Aspects of $\mathcal{N} = 4$ Chern-Simons Theories and Their Gravity Duals

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Abstract

We discuss aspects of $\mathcal{N} = 4$ Chern-Simons theories and their gravity duals. Our study is summarized as that of the relation between wrapped M-branes and their dual objects.

We propose the correspondence between wrapped M2-branes and monopole operators in the twisted sector. To confirm this proposal, we compute a superconformal index for the Chern-Simons theories including monopole contribution. We compare it to the corresponding multi-particle index for M-theory on the dual geometry in the large N limit. The dual geometry has non-trivial two-cycles in the internal space. M2-branes wrapped on them contribute to the multi-particle index. We derive the contribution of wrapped M2-branes by constructing the action of a seven-dimensional $\mathcal{N} = 2$ vector multiplet on the singular locus and performing Kaluza-Klein analysis. We establish one-to-one map between the data of both theories such as magnetic charges of twisted monopole operators and the wrapping number of wrapped M2-branes. By using the mapping, we confirm the agreement of the indices for many sectors by using analytic and numerical methods.

We establish the relation between an equivalent class of ranks and the three-cycle homology of the dual geometry. We obtain the equivalent class of ranks by classifying fractional D3-brane charges defined with taking account of D3-brane creation due to Hanany-Witten effect. The third homology describes the charges of fractional M2-branes, or M5-branes wrapped on three-cycles.

We also discuss the duality between baryonic operators and M5-branes wrapped on five-cycles. Their degeneracy, the conformal dimension, decomposability to mesons are reproduced in the gravity side. If the gauge group is the product of unitary gauge groups, the baryonic operators cannot be gauge invariant. We discuss that the gauge invariance cannot be imposed on all the counterparts corresponding to wrapped M-branes in $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence.

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Chapter 1

Introduction

M means Membrane, Miracle, Mystery, Mother, Matrix, $\cdots$.

1.1 M-theory

Why does the author study M-theory? His brief answer is that M-theory has possibilities beyond superstring theory.

Superstring theory also has big possibilities: it might unify gauge interaction and gravity in quantum level, reveal the thermodynamic property of black holes, elucidate the dynamics of the origin of our universe. Superstring theory is well-defined in ten-dimensional space-time and the rich dimension opens a possibility to explain the structure of gauge interaction assumed in the standard model. However, since well-known superstring theory is only defined perturbatively, non-perturbative aspects of superstring theory remain as mysteries. One of such mysteries is that superstring theory can be defined in at least five ways: type I, type IIA, type IIB, SO(32) Heterotic, $E_8 \times E_8$ Heterotic superstring theories. To elucidate the relation to each other, it is necessary to study non-perturbative effects in superstring theory.

M-theory has a possibility to solve such problems included in superstring theory. M-theory is an eleven-dimensional theory including gravity to appear when superstring theory goes to the strongly coupled region [1]. A new direction appearing in this limit is often called "M-direction" or "M-circle." Due to the discovery of M-theory, the five superstring theories mentioned above have been expected to be unified in the strongly coupled regime through duality. There is more than that in M-theory. It unifies not only the five superstring theories, but also extended objects appearing in superstring theory. In superstring theory, there also exist NS-fivebranes and D-branes, which couple to NS-NS fields and R-R fields [2], respectively. They are integrated into "M-branes" [3]: fundamental strings and D-twobranes are integrated into membranes, which is extended to two-dimensional space, and D-fourbranes and NS-fivebranes are into M-fivebranes, which are objects extending to five-dimensional space. D-zerobranes are dissolved into the M-circle as the Kaluza-Klein tower.

This fact implies that membranes should play a fundamental role in M-theory, as in the case of strings in superstring theory. Unfortunately, however, the world-volume theory of a membrane, which is also formulated to satisfy the general coordinate transformation invariance on it, is not understood well as a quantum theory compared to the world-sheet theory of a string [4]. For example, it has been successful to quantize a string if the background has an enough big symmetry. In addition, the guiding principle of the quantum field theory on a string world-sheet is established. That is, all gauge symmteries existing in the classical action should be respected in the quantum theory. In fact, this principle is so strong that we can determine the target space-time dimension as ten. As for membranes, however, both have not been established yet.

So can we proceed to the study of membranes and reveal mysteries of quantum aspects of membranes? The answer is yes. The recent progress elucidates a certain aspect of a multi-body system of membranes and the interaction in the system.

1.2 Interaction between membranes

How is the interaction between membranes in M-theory described? It is instructive to see that the question is not so simple compared to the case of D-branes in string theory. Let us consider the situation in which there are multiple D-branes in the space-time. In this situation, the interaction occurs between D-branes in general. This interaction is carried by open strings. Since open strings become gauge field in their massless mode, the interaction is described by a certain gauge theory. Open strings can attach any D-branes at their endpoints. This degree of freedom of the end points (Chan-Paton degree of freedom) can be described by a $N \times N$ matrix when there exist N D-branes. It accounts for non-abelian gauge symmetry on the brane system. In the low energy scale, the interaction is effectively described by supersymmetric Yang-Mills theories.

Probably this picture is also the case in M-theory, but it is more complicated. For example, it is natural to consider that the interaction between M-branes should be carried by membranes ending on them. The gauge degrees of freedom should be realized by the boundaries of membranes. This implies, however, that it might not be appropriate to describe the gauge degrees of freedom by a matrix and the gauge symmetry realized between membranes might not be usual Lie-algebra any more. If this is the case, is it possible to construct such a theory describing the interaction between membranes?

Recent progress in three-dimensional interacting conformal field theories (CFT) enables us to answer the question in the low energy limit. The answer is that the low energy effective actions describing gauge interaction between membranes are given by superconformal Chern-Simons-matter theories.

1.3 Membrane field theories

One of non-trivial guiding principles to construct an effective field theory of membranes is to possess the maximal supersymmetry or $SO(8)$ R-symmetry [5]. This is expected from AdS/CFT correspondence via membranes [6]. This obstacle is first overcome by Bagger, Lambert and Gustavsson [7, 8, 9, 10]. To describe the gauge interaction between membranes, they use a curious mathematical structure, so-called 3-bracket, which is a natural generalization of usual Lie-bracket. The gauge invariance in their model (BLG model) requires 3-bracket to satisfy the constraints including the generalized Jacobi identity. These rules are too restrictive to describe finitely many membranes more than two [11, 12].

Their breakthrough opens new arena for study of M-theory. Many important facts revealed in the past are reconfirmed: D-two and M-two relation [13, 14, 15], M-two and M-five bound system [16, 17], relation between infinite M-two and an M-five [18, 19, 20] by Myers-effect [21].

The study of superconformal Chern-Simons-matter systems is also highly stimulated. A new class of $\mathcal{N} = 4$ Chern-Simons theories coupling to hyper-multiplets, which is characterized by a quiver diagram, is constructed in [22]. This work is generalized in [23] preserving $\mathcal{N} = 4$ supersymmtries by introducing another type of hyper-multiplets, called twisted hyper-multiplets. Their model (HLLLP model) is given by a circular quiver diagram. These theories can be deformed by adding a mass term [24, 25, 26].

Following their work, Aharony, Bergman, Jafferis and Maldacena first construct $\mathcal{N} = 6$ Chern-Simons theories [27], which can be understood both as a generalization of the BLG model by relaxing anti-symmetricity on 3-bracket [28] and as a special case in [23] with one hyper and one twisted hyper-multiplet [29]. The crucial point in their model (ABJM model) is that it can describe finitely many membranes. Their insight enables us to take the large N limit and study the proposal of membrane field theory by AdS/CFT duality via membranes.

1.4 AdS/CFT duality via membranes

After the discovery of the ABJM model, their model has been generalized to various quiver superconformal Chern-Simons-matter theories by using techniques such as orbifolding method [30], brane construction, brane-tiliing [31]. Such techniques have been developed in constructing four-dimensional quiver gauge theories.

Such models are constructed to have their moduli space as eight-dimensional manifolds with a conical singularity. They are expected to be realized on a stack of membranes at the conical singularity. This geometric interpretation can be confirmed by using the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence. The dual geometry of a Chern-Simons-matter theory whose moduli space is an eight-dimensional cone C^8 is given by $\mathrm{AdS}_4 \times X^7$, where X^7 is the fiber of the cone. We give a list with respect to the study after the ABJM model in Table 1.1.

Table 1.1: A list of $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence after the discovery of the ABJM model. C^8 means an eight-dimensional cone and X^7 is the fiber of the cone.

$\mathcal{N}$	C^8	X^7	CSM theory
6	$\mathbf{C}^4$ orbifold	$\mathbf{S}^7$ orbifold	[32]
5			[29]
4			[33, 34, 35, 36]
3	Hyper-Kähler	Tri-Sasakian	[37]
2	Calabi-Yau	Sasaki-Einstein	[38, 39, 40, 41, 42]
1	Spin(7)	Weak G_2	[43, 44]

1.5 Why $\mathcal{N} = 4$?

In Table 1.1, the study of the author has been focused on $\mathcal{N} = 4$ Chern-Simons theories given by [35] in the period of his doctor's course. One might ask,

1. Why did the author start to study the very specific model? Was there any physical motivation of the study?

2. Why has he continued the study of it? Is there any physical importance in $\mathcal{N} = 4$ Chern-Simons theories compared to other model?

To answer the first question, it is expected to understand the fact that there was an obstacle to construct three-dimensional $\mathcal{N} = 4$ Chern-Simons theories. Of course, it is not difficult to construct just three-dimensional $\mathcal{N} = 4$ gauge theories with manifest $SO(4)$ R-symmetry by performing dimensional-reduction of four-dimensional $\mathcal{N} = 2$ gauge theories. However, if we add Chern-Simons interaction in this system as the supersymmetric completion, then the $SO(4)$ R-symmetry breaks to the diagonal $SO(3)$. This fact suggests that supersymmetry reduces to $\mathcal{N} = 3$ and it does [45]. In other words, Chern-Simons theories with eight supercharges cannot coexist with Yang-Mills kinetic terms.

