generalised Ishibashi states which can substitute for the coherent states used here, and indeed, a candidate of such states for the $c = -2$ triplet model is proposed in [129–131]. It is however not easy to discuss boundary theory of general LCFTs because representations and especially characters are known in only a very few models. Before discussing boundary behaviour, we need to know more about bulk representations of such theories.

Regarding the applications of LCFTs, there are numerous examples of systems in statistical models and string theory which have been claimed to be modelled by LCFTs, and boundary may become important in many cases. For example, the $O(n)$ model with $n < 1$ has a 'special' transition, as well as 'ordinary' and 'extraordinary' transitions, and such behaviour may somehow be related to the boundary states of LCFT. The correspondences of these models and LCFTs are however all quite speculative and it is difficult to make direct connections between LCFT results and what is happening in a system which is believed to be modelled. If a relation between some well-defined system and LCFT is established at the operator-content level, we can expect feed back from e.g. numerical simulations and our understanding of LCFT would be accelerated enormously.

Chapter 4

Conclusions

In this thesis we have employed the free-field construction as powerful tools to investigate boundary states of CFTs.

In Chap.2 we presented a construction of boundary states in the Coulomb-gas formalism of Dotsenko and Fateev [27]. We constructed coherent states on the charged bosonic Fock space realising $c < 1$ minimal models and found that they preserve conformal symmetries under certain conditions. We then calculated the closed string amplitudes of a cylinder between such boundary states and showed that linear sums of coherent boundary states satisfy Cardy's consistency conditions. In particular, we discussed the Ising model as an example and wrote its consistent boundary states associated with the fixed (up and down) and free boundary conditions using the coherent states we have constructed. Such a construction of boundary states is potentially quite powerful; as they are expressed on the Fock space, we can in principle calculate any correlation functions involving such boundaries, making a direct connection between algebraically classified boundary conditions and correlation functions which are observable.

In Chap.3 we discussed the behaviour of logarithmic CFT near boundaries, and the $c = -2$ triplet model was studied in detail. The character functions of this model do not themselves close under the modular S transformation and therefore the modular matrix S cannot be defined. The fusion rules found by Kausch and Gaberdiel [47] are not diagonalisable and the Verlinde formula fails. Due to these features which are absent in conventional rational CFTs, it is not possible to find consistent boundary states of this model by applying the standard Cardy method based on the Ishibashi construction of boundary states.

In particular, even the existence of consistent boundary states was not clear in this model. We studied this problem following the procedure similar to the one in Chap.2, by using the symplectic fermion representation of the $c = -2$ triplet model. We found, despite the pathological features mentioned above, a set of bona-fide boundary states with consistent modular properties. These consistent boundary states are expressed as linear sums of coherent states built on the Fock space. There is however a difficulty defining Ishibashi states, which is related to the fact that the cylinder amplitudes between boundaries are irregular. We concluded that the difficulty we met when we try to apply the Cardy method to this model is due to the absence of well-defined Ishibashi states.

The material discussed in this thesis may be regarded as generalisations of the boundary CFTs for free bosons and fermions, which have been established for many years and well understood. Compared to these theories, we must unfortunately admit that the boundary theories of Coulomb-gas systems and logarithmic CFTs are still in the elementary stage of developments. There are indeed many things to be understood, e.g. the treatment of the zero-mode and truncation of non-unitary representations in the Coulomb-gas, completeness of boundary conditions in LCFT, etc. We close this thesis by hoping that such issues will be clarified in the near future and the free-field construction of boundary CFTs presented in this thesis will become a truly useful tool in various physics applications.

Appendix A

Summary of conventions

1 Geometric conventions

For two-dimensional real coordinates (z^1, z^2) in the conformal gauge

$$g_{\mu\nu}(z^1, z^2) = \rho(z^1, z^2)\delta_{\mu\nu}, \tag{A.1}$$

we define complex coordinates $(z, \bar{z})$ as

$$z = z^1 + iz^2, \quad \bar{z} = z^1 - iz^2. \tag{A.2}$$

Derivatives with respect to these complex coordinates are

$$\partial = \partial_z = \frac{1}{2}(\partial_1 - i\partial_2), \quad \bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2). \tag{A.3}$$

The metrics for the complex coordinates $(z, \bar{z})$ are

$$g_{\mu\nu}(z, \bar{z}) = \rho(z, \bar{z}) \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad g^{\mu\nu}(z, \bar{z}) = \frac{1}{\rho(z, \bar{z})} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \tag{A.4}$$

where $\mu = (z, \bar{z})$.

The scalar curvature R is defined such that $R = 2/r^2$ for a sphere of radius r. More explicitly, we have used the Misner-Thorne-Wheeler [132] convention,

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\delta\beta} - \partial_\delta g_{\beta\gamma}), \tag{A.5}$$

$$R^{\alpha}{}_{\beta\gamma\lambda} = \partial_\gamma \Gamma^{\alpha}{}_{\beta\lambda} - \partial_\lambda \Gamma^{\alpha}{}_{\beta\gamma} + \Gamma^{\alpha}{}_{\delta\gamma}\Gamma^{\delta}{}_{\beta\lambda} - \Gamma^{\alpha}{}_{\delta\lambda}\Gamma^{\delta}{}_{\beta\gamma}, \tag{A.6}$$

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}, \tag{A.7}$$

$$R = g^{\mu\nu} R_{\mu\nu}, \tag{A.8}$$

except that we work in a two-dimensional Euclidean space.

Throughout this thesis we used the anharmonic ratio defined as

$$\eta = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}, \tag{A.9}$$

except in Sec.3.2 where we used

$$z = \frac{\eta}{\eta - 1} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}, \tag{A.10}$$

with $z_3 = z_1^*$, $z_4 = z_2^*$.

2 Elliptic modular functions

We summarise the definitions of elliptic modular functions and list formulas used in the main text.

The Dedekind eta function is defined as

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{A.11}$$

We have used the basic Jacobi theta functions defined by

$$\theta_2(\tau) = \sum_{k\in\mathbb{Z}} q^{(k+1/2)^2/2}, \tag{A.12}$$

$$\theta_3(\tau) = \sum_{k\in\mathbb{Z}} q^{k^2/2}, \tag{A.13}$$

$$\theta_4(\tau) = \sum_{k\in\mathbb{Z}} (-1)^k q^{k^2/2}, \tag{A.14}$$

as well as the generalised theta functions

$$\Theta_{\lambda,\mu}(\tau) = \sum_{k\in\mathbb{Z}} q^{(2\mu k+\lambda)^2/4\mu}, \tag{A.15}$$

where $q = e^{2\pi i\tau}$. From this definition it is obvious that $\Theta_{\lambda,\mu}(\tau)$ has the following symmetries,

$$\Theta_{\lambda,\mu}(\tau) = \Theta_{\lambda+2\mu,\mu}(\tau) = \Theta_{-\lambda,\mu}(\tau). \tag{A.16}$$

These two definitions of theta functions are related to each other. For example, as is easily verified,

$$\theta_2(\tau) = 2\Theta_{1,2}(\tau), \tag{A.17}$$

$$\theta_3(\tau) = \Theta_{0,2}(\tau) + \Theta_{2,2}(\tau), \tag{A.18}$$

$$\theta_4(\tau) = \Theta_{0,2}(\tau) - \Theta_{2,2}(\tau), \tag{A.19}$$

and

$$\sqrt{\eta(\tau)\theta_2(\tau)/2} = \Theta_{2,12}(\tau) - \Theta_{10,12}(\tau), \tag{A.20}$$

$$\sqrt{\eta(\tau)\theta_3(\tau)} = \Theta_{1,12}(\tau) + \Theta_{5,12}(\tau) - \Theta_{7,12}(\tau) - \Theta_{11,12}(\tau), \tag{A.21}$$

$$\sqrt{\eta(\tau)\theta_4(\tau)} = \Theta_{1,12}(\tau) - \Theta_{5,12}(\tau) - \Theta_{7,12}(\tau) + \Theta_{11,12}(\tau). \tag{A.22}$$

We have also used the notation

$$\Lambda_{\lambda,\mu}(\tau) = \frac{\Theta_{\lambda,\mu}(\tau)}{\eta(\tau)}. \tag{A.23}$$

Under the modular S ($\tau \to -1/\tau$) and T ($\tau \to \tau + 1$) transformations, $\Theta_{\lambda,\mu}(\tau)$ and $\eta(\tau)$

transform as

$$\Theta_{\lambda,\mu}(-1/\tau) = \sqrt{\frac{-i\tau}{2\mu}} \sum_{\nu=0}^{2\mu-1} e^{\lambda\nu\pi i/\mu} \Theta_{\nu,\mu}(\tau),$$
$$\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau), \tag{A.24}$$

and

$$\Theta_{\lambda,\mu}(\tau+1) = e^{\lambda^2\pi i/2\mu} \Theta_{\lambda,\mu}(\tau),$$
$$\eta(\tau+1) = e^{\pi i/12}\eta(\tau). \tag{A.25}$$