The situation drastically changes in the infra-red region. Since in three dimension the Yang-Mills coupling constants have the positive dimension $1/2$ under the canonical normalization, the theory goes to so strongly coupled region that the supersymmetric Yang-Mills kinetic terms disappear from the action. As a result, adjoint fields except for gauge fields in the vector multiplets become auxiliary fields. After integrating them out, Chern-Simons terms and matter multiplets remain. The remaining theories are nothing but superconformal Chern-Simons-matter theories developed recently. The infra-red theory obtained in this way often has new global symmetry, which cannot be seen from the ultra-violet theory [22]. Global symmetry in the IR theory is often enhanced compared to that in the UV theory. Indeed, eight supercharges of the $\mathcal{N} = 4$ Chern-Simons theories are realized in this way.[1] It is quite natural to expect that there are attractive issues in gauge theories attained by overcoming such difficulties.

The answer of the second question is the main issue in the thesis. He is sure that there are new issues which cannot be investigated in higher supersymmetric Chern-Simons-matter theories. One of new issues is the correspondence between the monopole operators in the twisted sector and M2-branes wrapped on non-trivial two-cycles in the internal manifold. He believes that this proposal has been exactly confirmed in the BPS sector and a key to success of the exact result might be eight supercharges. These issues are included in the dissertation.

[1] As guessed from these facts, eight supercharges of Chern-Simons theories require matter fields. That is why we often abbreviate the word “matter” and assume the word implicitly in $\mathcal{N} = 4$ Chern-Simons theories.

1.6 Organization of the thesis

The rest of the thesis is organized as follows. Our papers [46, 47, 48] are devoted to the dissertation. For convenience, we give a list of key words or key phrases explained in each section in each chapter.

In Chapter 2, we review aspects of $\mathcal{N} = 4$ Chern-Simons theories. After a brief review on a quiver gauge theory, we explain the fact mentioned in §1.5 (§2.1). In other words, by adding Chern-Simons interaction an super Yang-Mills theory with $SO(4)_{YM}$ R-symmetry, the $SO(4)_{YM}$ R-symmetry is broken to the diagonal $SO(3)$. We present an $\mathcal{N} = 4$ Chern-Simons theory as a circular quiver gauge theory (§2.2). The broken $SO(4)_{YM}$ R-symmetry is recovered into new $SO(4)_R$ R-symmetry in the model. We recapture these model by type IIB brane configurations (§2.3). Such a brane setup is helpful to understand the theory and physics intuitively. However, it is focused only in the UV region and thus not suitable to see global symmetries only realized in the IR region.

Generalized by increasing the number of nodes compared to the ABJM model, $\mathcal{N} = 4$ Chern-Simons model have more monopole operators (§2.4). One is so-called the diagonal monopole operator (§2.4.1). It also plays an important role in other quiver Chern-Simons theories, for example in the BLG model [11, 12], in the ABJM model [49, 50, 39, 51, 52], in more general quiver Chern-Simons theories [53, 54, 31, 37, 41, 55, 56]. We see its importance through the analysis of the moduli space and its geometric interpretation of M-circle. Moduli spaces of $\mathcal{N} = 2$ supersymmetric quiver Chern-Simons theories are studied in [40, 55, 56, 41]. In general, an $\mathcal{N} = 4$ Chern-Simons theory has other monopole operators. We call them non-diagonal or twisted monopole operators (§2.4.2). They are new ingredients compared to higher supersymmetric model. We discuss that they belong to the twisted sector and also have their geometric counterparts in the dual M-theory.

- §2.1 Quiver diagrams, Quiver gauge theories, $N_{3D} = 4$ super Yang-Mills theories, $SO(4)_{YM}$ R-symmetry, $SO(4)_{YM}$ breaking.

- §2.2 Lagrangian of $\mathcal{N} = 4$ Chern-Simons theories, Global symmetries, Parameters characterizing a model.

- §2.3 Type IIB brane setup by D3-, NS5-, $(1, k)$5-branes.

- §2.4 Definition of magnetic charges, Diagonal and non-diagonal monopole operators, Relative magnetic charges, Dual photon, Moduli space and its geometric interpretation, Linking number, M-theory brane setup, Twisted sector.

In Chapter 3, we study gravity duals of $\mathcal{N} = 4$ Chern-Simons theories. First, we briefly review AdS_5/CFT_4 duality via D3-branes (§3.2). Next, we give the eleven-dimensional supergravity and explain how to treat membranes in the gravity theory (§3.2). Then, we derive dual geometries of $\mathcal{N} = 4$ Chern-Simons theories (§3.3). Finally, we discuss the homology of the dual geometry (§3.4).

§3.1 AdS_5/CFT_4 correspondence, Maldacena limit, 't Hooft limit, Holography, Bulk-boundary correspondence, Strong/Weak duality, Exact agreement between the BPS sectors.

§3.2 Lagrangian of $D = 11$ supergravity, 1/2 BPS black membrane solution, Near brane geometry, Enhanced to maximal supersymmetry.

§3.3 Dual geometry of $\mathcal{N} = 4$ Chern-Simons theories, Global symmetries, Orbifold singularities, Localized vector-multiplets.

§3.3 Homology, A-type singularities, Exact sequence.

In Chapter 4, we confirm the correspondence between the non-diagonal monopole operators and the M2-branes wrapped on two-cycles in the dual geometry. For this purpose, we confirm the exact agreement of both indices including monopole or wrapped M2-branes contribution.

In §4.2, we compute a gauge theory index for $\mathcal{N} = 4$ Chern-Simons theories with taking account of monopole contribution. To compute a gauge theory index, we use several technical tools such as localization (§4.2.2) and radial quantization method (§4.2.3). Combining the techniques, we compute an index I^{gauge} in quiver $\mathcal{N} = 4$ Chern-Simons theories. We calculate the gauge theory index by taking the large N limit and keeping the Chern-Simons level k fixed (§4.2.4). After the large N calculation, we factorize the index three parts with respect to magnetic charges (§4.2.5). This factorization property suggests that only BPS monopole operators contribute to indices. We derive selection rules in the gauge theory side (§4.2.6). They are useful in relating both data in the field theory side and in the geometry sides.

In §4.3, we compute an AdS_4 multi-particle index for a dual geometry. We derive the contribution of the bulk sector by projecting the graviton index for the universal cover, $AdS_4 \times \mathbf{S}^7$ (§4.3.1). To determine the contribution of the twisted sector, we construct the action of a seven-dimensional $\mathcal{N} = 2$ vector multiplet in $AdS_4 \times \mathbf{S}^3$ (§4.3.2) and carry out the Kaluza-Klein analysis (§4.3.3). From the spectrum, we derive the contribution of the twisted sectors by the orbifold projection (§4.3.4). Performing the projection, we take account of the fact that the gauge charges couple to the Wilson lines turned on the singular loci. Due to this coupling, the Kaluza-Klein momenta are shifted along the cycles through the Aharanov-Bohm effect. In M-theoretical points of view, this means that wrapped M2-branes couple to the three-form torsion. We combine them to obtain the multi-particle index.

In §4.4, we establish the relation between the independent magnetic charges and the same number of charges on the gravity side: the M-momentum and wrapping numbers (§4.4.1). We reproduce the relation between the discrete torsions and the linking numbers [57] by requiring the agreement of indices for both sides (§4.4.2). The AdS_4 multi-particle index is also factorized into three parts (§4.4.3). We analytically prove the agreement of the neutral part of the both indices (§4.4.4). The agreement of the negative part follows from that of the positive part. To compute the positive part of the gauge theory index, we use numerical methods for several non-trivial examples (§4.4.5).

§4.1 Agreement of independent numbers, Agreement of spectra, Agreement of indices.

§4.2 Large k and fixed k, Localization, Q-exact deformation, Decomposition into each parts of a quiver diagram, Radial quantization, Large N computation, Factorization, Selection rule.

§4.3 Action of a localized vector-multiplet, Kaluza-Klein analysis and KK-spectra, Contributions from singularities, Wrapping number and magnetic charges, Wilson lines and linking numbers, Factorization of multi-paricle index, Comparison of both data.

In Chapter 5, we perform a classification of the class of $\mathcal{N} = 4$ Chern-Simons theories. The principle is whether or not they flow to the same infra-red fixed point [46]. Such a study in the ABJM model is carried in [58]. In four-dimensional case, the relation between two gauge theories which flow to the same IR CFT is called Seiberg duality [59]. The two theories have different ranks of gauge groups. In this sense, we classify the $\mathcal{N} = 4$ Chern-Simons theories by Seiberg-duality in three dimension.

For the purpose, we use the type IIB brane system given in §2.3. We assume that two theories related by a continuous deformation of a fivebrane are dual to each other and flow to the same infra-red fixed point. The close relation between interchange of branes and Seiberg duality is first pointed out in [60]. Indeed, under the deformation the parameter except for the ranks are invariant by taking into account the Hanany-Witten effect. The Hanany-Witten effect is the phenomenon that when a fivebranes pass through another D3-branes are produced or annihilated for the theory to be consistent or anomaly-free [61, 62]. Such a brane exchange procedure is applied to three-dimensional Chern-Simons theories in [63, 64, 65].

Through this classification, we obtain an equivalent class of the sets of the ranks or the fractional D3-brane charges (§5.1). This equivalent class can be seen in the dual geometry as discrete torsion described by the third homology (§5.2). Actually, through the IIB/M duality, fractional D3-branes correspond to M5-branes wrapped on three-cycles. Such wrapped M5-branes are called fractional M2-branes. They couple to 3-form potential in eleven-dimensional supergravity. We discuss the relation between the period integral of the three-form field on three-cycles and the fractional D3-brane charges (§5.3). We also establish the relation between the fractional brane charge and the torsion of the three-form field up to the ordering dependent constants.

§5.1 Ranks of gauge groups, Fractional D3-brane charges, Hanany Witten effect.

§5.2 Three-cycles, Unwrapped by removing segments.

§5.3 Period of 3-form potential, Fractional M2-branes and fractional D3-branes.

In Chapter 6, we discuss the relation between M5-branes wrapped on five cycles and baryonic operators. We define baryonic operators which carry baryonic charges G_B (§6.1). We identify them with M5-branes wrapped on five-cycles (§6.2). We will see that homology relation is consistent with the decomposability of products of baryonic operators into mesonic ones on the field theory side. We also find that the conformal dimension of baryonic operators are consistent with the mass of the wrapped M5-branes. This analysis itself is essentially the same as the four-dimensional case [66], in which the baryonic operators [67] in the Klebanov-Witten theory [68] was investigated.

We discuss this correspondence from the viewpoint of IIB/M duality through the quark-baryon transition (§6.3). We show that in the type IIB setup N open strings representing constituent bi-fundamental quarks can be continuously deformed into a D3-brane disk, which is dual to a wrapped M5-brane.

Note that it needs careful treatment in the case of $p = q = 1$, *i.e.* the ABJM model, in which case there are only torsion 5-cycles in the internal space. In this case, the only baryonic symmetry $G_B \simeq U(1)$ is spontaneously broken by getting a vev of the dual photon field. The baryonic operators can be defined in the gauge-invariant way by multiplying appropriate functions of the dual photon field [69].

§6.1 G_B charge, Decomposability to mesons, Degeneracy, Conformal dimension.

§6.2 Homology relation, Collective motion of the wrapped M5-branes, Mass of wrapped M5-branes.

§6.3 Quark-baryon transition, Wrapped M5-branes and wrapped D3-branes.

In Chapter 7, we discuss the relation between monopole operators and baryonic operators. They cannot be gauge invariant simultaneously. Actually, different from four-dimensional case, the baryonic symmetry in three-dimension does not decouple in the infra-red region. As a result, it often remains gauge symmetry and we have a choice to specify the gauge groups as unitary type or special unitary type. This is the reason why baryonic operators in $\mathcal{N} = 4$ model are not always gauge invariant operators. We discuss that the choice of gauge groups corresponds to that of boundary conditions of AdS_4.

In Chapter 8, we make a brief summary and discussion of each chapter.

In Appendix, we collect up convention (§A), $\mathcal{N} = 2$ superspace formulation (§B), $OSp(\mathcal{N}|4)$ superconformal algebra (§C), 1/2 BPS representations of $OSp(8|4)$ and $OSp(4|4)$ (§D). They can be read independently from the main text.

Chapter 2

$\mathcal{N}=4$ Chern-Simons theory

A quiver is a case for carrying arrows. *From a dictionary.*

2.1 Quiver gauge theory

In this section, we briefly review basics of a supersymmetric quiver gauge theory. Our convention of $\mathcal{N}_{3D}=2$ superspace is given in Appendix B.[1]

A quiver gauge theory is characterized by a quiver diagram. A quiver diagram is constituted by nodes and arrows between them. In general, a node represents both a gauge group and a gauge-multiplet, and an arrow represents a bi-fundamental matter-multiplet. We use indices a for vertices and I for edges. We give an example of a quiver diagram in Figure 2.2.

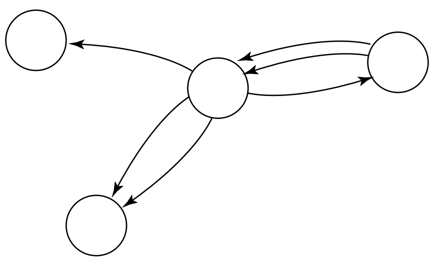

Figure 2.1: A generic quiver diagram.

Let us construct an $\mathcal{N}_{3D}=4$ quiver Yang-Mills theory. Such a Lagrangian can be obtained by giving each contribution by parts in a quiver diagram and summing up all contributions.

First, we consider nodes in the quiver diagram. For each node, we attach an $\mathcal{N}_{3D}=4$ vector-multiplet and its Yang-Mills coupling constant. An

[1]The notation $\mathcal{N}_{3D}$ means the number of supersymmetries in three dimension. For example, $\mathcal{N}_{3D}=2$ means $\mathcal{N}=2$ in $D=3$ or four real supercharges. In this convention, $\mathcal{N}_{3D}=2\mathcal{N}_{4D}=4\mathcal{N}_{6D}$.

$\mathcal{N}_{3D} = 4$ vector-multiplet V is constituted by an $\mathcal{N}_{3D} = 2$ vector-multiplet $v = (\widetilde{A}_\mu, \sigma, \lambda, D)$ and a chiral multiplet $\Phi = (\phi, \chi, F_\phi)$. Let us denote the vector-multiplet and the coupling constant for the a-th node by V_a and g_a, respectively. The Lagrangian for the a-th node is given by $\frac{1}{g_a^2}\mathcal{L}^{YM}_{V_a}$, where

$$\begin{aligned}
-\mathcal{L}^{YM}_{V_a} &= \mathrm{Tr}\Big[\int d^4\theta\, \Phi_a^\dagger e^{v_a}\Phi_a e^{-v_a} + \int d^2\theta\,(-F_a^2)\Big] \\
&= \mathrm{Tr}\Big[-\frac{1}{4}F_{a\mu\nu}F_a{}^{\mu\nu} + \frac{1}{2}\lambda_a{}^{A\dot{B}}[\not{D}, \lambda_{a\dot{B}A}] - \frac{1}{4}D_\mu\sigma_{a\dot{A}}{}^{\dot{B}}D^\mu\sigma_{a\dot{B}}{}^{\dot{A}} \\
&\quad + \frac{1}{2}F_a{}^A{}_B F_a{}^B{}_A - \frac{1}{2}\lambda_a{}^\dagger_{\dot{A}B}[\lambda_a{}^{B\dot{C}}\sigma_{a\dot{C}}{}^{\dot{A}}] + \frac{1}{4}[\sigma_{a\dot{A}}{}^{\dot{B}}, \sigma_{a\dot{C}}{}^{\dot{D}}][\sigma_{a\dot{B}}{}^{\dot{A}}, \sigma_{a\dot{D}}{}^{\dot{C}}]\Big]
\end{aligned} \tag{2.1}$$

Here we set $\widetilde{A} = iA$ and

$$(q^A) = \begin{pmatrix} q \\ \widetilde{q}^\dagger \end{pmatrix}, \quad (\psi_{\dot{A}}) = \begin{pmatrix} \psi \\ \widetilde{\psi}^\dagger \end{pmatrix}, \tag{2.2}$$

$$(\sigma_{\dot{A}}{}^{\dot{B}}) = \begin{pmatrix} \sigma & \sqrt{2}\phi \\ \sqrt{2}\phi^\dagger & -\sigma \end{pmatrix}, \quad (\lambda^{A\dot{B}}) = \begin{pmatrix} \lambda & \frac{1}{\sqrt{2}}\bar{\chi} \\ \frac{1}{\sqrt{2}}\chi & -\bar{\lambda} \end{pmatrix}, \tag{2.3}$$

$$(F^A{}_B) = (\frac{1}{\sqrt{2}}D^A{}_B) = \begin{pmatrix} \frac{1}{\sqrt{2}}D' & \bar{F}_\phi \\ F_\phi & -\frac{1}{\sqrt{2}}D' \end{pmatrix}, \quad (D' = D + [\phi, \phi^\dagger]). \tag{2.4}$$

Next, we consider arrows. For each edge, we attach a bi-fundamental hyper-multiplet H_I. A hyper-multiplet is constituted by two chiral multiplets $Q = (q, \psi, F_q), \widetilde{Q} = (\widetilde{q}, \widetilde{\psi}, F_{\widetilde{q}})$. Let us denote the head of I-th of edge as $h(I)$ and its tail as $t(I)$. Then the Lagrangian $\mathcal{L}^B_{H_I}$ is given by

$$\begin{aligned}
-\mathcal{L}^B_{H_I} &= \mathrm{Tr}\Big[\int d^4\theta\left(Q_I^\dagger e^{v_{h(I)}}Q_I e^{-v_{t(I)}} + \widetilde{Q}_I e^{v_{h(I)}}\widetilde{Q}_I^\dagger e^{-v_{t(I)}}\right) + \int d^2\theta\left(\widetilde{Q}_I\Phi_{h(I)}Q_I - Q_I\Phi_{t(I)}\widetilde{Q}_I\right)\Big] \\
&= \mathrm{Tr}\Big[-D_\mu q_I{}^\dagger_A D^\mu q_I{}^A + \frac{1}{2}\psi_I{}^{\dagger\dot{A}}\not{D}\psi_{I\dot{A}} + \frac{1}{2}(\psi_I{}^{\dagger\dot{A}}\sigma_{h(I)\dot{A}}{}^{\dot{B}}\psi_{I\dot{B}} + \psi_{I\dot{B}}\sigma_{t(I)\dot{A}}{}^{\dot{B}}\psi_I{}^{\dagger\dot{A}}) \\
&\quad + (\psi_I{}^{\dagger\dot{B}}\lambda_{h(I)}{}^\dagger_{\dot{B}A}q_I{}^A + q_I{}^\dagger_A\lambda_{h(I)}{}^{A\dot{B}}\psi_{I\dot{B}}) + q_I{}^\dagger_A(D_{h(I)}{}^A{}_B - \delta^A{}_B\frac{1}{2}\sigma_{h(I)\dot{D}}{}^{\dot{C}}\sigma_{h(I)\dot{C}}{}^{\dot{D}})q_I{}^B \\
&\quad - (q_I{}^A\lambda_{t(I)}{}^\dagger_{\dot{B}A}\psi_I{}^{\dagger\dot{B}} + \psi_{I\dot{B}}\lambda_{t(I)}{}^{A\dot{B}}q_I{}^\dagger_A) + q_I{}^B(-D_{t(I)}{}^A{}_B - \delta^A{}_B\frac{1}{2}\sigma_{t(I)\dot{D}}{}^{\dot{C}}\sigma_{t(I)\dot{C}}{}^{\dot{D}})q_I{}^\dagger_A \\
&\quad + q_I{}^\dagger_A\sigma_{t(I)\dot{C}}{}^{\dot{B}}q_I{}^A\sigma_{h(I)\dot{B}}{}^{\dot{C}}\Big].
\end{aligned} \tag{2.5}$$

In this attachment, an orientation for each arrow is only a convention of Q and $\widetilde{Q}$ in the $\mathcal{N}=2$ notation and it is not important.

The total Lagrangian is obtained by summing all contributions.

$$\sum_a \frac{1}{g_a^2}\mathcal{L}^{YM}_{V_a} + \sum_I \mathcal{L}^B_{H_I}. \tag{2.6}$$

It can be easily seen that undotted indices (dotted indices) are contracted with undotted (dotted) ones in the action. Therefore, this action has an $SU(2)\times$

Table 2.1: $SO(4)_{YM}$ R-symmetry and field contents.

	susy param ($\epsilon^{B\dot{C}}$)	vector-multiplet $A_{a\mu}$	($\sigma_{a\dot{A}}{}^{\dot{B}}$)	($\lambda_a{}^{B\dot{C}}$)	($D_a{}^A{}_B$)	hyper-multiplet $q_I{}^A$	$\psi_{I\dot{A}}$
$SU(2)$	**2**	**1**	**1**	**2**	**3**	**2**	**1**
$SU(2)_\bullet$	**2**	**1**	**3**	**2**	**1**	**1**	**2**

$SU(2)_\bullet$ R-symmetry and thus $\mathcal{N}=4$ supersymmetry. Let us denote this R-symmetry as $SO(4)_{YM}$. Charge assignment is summarized in Table 2.1.