The three functions $\theta_2(\tau)$, $\theta_3(\tau)$ and $\theta_4(\tau)$ transform into each other under S and T, as

$$\theta_2(-1/\tau) = \sqrt{-i\tau}\theta_4(\tau), \tag{A.26}$$

$$\theta_3(-1/\tau) = \sqrt{-i\tau}\theta_2(\tau), \tag{A.27}$$

$$\theta_4(-1/\tau) = \sqrt{-i\tau}\theta_3(\tau), \tag{A.28}$$

and

$$\theta_2(\tau+1) = \sqrt{i}\theta_2(\tau), \tag{A.29}$$

$$\theta_3(\tau+1) = \theta_4(\tau), \tag{A.30}$$

$$\theta_4(\tau+1) = \theta_3(\tau). \tag{A.31}$$

Appendix B

Ising model boundary states

As one of the simplest statistical models of spin systems, the Ising model has been studied extensively for more than three quarters of a century. The critical Ising model is described by $c = 1/2$ Virasoro minimal conformal field theory, and via Jordan-Wigner transformation it is also related to free Majorana fermions. It is therefore possible to construct consistent boundary states from free fields, apart from the standard Cardy's method based on Ishibashi states. Although this has been studied by many authors [85, 106, 107], here we summarise the result emphasizing the robustness of the procedure in a simple well-known example.

1 Boundary conditions and Jordan-Wigner transformation

We consider a one-dimensional spin chain with the Hamiltonian [133],

$$H = -\frac{1}{2}\lambda \sum_m \sigma_1(m) - \frac{1}{2}\sum_m \sigma_3(m)\sigma_3(m+1), \tag{B.1}$$

where we choose $\lambda = 1$ for the Ising model at criticality, and m indicates the location of a spin. The matrices $\sigma_i(m)$ are the Pauli spin matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.2}$$

In this notation, a state of each spin is denoted by a two-component vector. In particular, the spin up, down, and free states are respectively written as

$$(\uparrow) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (\downarrow) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (F) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{B.3}$$

The boundary conditions with all spins up, down, and free along the boundary are then written as

$$|\uparrow\rangle = \prod_m \begin{pmatrix} 1 \\ 0 \end{pmatrix}_m, \tag{B.4}$$

$$|\downarrow\rangle = \prod_m \begin{pmatrix} 0 \\ 1 \end{pmatrix}_m, \tag{B.5}$$

$$|F\rangle = \prod_m \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]_m. \tag{B.6}$$

The Ising spin chain is mapped to a pair of fermions by the Jordan-Wigner transformation,

$$\psi_1(n) = \frac{1}{\sqrt{2}} \left(\prod_{m=-\infty}^{n-1} \sigma_1(m) \right) \sigma_2(n), \tag{B.7}$$

$$\psi_2(n) = \frac{1}{\sqrt{2}} \left(\prod_{m=-\infty}^{n-1} \sigma_1(m) \right) \sigma_3(n), \tag{B.8}$$

which satisfy the ordinary anticommutation relation of Majorana fermions, $\{\psi_i(m), \psi_j(n)\} = \delta_{ij}\delta_{mn}$, where $i, j = \{1, 2\}$. The holomorphic and antiholomorphic fermions defined by

$$\psi(n) = \frac{1}{\sqrt{2}}(\psi_1(n) - \psi_2(n)), \tag{B.9}$$

$$\bar{\psi}(n) = \frac{1}{\sqrt{2}}(\psi_1(n) + \psi_2(n)), \tag{B.10}$$

then satisfy the anticommutation relations $\{\psi(m), \psi(n)\} = \delta_{mn}$, $\{\bar{\psi}(m), \bar{\psi}(n)\} = \delta_{mn}$,