Next, let us consider to add the Chern-Simons term in this Lagrangian (2.6) in a supersymmetric form. To do this, we attach a supersymmetric Chern-Simons Lagrangian given by (B.74) for each edge:

$$\begin{aligned}
-\mathcal{L}^{CS}_{V_a} &= -\mathrm{Tr}\left[\frac{1}{4}\int_0^1 dt \int d^4\theta\,(-v_a)\bar{D}_\alpha(e^{-tv_a}D^\alpha e^{tv_a}) + \int d^2\theta\,\frac{1}{2}\Phi_a^2\right] \\
&= -\mathrm{Tr}\Bigg[\varepsilon^{\mu\nu\rho}\left(\frac{1}{2}A_{a\mu}\partial_\nu A_{a\rho} - \frac{i}{3}A_{a\mu}A_{a\nu}A_{a\rho}\right) \\
&\qquad +\frac{1}{2}(\lambda_a{}^{A\dot{B}}\lambda^\dagger_{a\dot{A}B} + \sigma_{a\dot{A}}{}^{\dot{B}}D_a{}^B{}_A) - \frac{1}{6}\sigma_{a\dot{A}}{}^{\dot{B}}\sigma_{a\dot{B}}{}^{\dot{C}}\sigma_{a\dot{C}}{}^{\dot{A}}\Bigg],
\end{aligned} \tag{2.7}$$

where k_a is the Chern-Simons coupling constant associated of the a-th node. Then, the contribution of the a-th node is totally

$$\frac{1}{g_a^2}\mathcal{L}^{YM}_{V_a} + \frac{k_a}{2\pi}\mathcal{L}^{CS}_{V_a}. \tag{2.8}$$

And the total Lagrangian is given by

$$\sum_a\left(\frac{1}{g_a^2}\mathcal{L}^{YM}_{V_a} + \frac{k_a}{2\pi}\mathcal{L}^{CS}_{V_a}\right) + \sum_I \mathcal{L}^B_{H_I}. \tag{2.9}$$

This time, due to existence of the Chern-Simons terms, some undotted indices (dotted indices) are contracted with dotted (undotted) ones in the Lagrangian. This is the reason why $SO(4)_{YM}$ is broken to the diagonal $SO(3)$ and $\mathcal{N}=4$ supersymmetry is broken to $\mathcal{N}=3$.

This situation changes if we consider the infra-red region of the quiver gauge theory. As mentioned in §1.5, the Yang-Mills coupling constants g_a are relevant in the IR limit. Hence, as the energy scale goes to the infra-red region, they go to blow up and the Yamg-Mills terms vanish from the action:

$$\sum_a \frac{k_a}{2\pi}\mathcal{L}^{CS}_{V_a} + \sum_I \mathcal{L}^B_{H_I}. \tag{2.10}$$

In the Lagrangian (2.10) adjoint fields included except for gauge fields in the $\mathcal{N}_{3D}=4$ vector-multiplets are all auxiliary fields. By integrating them out with the auxiliary fields in the hyper-multiplets, we obtain the on-shell Lagrangian realized in the IR limit constituted by the hyper-multipelts H_I and the gauge fields A_a.

2.2 IR symmetry

Let us give an $\mathcal{N}=4$ Chern-Simons theory with unitary-type gauge group. As mentioned at the beginning of this section, such a Chern-Simons theory is described by a circular quiver diagram. For brevity, we fix one oriention in the circular quiver diagram: the left is bigger than the right modulo the number of the nodes. This enables us to write the head and tail of the I-th arrow as

$$h(I)=L(I), \quad t(I)=R(I), \tag{2.11}$$

where $L(I)$ and $R(I)$ represent the vertices at the left and the right ends of an edge I, respectively. Similarly, we define $L(a)$ and $R(a)$ for the edges on the left and the right side of a vertex a.

$\mathcal{N}=4$ supersymmetries require the Chern-Simons coupling constant associated with the a-th node to be given by

$$k_a=k(s_{L(a)}-s_{R(a)}), \quad k\in\mathbf{Z}, \quad s_I=0,1, \tag{2.12}$$

where s_I are integers assigned to edges in the quiver diagram, and they take only two values 0 and 1. Here we define the Chern-Simons level k. Corresponding to these two values, we classify the hyper-multiplets into two groups, untwisted and twisted hyper-multiplets. If $s_I=0$ ($s_I=1$) the hyper-multiplet is called untwisted (twisted) hyper-multiplet. Then, a function s_I tells us the order of two kinds of hyper-multiplets and we call a map s_I an ordering. When we want to distinguish these two kinds of hyper-multiplets, we use index i for untwisted hyper-multiplets, and i' for twisted ones. Let p and q be the numbers of untwisted and twisted hyper-multiplets, respectively. Because the quiver diagram is circular, the total number of hyper-multiplets and the number of vector-multiplets are the same:

$$r=p+q. \tag{2.13}$$

We give an example of a circular quiver diagram in Figure 2.2. Note that the ABJM model is realized by setting $p=q=1$ and the HLLLP model is also included as a special case with $p=q$ and $s_I=0,1$ for $I\in$ even, odd, respectively.

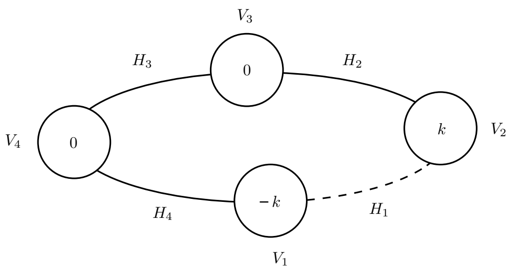

Figure 2.2: A circular quiver diagram of an $\mathcal{N}=4$ supersymmetric Chern-Simons theory with $(p,q)=(3,1)$. The solid lines describe untwisted hyper-multiplets and the broken line means a twisted hyper-multiplet.

The off-shell Lagrangian is given by (2.10) with suitable substitutions.

$$\mathcal{L}^{CS}_{\mathcal{N}_{3D}=4} = \sum_{a=1}^{r} \frac{k_a}{2\pi} \mathcal{L}^{CS}_{V_a} + \sum_{I=1}^{r} \mathcal{L}^{B}_{H_I}. \tag{2.14}$$

The Chern-Simons Lagrangian is given by

$$\begin{aligned} -\mathcal{L}^{CS}_{V_a} &= -\mathrm{Tr}\left[\frac{1}{4}\int_0^1 dt \int d^4\theta\,(-v_a)\bar{D}_\alpha(e^{-tv_a}D^\alpha e^{tv_a}) + \int d^2\theta\,\frac{1}{2}\Phi_a^2\right] \\ &= -\mathrm{Tr}\left[\varepsilon^{\mu\nu\rho}\left(\frac{1}{2}A_{a\mu}\partial_\nu A_{a\rho} - \frac{i}{3}A_{a\mu}A_{a\nu}A_{a\rho}\right)\right. \\ &\qquad \left. + \frac{1}{2}(\lambda_a{}^{AB}\lambda^\dagger_{aAB} + \sigma_{aA}{}^B D_a{}^B{}_A) - \frac{1}{6}\sigma_{aA}{}^B\sigma_{aB}{}^C\sigma_{aC}{}^A\right], \end{aligned} \tag{2.15}$$

and the Lagrangian of the huper-multiplet H_I is

$$\begin{aligned} -\mathcal{L}^B_{H_I} &= \mathrm{Tr}\left[\int d^4\theta\left(Q_I^\dagger e^{v_{L(I)}}Q_I e^{-v_{R(I)}} + \widetilde{Q}_I e^{v_{L(I)}}\widetilde{Q}_I^\dagger e^{-v_{R(I)}}\right) + \int d^2\theta\left(\widetilde{Q}_I\Phi_{L(I)}Q_I - Q_I\Phi_{R(I)}\widetilde{Q}_I\right)\right] \\ &= \mathrm{Tr}\left[-D_\mu q^\dagger_{IA}D^\mu q_I{}^A + \frac{1}{2}\psi_I{}^{\dagger A}\not{D}\psi_{IA} + \frac{1}{2}(\psi_I{}^{\dagger A}\sigma_{L(I)A}{}^B\psi_{IB} + \psi_{IB}\sigma_{R(I)A}{}^B\psi_I{}^{\dagger A})\right. \\ &\quad + (\psi_I{}^{\dagger B}\lambda^\dagger_{L(I)BA}q_I{}^A + q^\dagger_{IA}\lambda_{L(I)}{}^{AB}\psi_{IB}) + q^\dagger_{IA}(D_{L(I)}{}^A{}_B - \delta^A{}_B\frac{1}{2}\sigma_{L(I)D}{}^C\sigma_{L(I)C}{}^D)q_I{}^B \\ &\quad - (q_I{}^A\lambda^\dagger_{R(I)BA}\psi_I{}^{\dagger B} + \psi_{IB}\lambda_{R(I)}{}^{AB}q^\dagger_{IA}) + q_I{}^B(-D_{R(I)}{}^A{}_B - \delta^A{}_B\frac{1}{2}\sigma_{R(I)D}{}^C\sigma_{R(I)C}{}^D)q^\dagger_{IA} \\ &\quad \left. + q^\dagger_{IA}\sigma_{R(I)C}{}^B q_I{}^A\sigma_{L(I)B}{}^C\right], \end{aligned} \tag{2.16}$$

where we omit the dots.

One of important facts in a circular quiver gauge theory is that the diagonal $U(1)$ subgroup does not couple to any bi-fundamental matter fields. Let us denote the group as $U(1)_d$. This fact enables us to take the dual of the gauge field and obtain its dual scalar field, which is called the dual photon. This scalar field plays an important role in the membrane field theories. It is discussed in detail in §2.4.1.

To specify gauge group of an $\mathcal{N}=4$ Chern-Simons theory, it is sufficient to specify that of each node. In general, we can consider unitary or special unitary gauge group for each node.[2] One of the ways is to assign the unitary gauge group $U(N_a)_a$ for the a-th node. In this case, the gauge group is given by

$$G = \left(\prod_{a=1}^{r} U(N_a)_a\right)/U(1)_d = G_{SU} \times G_B, \tag{2.17}$$

where G_{SU} and G_B are defined by

$$G_{SU} = \prod_{a=1}^{r} SU(N_a)_a, \quad G_B = \left(\prod_{a=1}^{r} U(1)_a\right)/U(1)_d. \tag{2.18}$$

The abelian part $G_B \simeq U(1)^{r-1}$ is called baryonic symmetry. This realization of gauge group is used to discuss monopole operators in §2.4 and §4.

Another way to specify gauge group is to assign the special unitary gauge group $SU(N_a)_a$ for the a-th node. We also consider the $U(1)_d$ gauge field to keep the dual photon. In this case, the gauge group is

$$G_{SU} = \prod_{a=1}^{r} SU(N_a)_a. \tag{2.19}$$

This realization of gauge group is used to discuss baryonic operators in §6. Note that in this gauge group almost all Chern-Simons terms (2.7) vanish and gauge fields become auxiliary fields. This suggests that $\mathcal{N}=4$ supersymmetries might not be kept any more.

This model has the three dimensional conformal symmetry

$$SO(2,3) \simeq Sp(4,\mathbf{R}). \tag{2.20}$$

To confirm this classically, it is sufficient to check that there does not exist any dimensionful parameter. Furthermore, we expect that this symmetry is preserved in the quantum level. The reason why we expect this is because the family of the $\mathcal{N}=4$ Chern-Simons theories is totally discretized. This fact expects us that $\mathcal{N}=4$ Chern-Simons theories do not have any exactly marginal operator and have completely flown to the conformal fixed point [70].

The R-symmetry of this model is

$$SO(4)_R \simeq SU(2)_R \times SU(2)'_R. \tag{2.21}$$

[2]We can also consider gauge group as an orthogonal group and a unitary symplectic group for a pair of nodes adjacent to each other [29, 58].

We also use the notation $SU(2)_U, SU(2)_T$ instead of $SU(2)_R, SU(2)'_R$, respectively. This is because two complex scalar fields which are included in untwisted and twisted hyper-multiplets belong to a doublet of $SU(2)_U, SU(2)_T$, respectively. Fermions are transformed in the opposite way from the scalar fields in the same multiplet. See Table 2.2.

Table 2.2: $SO(4)_R$ R-symmetry and its charge assignment.

	untwisted hyper-multiplet		twisted hyper-multiplet	
	${q_i}^A = (q_i, \tilde{q}_i^\dagger)$	$\psi_{iA'} = (\psi_i, \tilde{\psi}_i^\dagger)$	${q_{i'}}^{A'} = (q_{i'}, \tilde{q}_{i'}^\dagger)$	$\psi_{i'A} = (\psi_{i'}, \tilde{\psi}_{i'}^\dagger)$
$SU(2)_R$	**2**	**1**	**1**	**2**
$SU(2)'_R$	**1**	**2**	**2**	**1**
$U(1)_P$	1	1	0	0
$U(1)'_P$	0	0	1	1

As often mentioned, this R-symmetry is peculiar in the IR region and thus becomes manifest after integrating out the adjoint fields except for gauge fields in the $\mathcal{N}_{3D} = 4$ vector-multiplets. This fact has been confirmed in [71].

$SO(4)_R$ R-symmtry tells us that these models have the $\mathcal{N} = 4$ supersymmtries. In fact, due to the action (2.10) written in the $\mathcal{N} = 2$ superfield formalism, a supersymmetry is attained manifestly in these models. Since the supercharge is charged under the R-symmetry, supersymmetry is enhanced to $\mathcal{N} = 4$. The supersymmetry variation is given in the literature.

These global symmetries are not always commutable to each other. This suggests that these symmetries are enhanced to the algebraically closed one. The symmetry algebra is $OSp(4|4)$, which is given in Appendix C in a more generalized form. The algebra $OSp(4|4)$ includes a so-called special conformal supercharge, denoted by $\mathcal{S}$ in Appendix C. It is not included in the three global symmetry algebras above.

There also exists the flavor symmetry

$$U(1)_P \times U(1)'_P. \tag{2.22}$$

The component fields in the untwisted hyper-multiplets H_i and those of the twisted hyper-multiplets $H_{i'}$ are transformed in different ways under the global symmetries. See Table 2.2. Note that in $p = q = 1$ case, which is the ABJM model, this flavor symmetry mixes with (2.21), and enhanced to $SU(4)$ R-symmetry.

Let us make a summary. $\mathcal{N} = 4$ Chern-Simons theories have the following global symmetry

$$OSp(4|4) \times U(1)_P \times U(1)'_P. \tag{2.23}$$

We specify unitary or special unitary gauge group for each node. The theories are characterized by a set of ranks of gauge groups N_a, two numbers of untwisted and twisted hyper-multiplets p, q, an ordering s_I and a Chern-Simon level k. Given a set of these parameters, we have an $\mathcal{N} = 4$ Chern-Simons theory.

2.3 Type IIB brane setup

In this subsection, we give a type IIB brane configuration corresponding to an $\mathcal{N} = 4$ Chern-Simons theory with unitary or special unitary gauge group. As mentioned before, it is characterized by the following data $\{N_a, p, q, s_I, k\}$. We set $r = p + q$.

Let us take an $\mathcal{N} = 4$ Chern-Simons theory with the same rank of the gauge groups $N_a = N$. The rank N is realized by N coincident D3-branes wrapped on $\mathbf{S}^1$, which is the compactified direction, say x^6. $U(N)^r$ gauge group can be realized by introducing r 5-branes intersecting with the D3-brane worldvolume at distinct points in x^6 direction and divide the $\mathbf{S}^1$ into r intervals. Let us label the intervals by $a = 1, 2, \ldots, r$. The vector multiplet V_a is realized by an open string ending on D3-branes at the a-th interval. The Yang-Mills coupling of V_a is proportional to the inverse root length of the a-th interval. Hence in the sufficiently low energy scale compared to the lengths of the intervals, the Yang-Mills kinetic terms die out and the theory can be described well in three-dimension. Let us label the intersection points of D3-branes and 5-branes by $I = 1, 2, \ldots, r$ in order along $\mathbf{S}^1$. We emphasize that when we use I as a label of 5-branes, it represents the position of the fivebrane along $\mathbf{S}^1$.

In type IIB string theory, 5-branes are specified by two charges: the NS-NS charge and the R-R charge. Let us call a 5-brane with α NS-NS charge and β R-R charge as a (α, β)5-brane. Notice that a $(1, 0)$5-brane is an NS5-brane and a $(0, 1)$5-brane is a D5-brane. In ten-dimensional space-time, a set of 5-branes preserve $\mathcal{N} = 3$ supersymmtries if a (α, β)5-brane is inclined with the definite angle $\theta = \arctan(\alpha/\beta)$ [72].

The two kinds of hyper-multiplets are obtained by restricting the kinds of 5-branes to two: $(1, k_I)$5-branes, where $k_I = ks_I$. We introduce p NS5-branes labeled by $i = 1, 2, \ldots, p$, and q $(1, k)$5-branes labeled by $i' = 1, 2, \ldots, q$. We place them around the $\mathbf{S}^1$ according to a given ordering s_I. Then the I-th hypermultiplet H_I arises from massless modes of an open string stretched between a $L(I)$-th D3-brane and a $R(I)$-th D3-brane, which steps over the I-th 5-brane. See also Figure 2.3.

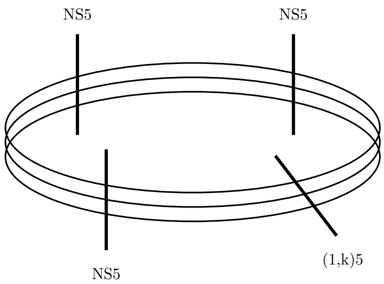

Figure 2.3: The type IIB brane configuration corresponding to an $\mathcal{N}=4$ Chern-Simons theory with $(N,p,q)=(3,3,1)$. There are 3 D3-branes, 3 NS5-branes and 1 $(1,k)$5-brane inclined with the definite angle $\theta=\arctan(k)$ relative to NS5-branes.

In this setup, the Chern-Simons terms are induced from the boundary interaction of D3-branes ending on 5-branes. The Chern-Simons coupling is given by the difference of the R-R charges of 5-branes on the boundaries [72, 73]. As a result, we realize the Chern-Simons coupling k_a as (2.12).

Finally, we obtain the type IIB brane setup corresponding to an $\mathcal{N}=4$ model whose gauge groups are unitary or special unitary groups with the same rank.[3] Let us summarize the type IIB brane configurations in Table 2.3.

Table 2.3: The type IIB brane configuration. $(1,k)$5-branes are inclined with the definite angle $\theta=\arctan(k)$ in the 37, 48 and 59 planes.

	0	1	2	3	4	5	6	7	8	9
N D3-branes	○	○	○				○			
p NS5-branes	○	○	○					○	○	○
q (1,k)5-branes	○	○	○		$[37]_\theta$		$[48]_\theta$		$[59]_\theta$	

Once we obtain the type IIB brane setup corresponding to an $\mathcal{N}=4$ model with the same rank of gauge groups, we can easily obtain that for one with the different ranks of gauge groups N_a by permitting fractional D3-branes, which are ending on different 5-branes. Let N be the minimum in N_a. Then the setup of the different ranks can be realized by adding (N_a-N) D3-branes at the a-th interval into the former setup.

[3]To realize an orthogonal group and unitary symplectic group for a pair of nodes, we add an orientifold and an anti-orientifold three-plane for the corresponding intervals [58].

2.4 Monopole operators

In three-dimensional euclidean space, local operators in general carry magnetic charges. Such operators are called monopole operators. Monopole operators correspond to instantons in three-dimensional space-time and become key objects to understand a rich structure of effective field theories of membranes.