$\{\psi(m), \bar{\psi}(n)\} = 0$, and in the continuum limit they coincide with the Majorana fermion fields $\psi(w)$, $\bar{\psi}(\bar{w})$ of Subsec.1.3.2. The complex coordinates are defined by $w = t - ix$, $\bar{w} = t + ix$, where x is the position and t is the Euclidean time. As these fermions are directly constructed from the Pauli matrices, we may find the action of fermions on the boundary states (B.4) - (B.6). Such expressions are simplified by using equations of motion, and in terms of the mode operators defined by

$$\psi(w) = \sqrt{\frac{2\pi}{L}} \sum_n b_n e^{-2\pi n w/L}, \tag{B.11}$$

$$\bar{\psi}(\bar{w}) = \sqrt{\frac{2\pi}{L}} \sum_n \bar{b}_n e^{-2\pi n \bar{w}/L}, \tag{B.12}$$

we find

$$\begin{aligned} &(b_n - i\bar{b}_{-n})|\uparrow,\downarrow\rangle = 0, \\ &(b_n + i\bar{b}_{-n})|F\rangle = 0. \end{aligned} \tag{B.13}$$

2 Majorana fermion boundary states

Now let us consider the system of the Majorana fermions with action

$$\mathcal{S} = \frac{1}{2\pi} \int d^2x (\bar{\psi}\partial\bar{\psi} + \psi\bar{\partial}\psi), \tag{B.14}$$

and consider the boundary states by a top-down approach. As we have been doing in the main text, we consider a cylinder of length T and circumference L. The cylinder is mapped to the complex z-plane by $z = e^{2\pi(t-ix)/L}$, where t and x are now coordinates along and around the cylinder, respectively. We assume that the boundary term in this action vanishes, $\psi^2 + \bar{\psi}^2 = 0$, which reduces to

$$\psi = \pm i\bar{\psi}. \tag{B.15}$$

On the z-plane the mode expansions (B.11), (B.12) become

$$\psi(z) = \sum_{n\in\mathbb{Z}+\lambda} b_n z^{-n-1/2}, \tag{B.16}$$

$$\bar{\psi}(z) = \sum_{n\in\mathbb{Z}+\lambda} \bar{b}_n \bar{z}^{-n-1/2}, \tag{B.17}$$

where $\lambda = 0$ for R and $1/2$ for NS sectors. The anticommutation relations are $\{b_m, b_n\} = \delta_{m+n} = \{\bar{b}_m, \bar{b}_n\}$ and $\{b_m, \bar{b}_n\} = 0$. Then through the radial quantization on the z-plane the boundary condition (B.15) at the boundary $t = 0$ becomes

$$(b_n \mp i\bar{b}_{-n})|B_\pm\rangle = 0, \tag{B.18}$$

which is satisfied by the coherent states [126–128]

$$|B_\pm\rangle = \prod_{n>0} e^{\pm i b_{-n}\bar{b}_{-n}}|0_\pm\rangle, \tag{B.19}$$

where $|0_\pm\rangle$ is annihilated by $b_{n>0}$ and $\bar{b}_{n>0}$, and satisfies $(b_0 \mp i\bar{b}_0)|0_\pm\rangle_R = 0$ in the R sector. The NS ground state is unique, $|0_\pm\rangle_{NS} = |0\rangle_{NS}$. The two degenerate ground states $|0_\pm\rangle_R$ in the R sector are treated symmetrically by demanding [107]

$$b_0|0_\pm\rangle_R = \frac{1}{\sqrt{2}} e^{\pm i\pi/4}|0_\mp\rangle_R,$$
$$\bar{b}_0|0_\pm\rangle_R = \frac{1}{\sqrt{2}} e^{\mp i\pi/4}|0_\mp\rangle_R, \tag{B.20}$$

and the distinction is absorbed in the definition of $|B_\pm\rangle$ in (B.19). Thus we have four conformally invariant free fermion boundary states

$$|B_\pm\rangle_{NS} = \prod_{n\geq 1/2} e^{\pm i b_{-n}\bar{b}_{-n}}|0\rangle_{NS},$$
$$|B_\pm\rangle_R = \prod_{n\geq 1} e^{\pm i b_{-n}\bar{b}_{-n}}|0_\pm\rangle_R. \tag{B.21}$$

On the other end of the cylinder, the states satisfying the condition $\langle B_\pm|(b_n \pm i\bar{b}_{-n}) = 0$ are found similarly as

$$
\begin{aligned}
{}_{NS}\langle B_\pm| &= {}_{NS}\langle 0| \prod_{n\geq 1/2} e^{\pm i b_n \bar{b}_n}, \\
{}_{R}\langle B_\pm| &= {}_{R}\langle 0_\pm| \prod_{n\geq 1} e^{\pm i b_n \bar{b}_n}.
\end{aligned} \tag{B.22}
$$