Let us first classify monopole operators in a quiver gauge theory. For this purpose, we use gauge invariant magnetic charges defined by

$$m_a = \frac{1}{2\pi}\oint \mathrm{Tr}F_a, \quad a = 1, \ldots, r. \tag{2.24}$$

The integral is carried out on a two-cycle in the euclidean space. These magnetic charges are defined in each node in quiver gauge theories, so r coincides with the number of the nodes. In other words, a generic monopole has a magnetic charge specified by r integers, whose set forms a lattice:

$$(m_1, m_2, \cdots, m_r) \in \mathbf{Z}^r. \tag{2.25}$$

We call this lattice the magnetic charge lattice. For clearness, we consider a generic abelian quiver Chern-Simons-matter theory. We label verices by a and denote the corresponding gauge group by $U(1)_a$. We assume that the levels k_a satisfy

$$\sum_{a=1}^{r} k_a = 0.^4 \tag{2.26}$$

Let us consider a monopole operator with magnetic charges $m_a \in \mathbf{Z}$. The gauge invariance of the operator requires the Gauss law constraint

$$m_a k_a + Q_a = 0, \tag{2.27}$$

where Q_a is the $U(1)_a$ electric charge carried by matter fields included in the monopole operator. This guarantees the invariance of the operator under the gauge symmetry. By summing up this over all a, we obtain the constraint

$$\sum_a m_a k_a = 0. \tag{2.28}$$

[4] Chern-Simons-matter theories which do not satisfy the relation (2.26) have also been considered [74, 75, 76, 77]. It is discussed that the total sum of the Chern-Simons couplings corresponds to 0-form flux, or the Roman mass. Therefore, the holographic dual of such a theory is the massive IIA theory. It is pointed out the massive IIA theory cannot be lifted to M-theory in [78]. This corresponds to the fact that a diagonal monopole operator cannot be gauge-invariant in the boundary CFT language.

This relation decreases the rank of the magnetic charge lattice by one, which means that gauge-invariant monopole operators are labeled by $r-1$ independent magnetic charges. Let us denote the charge lattice of the gauge-invariant magnetic charges as Γ_M:

$$\Gamma_M = \mathbf{Z}^r/(2.28) \cong \mathbf{Z}^{r-1}. \tag{2.29}$$

It is easily seen that a magnetic charge which has the same component for each node always satisfies (2.28), due to (2.26). In other words, a vector with the same component always in the lattice Γ_M. A monopole operator with such a magnetic charge is called a *diagonal* monopole operator. Let us call the residual ones, which have a different magnetic charge for a pair of nodes, as *twisted* or *non-diagonal* monopole operators.

We show how to decompose a generic monopole charge into the contribution coming from the diagonal monopole operator and that of non-diagonal monopole operators. Actually, for later use, we only have to show it for the monopole operators whose components of magnetic charge are all positive. Let us denote the lattice of them as $\Gamma_M^{(+)}$. Let us take such an element from $\Gamma_M^{(+)}$, then there is the minimum in the components. We choose a node which take the minimum and denote it as $a = \bullet$. We identify $m_\bullet$ with the contribution of the diagonal monopole operator in the positive magnetic operator. Under this identification, $(m_a - m_\bullet)$ is an element of $\Gamma_M^{(+)}$, and this is the contribution coming from the non-diagonal monopole operators. In a circular quiver gauge theory, it is convenient to define the relative magnetic charges, which is independent of the diagonal magnetic charge, as

$$\mu_I = m_{L(I)} - m_{R(I)}. \tag{2.30}$$

By definition μ_I satisfy

$$\sum_{I=1}^{r} \mu_I = 0. \tag{2.31}$$

Combining this and the relation (2.12), (2.28), we obtain the gauge-invariant condition for the relative magnetic charge as

$$\sum_{i=1}^{p} \mu_i = 0, \quad \sum_{i'=1}^{q} \mu_{i'} = 0. \tag{2.32}$$

Hereafter, we often focus only on gauge-invariant monopole operators and assume gauge-invaricance of them implicitly.

2.4.1 Diagonal monopole operators

Let us discuss a diagonal monopole operator in more detail. In quiver Chern-Simons-matter theories, a diagonal monopole operator typically appears as a dual scalar field of the $U(1)_d$ gauge field, which is called a "dual photon". In other words, the $U(1)_d$ gauge field decouple to the matter fields. By taking the dual of the diagonal $U(1)$ gauge field strength F_d, we obtain a dual scalar field denoted by $\tilde{a}$ as

$$d\tilde{a} = \frac{1}{2\pi} * F_d, \tag{2.33}$$

where $*$ is the Hodge dual in three dimension. From the definition (2.33), $\tilde{a}$ is the canonical conjugate to F_d, so by the first quantization rule we obtain

$$[\,\widetilde{a}, \hat{m}_d\,] = i, \tag{2.34}$$

where $\hat{m}_d$ is given by (2.24) for the diagonal $U(1)_d$. This means that the operator $e^{im\tilde{a}}$ changes the flux $\hat{m}_d$ by m. Namely, $e^{im\tilde{a}}$ is an operator carrying a diagonal magnetic charge, which is the same magnetic charge $m_a = m$ for all the $U(1)_a$ gauge groups. The Hodge dual in (2.33) is calculated from (2.14) as

$$\frac{1}{2\pi} * F_d = \sum_{a=1}^{n} k_a A_a. \tag{2.35}$$

Therefore, under the gauge transformation $\delta A_a = d\lambda_a$ the dual photon field is transformed by

$$\delta\tilde{a} = k_a \lambda_a. \tag{2.36}$$

This means that the operator $e^{i\tilde{a}}$ carries the $U(1)_a$ electric charge by k_a.

Here let us place an important assumption with respect to the dual photon. That is, the dual photon is a periodic scalar field with the period 2π [40].

$$\widetilde{a} \sim \widetilde{a} + 2\pi. \tag{2.37}$$

This assumption is equivalent to requiring that the diagonal magnetic charge m_d is quantized by an integer. This assumption does not seem to be derived only by the circular quiver gauge theory given in §2.2. This is because the diagonal $U(1)_d$ does not couple to any matter fields in the theory and the Dirac quantization condition does not seem to be applied. We expect that this assumption can be derived beyond effective field theories of membranes by including higher derivative corrections. This assumption implies that the dual photon gets a non-trivial vev and one of $U(1)$s included in $U(1)_B$, precisely speaking $\sum_a k_a U(1)_a$, break to the discrete subgroup:

$$U(1)_B \to U(1)^{r-2} \times \mathbf{Z}_{(k_a)_a}, \tag{2.38}$$

where $(k_a)_a$ is the greatest common devisor of k_a.

Under this assumption, the dual photon field has a clear interpretation in M-theory. That is, the dual photon describes the "M-circle" mentioned in §1.1 and the diagonal magnetic charge is identified with the Kaluza-Klein momentum along the M-direction or "M-momentum". In this picture, the diagonal monopole operators, which are combined by e^{ima} and matter fields in the gauge-invariant way, correspond to the Kaluza-Klein mode with the M-momentum proportional to m. This interpretation is quite natural since this is also the case in just one membrane world-volume theory obtained from one D2-brane theory in the flat background [3].

Let us discuss the interpretation of the dual photon by studying the vacuum moduli space [35, 34, 36], since the vacuum moduli space has also geometric information. That is, the vacuum moduli space can be identified with the transverse direction of a stack of membranes. In general, in a supersymmeteric gauge theory, the moduli space can be obtained by solving F-term and D-term conditions and fixing gauge symmetry. Actually, it is known that the moduli space can be also obtained by solving only F-term conditions and fixing complexified gauge symmetry [79]. Here we take the latter procedure. From the F-term conditions of Q_I and $\widetilde{Q}_I$, we obtain

$$\phi_{L(I)} = \phi_{R(I)}. \tag{2.39}$$

Here we consider the Higgs branch, in which q_I and $\widetilde{q}_I$ get non-trivial vacuum expectation values. This means that all ϕ_a take the same value. We denote it by ϕ. The F-term condition for Φ_a is

$$q_{L(a)}\widetilde{q}_{L(a)} - ks_{L(a)}\phi = q_{R(a)}\widetilde{q}_{R(a)} - ks_{R(a)}\phi. \tag{2.40}$$

This means that $q_I\widetilde{q}_I - ks_I\phi$ is a constant independent of the index I. In other words, the product $q_I\widetilde{q}_I$ takes two values according to s_I. We can define "meson operators" M and M' by

$$M = q_i\widetilde{q}_i, \quad M' = q_{i'}\widetilde{q}_{i'}. \tag{2.41}$$

The suffixes i and I' are associated with untwisted and twisted hyper-multiplets, respectively. Now, we have $2r$ complex variables q_I and $\widetilde{q}_I$ constrained by (2.41). ϕ_a are dependent fields. The number of independent complex variables is $r + 2$. In addition to these, we need to take account of the dual photon field a. The dual photon field is combined with the scalar field σ in the diagonal $U(1)$

vector-multiplet into a complex scalar field belonging to a chiral multiplet. It is convenient to define $e^{i\tilde{a}+\sigma}$.

Now we have $r+3$ independent complex variables. We have to divide this space by complexified gauge symmetry $U(1)_C^{r-1}$ to obtain a complex 4-dimensional moduli space. Under the gauge transformation, the complex scalar fields transform as

$$q_I \to e^{i\lambda_I} q_I, \quad e^{i\tilde{a}+\sigma} \to e^{-ik\sum_I s_I \lambda_I} e^{i\tilde{a}+\sigma}, \tag{2.42}$$

where we defined

$$\lambda_I = \lambda_{L(I)} - \lambda_{R(I)}. \tag{2.43}$$

By definition, parameters λ_I are constrained by

$$\sum_{I=1}^{r} \lambda_I = 0. \tag{2.44}$$

Let us rewrite the parameters λ_I by φ, θ_i, $\theta_{i'}$ as

$$\lambda_i = \frac{\varphi}{p} + \theta_i, \quad \lambda_{i'} = -\frac{\varphi}{q} + \theta_{i'}, \tag{2.45}$$

where θ_i and $\theta_{i'}$ satisfy

$$\sum_{i=1}^{p} \theta_i = \sum_{i'=1}^{q} \theta_{i'} = 0. \tag{2.46}$$

Then, the gauge transformation becomes

$$q_i \to e^{i\varphi/p} e^{i\theta_i} q_i, \quad q_{i'} \to e^{-i\varphi/q} e^{i\theta_{i'}} q_{i'}, \quad e^{ia+\sigma} \to e^{-ik\varphi} e^{i\tilde{a}+\sigma} \tag{2.47}$$

We can fix the continuous part of this gauge symmetry by

$$e^{ia+\sigma} = 1 \tag{2.48}$$

and

$$q_{i=1} = \cdots = q_{i=p} \equiv z_1, \quad q_{i'=1} = \cdots = q_{i'=q} \equiv z_3. \tag{2.49}$$

(2.48) fixes φ transformation and two equations in (2.49) fix the θ_i and $\theta_{i'}$ transformations. If (2.49) hold, the relations in (2.41) guarantee

$$\widetilde{q}_{i=1} = \cdots = \widetilde{q}_{i=p} \equiv z_2, \quad \widetilde{q}_{i'=1} = \cdots = \widetilde{q}_{i'=q} \equiv z_4. \tag{2.50}$$

After the gauge fixing, we have four independent complex variables. Even after the gauge fixing above, we still have residual gauge symmetry with the parameters

$$\lambda_i = \frac{2\pi N}{kp} + \frac{2\pi m}{p}, \quad \lambda_{i'} = -\frac{2\pi N}{kq} + \frac{2\pi n}{q}, \tag{2.51}$$

Table 2.4: Actions of generators of global symmetries on the coordinates z_1, z_2, z_3, and z_4.

	z_1	z_2	z_3	z_4
T_3	$+1/2$	$+1/2$	0	0
T'_3	0	0	$+1/2$	$+1/2$
P	$+1$	-1	0	0
P'	0	0	$+1$	-1

where N, m, n are arbitrary integers. Due to this residual gauge symmetry the global rotations

$$\exp(2\pi i P/p), \quad \exp(2\pi i P'/q), \quad \exp(2\pi i P_M), \tag{2.52}$$

are gauge equivalent to 1, where P and P' are the generators of $U(1)_P$ and $U(1)'_P$, respectively. Their actions on the coordinates are shown in Table 2.4. P_M is the linear combination of P and P';

$$P_M = \frac{1}{kq}P' - \frac{1}{kp}P. \tag{2.53}$$

The three actions (2.52) are explicitly given by

$$(z_1, z_2, z_3, z_4) \mapsto (e^{\frac{2\pi i}{p}} z_1, e^{-\frac{2\pi i}{p}} z_2, z_3, z_4), \tag{2.54}$$

$$(z_1, z_2, z_3, z_4) \mapsto (z_1, z_2, e^{\frac{2\pi i}{q}} z_3, e^{-\frac{2\pi i}{q}} z_4), \tag{2.55}$$

$$(z_1, z_2, z_3, z_4) \mapsto (e^{-\frac{2\pi i}{pk}} z_1, e^{\frac{2\pi i}{pk}} z_2, e^{\frac{2\pi i}{qk}} z_3, e^{-\frac{2\pi i}{qk}} z_4), \tag{2.56}$$

respectively. The shift generated by P_M is gauge equivalent to the shift of dual photon field up to the gauge symmetry associated with the parameter φ, and we regard P_M as the M-momentum. By taking account of the discrete residual gauge symmetry (2.52), we obtain the moduli space

$$\mathbf{C}^4_{p,q,k} := ((\mathbf{C}^2/\mathbf{Z}_p) \times (\mathbf{C}^2/\mathbf{Z}_q))/\mathbf{Z}_k. \tag{2.57}$$

Especially in the special case with $p = q = 1$, the moduli space reduces to

$$\mathbf{C}^4_{1,1,k} = \mathbf{C}^4/\mathbf{Z}_k, \tag{2.58}$$

which agrees with that of ABJM model. In non-abelian case with the same rank $N_a = N$, we can obtain the moduli space as the symmetric product of N copies of the orbifold.

It is instructive to see that the vacuum moduli space obtained in this way is spanned by a basic set of chiral gauge-invariant operators. Now the gauge groups are abelian, so we only take care of the baryonic symmetry G_B. First, we consider the $k=1$ case. We already constructed mesonic operators M, M' in (2.41). In addition to them, the dual photon enables us to construct the following mesonic operators

$$b = e^{-i\tilde{a}} \prod_{i=1}^{p} (q_i), \quad b' = e^{i\tilde{a}} \prod_{i'=1}^{q} (q_{i'}), \quad \tilde{b} = e^{i\tilde{a}} \prod_{i=1}^{p} (\tilde{q}_i), \quad \tilde{b}' = e^{-i\tilde{a}} \prod_{i'=1}^{q} (\tilde{q}_{i'}). \quad (2.59)$$

The mesonic operators are related by

$$b\tilde{b} = M^p, \quad b'\tilde{b}' = M'^q. \quad (2.60)$$

These are just the defining equations of A_p and A_q type singularities, respectively. They describe the space $(\mathbf{C}^2/\mathbf{Z}_p) \times (\mathbf{C}^2/\mathbf{Z}_q)$. In fact, the defining equations can be solved by using free complex variables z_l $(l = 1, 2, 3, 4)$ as

$$b = \tilde{T} z_1^p, \quad b' = T z_2^q, \quad \tilde{b} = T z_3^p, \quad \tilde{b}' = \tilde{T} z_4^q, \quad M = z_1 z_3, \quad M' = z_2 z_4, \quad (2.61)$$

where $T\tilde{T} = 1$. However, this parametrization has a redundancy, which is described by the transformation (2.54), (2.55). By identifying the points moved to each other by them, we obtain the space mentioned above. As for a general k, the period of the dual photon becomes $2\pi/k$ and $T, \tilde{T}$ are identified with $e^{2\pi i/k}T$, $e^{-2\pi i/k}\tilde{T}$, respectively. This change gives another constraint on z_l to be identified under the transformation (2.56). Dividing the space by it, we reproduce the vacuum moduli space above.

In the geometric point of view, the moduli space describes the transverse direction of a stack of membranes. In other words, $\mathcal{N}=4$ Chern-Simons theories are expected to describe multiple membranes at the orbifold singularity of (2.57). This fact is important to obtain the dual geometry in §3.3.

Let us see this fact from the brane setup. We start from the type IIB brane system in Table 2.3. We first perform T-duality transformation along x^6 and then lift the system to M-theory configuration. After a coordinate transformation which makes two-torus at infinity diagonal [27], we have the configuration shown in Table 2.5. Through this duality chain, D2-branes are transformed into membranes, and NS5-brane i and $(1, k)$-fivebrane i' are mapped to purely geometric objects, Kaluza-Klein (KK) monopoles. We have already assumed implicitly this correspondence between the 5-branes and singular loci when we used indices i and i' to label the singular loci. In general, Q coincident KK monopoles are described as an orbifold with A_{Q-1}-type singularity. The geometry shown in Table 2.5 is the product of A_{p-1} and A_{q-1} singularities, which is nothing but the moduli space obtained above.

Table 2.5: The dual M-theory geometry. "$\bowtie$" in the column means that $\mathbf{S}^1$ fibration shrinks at the core of the KK monopoles.

	0	1	2	3	4	5	6	7	8	9	M
N M2-branes	$\circ$	$\circ$	$\circ$								
p KK monopoles	$\circ$	$\circ$	$\circ$				$\bowtie$	$\circ$	$\circ$	$\circ$	$\circ$
q KK monopoles	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$				$\bowtie$

Notice that with respect to two kinds of hyper-multiplets, the moduli space depends only on their number and independent of their ordering. To classify two inequivalent theories with the same moduli space, it is useful to define the topological invariant, called the linking number [61], for each hyper-multiplet by

$$l_i = n_i + k \sum_{i<j'<\bullet} 1, \quad l_{i'} = n_{i'} - k \sum_{\bullet<j<i'} 1. \tag{2.62}$$

Here it is necessary to choose a reference node to define the linking numbers and we choose it as the node $\bullet$ for later convenience. Two theories with different sets of linking numbers are inequivalent to each other even though they share the same moduli space. A classification of the class of $\mathcal{N}=4$ Chern-Simons theories is studied in detail in §5.

In the end, we have a comment on another role of the diagonal monopole operators. That is the enhancement of global symmetry in the highly strongly-coupled region. To reveal the full global symmetry which cannot be seen from the Lagrangian, we need to take into account non-perturbative effects. Therefore, monopole operators play a key role in such an issue. Indeed, it is often discussed that the supersymmetry of the ABJM model is enhanced to $\mathcal{N}=8$ with the Chern-Simons level $k=1,2$ by taking diagonal monopole operators into account [96, 97, 98, 99, 100, 101].

2.4.2 Twisted monopole operators

In the previous subsection, we discussed that the diagonal monopole charge can be identified with the M-momentum or the D-particle charge from the standpoints of M-theory or type IIA theory, respectively. A natural question is what corresponds to the residual $r-2$ non-diagonal monopole charges, or twisted monopole operators.

We proposed the answer of this question in [46], and our proposal was confirmed in the BPS sector [80, 47, 48]. Our proposal is that twisted monopole operators correspond to M2-branes wrapped on non-trivial two-cycles in the internal space of the dual geometry. This is one of the main topics in the doctoral thesis.

Here, let me mention where the name of twisted monopole operators come from. We call operators with non-diagonal magnetic charges twisted monopole operators just since they are in the twisted sector. In the perspective of string theory, the twisted sector arises when the flat space-time is divided by a discrete group. In this case, the twisted sector is characterized by the set of the string states with "non-trivial" boundary conditions, which would not be imposed in the flat space-time. In the viewpoint of the field theory, a state in the twisted sector is specified by a non-vanishing charge under the discrete action.

Let us discuss the above things for an $\mathcal{N} = 4$ Chern-Simons theories, which is described by a circular quiver diagram. The discrete actions $\mathbf{Z}_p, \mathbf{Z}_q$ correspond to the shift symmetries in the quiver diagram. Namely, the $\mathbf{Z}_p$ or $\mathbf{Z}_q$ action shifts the part of the diagram including p untwisted hyper-multiplets or q twisted hyper-multiplets, respectively. It is easy to see that non-diagonal charges are not invariant under such shift symmetries and that non-diagonal monopole operators are in the twisted sector.

This situation with respect to the twisted sector is reminiscent of that in the four-dimensional case. In other words, the similar phenomena happen in $\mathcal{N}_{4D} = 2$ super Yang-Mills theory, which can be obtained as a $\mathbf{Z}_m$ orbifold of the $\mathcal{N}_{4D} = 4$ one. It is described by the A_{m-1} type quiver diagram, and has $\mathbf{Z}_m$ symmetry shifting the diagram. The twisted sector in $\mathcal{N}_{4D} = 2$ models was studied in the context of $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence [81, 82]. Its contribution in the field theory side was identified with that coming from the orbifold singularity in geometry side by using the field-operator correspondence. The above proposal in $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence is quite similar to their result in $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence. In fact, non-trivial two-cycles in the internal space of the dual geometry, on which M2-branes can be wrapped, come from its orbifold singularity. We will discuss this in detail in §4.