The Virasoro operators of Majorana fermions are written using the mode operators as

$$
\begin{aligned}
L_n &= \frac{1}{2}\sum_{k\in\mathbb{Z}+\lambda} (\frac{n}{2}-k) : b_k b_{n-k} : + a_\lambda \delta_{n0}, \\
\bar{L}_n &= \frac{1}{2}\sum_{k\in\mathbb{Z}+\lambda} (\frac{n}{2}-k) : \bar{b}_k \bar{b}_{n-k} : + \bar{a}_\lambda \delta_{n0},
\end{aligned}
$$

where $a_{1/2} = \bar{a}_{1/2} = 0$ (R) and $a_0 = \bar{a}_0 = 1/16$ (NS). It is easily verified that the states (B.21) are indeed conformally invariant, $(L_n - \bar{L}_{-n})|B_\pm\rangle_{NS,R} = 0$. Defining the modular parameters as $\tilde{\tau} = 2iT/L$ and $\tilde{q} = e^{2\pi i\tilde{\tau}}$, the cylinder partition functions $\langle a|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-1/24}|b\rangle$ for the boundary coherent states $|a\rangle = |B_\pm\rangle_{NS,R}$ are calculated (in the order of $|B_+\rangle$, $|B_-\rangle$) as

$$
\langle a|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|b\rangle = \begin{cases} \begin{pmatrix} \sqrt{\frac{\theta_3(\tilde{\tau})}{\eta(\tilde{\tau})}} & \sqrt{\frac{\theta_4(\tilde{\tau})}{\eta(\tilde{\tau})}} \\ \sqrt{\frac{\theta_4(\tilde{\tau})}{\eta(\tilde{\tau})}} & \sqrt{\frac{\theta_3(\tilde{\tau})}{\eta(\tilde{\tau})}} \end{pmatrix} & \text{in NS,} \\ \begin{pmatrix} \sqrt{\frac{\theta_2(\tilde{\tau})}{2\eta(\tilde{\tau})}} & 0 \\ 0 & \sqrt{\frac{\theta_2(\tilde{\tau})}{2\eta(\tilde{\tau})}} \end{pmatrix} & \text{in R.} \end{cases} \tag{B.23}
$$

We cannot have both the states of the R sector in a consistent (GSO projected) theory, and either of $|B_\pm\rangle_R$ has to be excluded. Which one is to be discarded is in fact a matter of choice, depending on how we define the Jordan-Wigner transformation. For definiteness, we discard $|B_-\rangle_R$ and choose $|B_\pm\rangle_{NS}$ and $|B_+\rangle_R$ as the basis of the boundary states. Then we obtain the three modular functions $\sqrt{\theta_2/\eta}$, $\sqrt{\theta_3/\eta}$, and $\sqrt{\theta_4/\eta}$ which transform among themselves under the modular transformations S and T.

Now we may follow the same procedure as Subsec.1.4.3, assuming the existence of the state $|\tilde{0}\rangle$ such that $n^i_{\tilde{0}\tilde{\alpha}} = n^i_{\tilde{\alpha}\tilde{0}} = \delta^i_{\tilde{\alpha}}$, and obtain consistent boundary states expressed in terms of the coherent states defined above. We find

$$|\tilde{0}\rangle = |\tilde{I}\rangle = \frac{1}{\sqrt{2}}|B_+\rangle_{NS} + \frac{1}{\sqrt[4]{2}}|B_+\rangle_R, \quad \text{(B.24)}$$

$$|\tilde{\epsilon}\rangle = \frac{1}{\sqrt{2}}|B_+\rangle_{NS} - \frac{1}{\sqrt[4]{2}}|B_+\rangle_R, \quad \text{(B.25)}$$

$$|\tilde{\sigma}\rangle = |B_-\rangle_{NS}. \quad \text{(B.26)}$$

We can compare this with the direct result (B.13) and confirm Cardy's identification

$$|\tilde{I}\rangle = |\uparrow\rangle, \quad \text{(B.27)}$$

$$|\tilde{\epsilon}\rangle = |\downarrow\rangle, \quad \text{(B.28)}$$

$$|\tilde{\sigma}\rangle = |F\rangle, \quad \text{(B.29)}$$

(up and down may be exchanged). The Ising model has two $\mathbb{Z}_2$ symmetries, namely, high-temperature-low-temperature symmetry (P) and spin up-down symmetry (Q). Actions of P and Q on the boundary states are

$$P: \quad \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}} \leftrightarrow |F\rangle, \quad \text{(B.30)}$$

$$Q: \quad |\uparrow\rangle \leftrightarrow |\downarrow\rangle,\ |F\rangle \leftrightarrow |F\rangle. \quad \text{(B.31)}$$

In terms of the Majorana fermions, P exchanges $|B_+\rangle$ and $|B_-\rangle$. Q is blind about $\pm$, but flips the sign of $|B\rangle_R$. The boundary conditions of the Majorana fermions (B.13) do not resolve the degeneracy of Q since the distinction of up and down is encoded in the partition function.

Appendix C

Vertex operator algebra

Expressions of normal ordering in the twisted vacua are conveniently calculated using the twisted Borcherds identity in the vertex algebras [121, 134]. We collect the necessary definitions and formulae leading to the expressions for the energy-momentum tensor and W-operators which are used in Chap.3.

Let $\mathcal{U}$ be a vector space. A *field* $f(z)$ is defined as a power series

$$f(z) \equiv \sum_{n \in \mathbb{Z}+\lambda} f_{(n)} z^{-n-1}, \tag{C.1}$$

where $f_{(n)}$ has the property that $f_{(n)} v = 0$ for $n \gg 0$, $\forall v \in \mathcal{U}$. The parenthesis in $f_{(n)}$ is to emphasize the difference from the ordinary expansion $f(z) = \sum_n f_n z^{-n-h}$ with h the conformal weight of f. A *state* $|f\rangle$ is defined on a $SL(2, C)$ invariant vector $|0\rangle$ (denoted Ω in Chap.3) as

$$|f\rangle = f_{(-1)}|0\rangle. \tag{C.2}$$

A vertex operator Y is defined as a map from a state $|f\rangle$ to a field $f(z)$,

$$Y : |f\rangle \mapsto f(z) \equiv Y(|f\rangle, z). \tag{C.3}$$

We define the twisted *module* as a vector accompanied by a vertex operator satisfying the following three axioms,

$$(M1) \qquad Y(|0\rangle, z) = 1\!\!1, \tag{C.4}$$

$$(M2) \qquad Y(L_{-1}|f\rangle, z) = \partial Y(|f\rangle, z), \tag{C.5}$$

$$(M3) \qquad \sum_{j=0}^{\infty} \begin{pmatrix} m \\ j \end{pmatrix} Y(f_{(n+j)}|g\rangle, z) z^{m-j}$$

$$= \sum_{j=0}^{\infty} (-1)^j \begin{pmatrix} n \\ j \end{pmatrix} \left[f_{(m+n-j)} Y(|g\rangle, z) z^j - (-1)^n \epsilon_{fg} Y(|g\rangle, z) f_{(m+j)} z^{n-j} \right], \tag{C.6}$$

where $n \in \mathbb{Z}$, $m \in \mathbb{Z} + \lambda$, and ϵ_{fg} is defined to be -1 if f and g are both fermionic, and 1 otherwise. $1\!\!1$ is the identity. (M1), (M2), (M3) are called the vacuum axiom, the translational covariance, and the twisted Borcherds identity, respectively. Putting $m = \lambda$ and $n = -1$ in (M3), we have a formula for the normal-ordered products in a twisted module

$$\begin{aligned} :Y(|f\rangle, z) Y(|g\rangle, z): \quad &\equiv \quad \sum_k \left[\sum_{m=-\infty}^{\lambda-1} f_{(m)} g_{(k-m-1)} + \epsilon_{fg} \sum_{m=\lambda}^{\infty} g_{(k-m-1)} f_{(m)} \right] z^{-k-1} \\ &= \quad \sum_{j=0}^{\infty} \begin{pmatrix} \lambda \\ j \end{pmatrix} Y(f_{(j-1)}|g\rangle, z) z^{-j}. \end{aligned} \tag{C.7}$$

Note that the last line of (C.7) gives only a finite number of terms. The energy-momentum tensor and the W-operators of our model are defined in vertex algebra as

$$T(z) = \frac{1}{2} d_{\alpha\beta} Y(\chi^{\alpha}_{(-1)} \chi^{\beta}_{(-1)} |0\rangle, z), \tag{C.8}$$

$$W^a(z) = t^a_{\alpha\beta} Y(\chi^{\alpha}_{(-2)} \chi^{\beta}_{(-1)} |0\rangle, z), \tag{C.9}$$

where $t^0_{\pm\mp} = -1/2$, $t^{\pm}_{\pm\pm} = \pm 1$. The Virasoro operators (3.27) are obtained by applying (C.7) to (C.8). The W-operator (C.9) similarly leads to the expressions (3.28).

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