Chapter 3

M-theory dual

Duality is often understood as equivalence of two different viewpoints of one object.

3.1 AdS/CFT duality

In this section, we briefly review AdS/CFT duality [6]. For details, see reviews [83, 84, 85, 86].

Let us discuss this statement in the novel case of $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality. The duality claims that two different points of view of a stack of D3-branes are equivalent under certain limits. One viewpoint of a pile of D3-branes is an object which open strings are attached. Since open strings can behave as gauge particles as the massless excitation, a stack of D3-branes is an object on which a non-abelian gauge theory is realized. The other viewpoint of multiple D3-branes is an object which emits and absorbs closed strings. Since closed strings include graviton as their massless mode, they have their own mass and curve the geometry. In this picture, the stack of D3-branes is described by a black threebrane solution in type IIB supergravity.

The key to realize such an idea is in how to take the certain limits. In general, open strings and closed strings couple to each other. To make two viewpoints of the branes independent, it is necessary to decouple their interaction, which is realized by turning off the string coupling constant g_s. It is possible to take this limit since in black threebrane solutions the dilaton and thus the string coupling constant are free parameters. To avoid making the theory trivial in this limit, Maldacena takes the large N limit with g_sN fixed. The large N limit means the number of D3-branes increases infinitely and it can be expected to keep non-trivial gravitational effect of them. Under this limit Maldacena studied the infra-red physics happening on D3-branes coming close to the D3-branes.

Under these limits, the gauge theory realized on the D3-branes flows to a certain conformal field theory. In the gravity side, the black three-brane solution become $\mathrm{AdS}_5 \times X^5$, where X^5 is a five dimensional compact manifold. He proposes that two theories obtained in these limits are equivalent.

We summarize the limits he proposed as follows:

$$g_s \to 0, \quad N \to \infty, \quad g_s N\text{: fixed.} \tag{3.1}$$

$$r \ll R \ll l_{IR}. \tag{3.2}$$

Here R is the curvature radius of the black three brane solution, l_{IR} is the length scale considered here and r describes the radial direction of the D3-branes. In a case with the string scale l_s much smaller than l_{IR}, we can ignore the stringy excitation and use supergravity approximation. This limit is sometimes called the "Maldacena limit" [86]. His claim is that under these limits the gauge/gravity duality conceived by 't Hooft [87] is realized via the D3-branes. Actually, the string coupling constant g_s and the gauge coupling constant g_{YM} are related by

$$g_s \sim g_{YM}^2. \tag{3.3}$$

From this fact the limit (3.1) is nothing but the 't Hooft limit and $\lambda = g_{YM}^2 N$ is just 't Hooft coupling constant [87].

This gauge/gravity duality can be considered as a kind of holography [88]. Here holography means the phenomenon that all the information of an object (a theory) can be encoded into a less dimensional one just by one. In this context, the hologram of the gravity theory is the theory realized on the boundary. Actually, the stack of probe D3-branes can be considered to exist on the boundary. The five-dimensional anti-de-Sitter space is asymptotically conformally flat. Therefore, a four-dimensional conformal field theory in the flat space-time can couple to gravity consistently on the boundary [89]. This four-dimensional CFT realized on the boundary is identified with that realized on a piles of D3-branes. In this sense, this correspondence is also called bulk-boundary correspondence.

The crucial point of the gauge/gravity correspondence lies in a strong/weak duality. Indeed, an effective coupling constant in the large N gauge theory is given by λ. This means that perturbation is perfomed by positive powers of λ and thus the region where perturbation is well-defined is $\lambda \sim 0$. On the other hand, an effective coupling constant in the gravity theory is given by inverse powers of λ, and thus the well-defined perturbative region is $\lambda \sim \infty$. To see this intuitively, remember that an effective coupling constant in the curved background is given by the inverse of its curvature radius. This is obvious from the fact that the curvature radius becomes smaller as the gravity force gets stronger. Black three brane solutions tell us the curvature radius of AdS_5 as follows.

$$R_{AdS_5}^4 \sim g_s N l_s^4 \quad \leftrightarrow \quad \left(\frac{l_s}{R_{AdS_5}}\right)^4 \sim \frac{1}{\lambda}. \tag{3.4}$$

Combining these facts, the effective coupling constant of supergravity on the AdS_5 background is given by some negative power of the 't Hooft coupling constant λ. Indeed, as seen from (3.2), the radial direction of D3-branes can describes the energy scale of the boundary field theory. It plays a role to connect the ultra-violet and infra-red region [90].

This means that the strong/weak duality always has the strong/weak point. The strong point is that by using the duality one theory in the strongly coupled region can be analyzed by the other theory in the weakly coupled regime. The weak point is that it is always difficult to prove the duality since we cannot take the perturbative approach by using the coupling constant λ in the proof. Since one of our motivations to study AdS/CFT duality is to confirm the proposals with respect to membranes, we need to avoid this difficulty in a certain way.

One of the ways to avoid this difficulty is to restrict the region of our interest to the BPS sector. In other words, our analysis is focused only on the states or operators which preserve a supersymmetry. Such states are called BPS states. Due to supersymmetry, quantum numbers of a BPS state are often free from quantum correction and become independent of the coupling constant λ. In the region, assuming that AdS/CFT duality is correct, we can expect the exact agreement of the spectrum in the BPS sector [91, 89].

We assume that the above discussion is also the case with membranes in M-theory, whose near-brane geometry is $\mathrm{AdS}_4 \times X^7$, where X^7 is a seven dimensional compact manifold. In the next section, we consider the eleven-dimensional supergravity and study how membranes are described as black objects.

3.2 Eleven-dimensional supergravity

In a sufficiently small energy scale, M-theory is well-described by eleven-dimensional supergravity [92]. The eleven-dimensional supergravity is constituted by $D = 11$ vielbein $e_{\hat{A}}{}^{A}$, Majorana gravitino Ψ_A and three-form gauge field C_3. Here

$A, B, \cdots$ are the indices of eleven-dimension and $\hat{A}, \hat{B}, \cdots$ are the local Lorentz indices. The action is given by

$$S_{11D} = \frac{1}{2\kappa_{11}^2}\left[\int d^{11}x\, e\Big(\mathcal{L}_e + \mathcal{L}_\Psi + \mathcal{L}_G + \mathcal{L}_{\Psi G}\Big) + \int \mathcal{L}_{WZ}\right], \tag{3.5}$$

where $2\kappa_{11}^2$ is the eleven-dimensional Newton constant given by $2\kappa_{11}^2 = \frac{(2\pi l_P)^9}{2\pi}$, e is the determinant of the vierbein and

$$\begin{aligned}
\mathcal{L}_e &= R, && (3.6)\\
\mathcal{L}_\Psi &= -\frac{1}{2}\bar{\Psi}_A \Gamma^{ABC} D_B \Psi_C, && (3.7)\\
\mathcal{L}_G &= -\frac{1}{2\cdot 4!} G_{4ABCD} G_4{}^{ABCD}, && (3.8)\\
\mathcal{L}_{\Psi G} &= \frac{1}{8}\bar{\Psi}_A \Gamma^{[A} \not{G}_4 \Gamma^{B]} \Psi_B, && (3.9)\\
\mathcal{L}_{WZ} &= -\frac{1}{3!} C_3 \wedge G_4 \wedge G_4. && (3.10)
\end{aligned}$$

Here G_4 is the field strength of the gauge field C_3, $G_4 = dC_3$, and $\not{G}_4$ is contracted by $D = 11$ Γ-matrices as $\frac{1}{4!}G_{4ABCD}\Gamma^A\Gamma^B\Gamma^C\Gamma^D$. This action is invariant under the following supersymmetry transformation.

$$\begin{aligned}
\delta e_{\hat{A}}{}^{B} &= \frac{1}{4}\bar{\Psi}_{\hat{A}} \Gamma^B \epsilon, && (3.11)\\
\delta \Psi_A &= D_A \epsilon + \frac{1}{4!}(\Gamma_A \not{G}_4 \epsilon - 3\not{G}_4 \Gamma_A)\epsilon, && (3.12)\\
\delta C_{3ABC} &= \frac{1}{4}(\bar{\epsilon}\Gamma_{AB}\Psi_C + \bar{\epsilon}\Gamma_{BC}\Psi_A + \bar{\epsilon}\Gamma_{CA}\Psi_B). && (3.13)
\end{aligned}$$

Let us construct a black membranes solution. We consider the case which membranes are extending to x^1, x^2 direction in the flat space-time. Such coincident membranes should be invariant under the three-dimensional Poincare transformations and under the rotations around the position. Such transformation groups are $SO(1,2) \ltimes \mathbf{R}^3$ and $SO(8)$, respectively. Such a metric is given by

$$H^{-\frac{2}{3}}\eta_{\mu\nu}dx^\mu dx^\nu + H^{\frac{1}{3}}(dr^2 + r^2 d\vec{\Omega}_{\mathbf{S}^7}^2). \tag{3.14}$$

For four-form field strength, we assume Freund-Rubin ansatz [93]

$$G_4 = dt \wedge dx \wedge dy \wedge dH^{-1}. \tag{3.15}$$

To respect the above symmetries, H should be a function of r. H is determined by the equation of motion of the four-form flux.

$$d * G_4 = -\frac{1}{2}G_4 \wedge G_4, \tag{3.16}$$

where $*$ is the $D = 11$ Hodge dual. From the ansatz, we can easily see that $dG_4 = G_4 \wedge G_4 = 0$. Therefore, from the equation of motion, G_4 becomes a harmonic four-form. This is easily solved and we obtain

$$H = 1 + \frac{R^6_{\mathbf{S}^7}}{r^6}. \tag{3.17}$$

Here $R_{\mathbf{S}^7}$ is a constant, and we set the boundary condition that $H \to 1$ as $r \to \infty$. We can determine the constant $R_{\mathbf{S}^7}$ by the flux quantization condition.

$$-\frac{1}{(2\pi l_p)^6} \int_{\mathbf{S}^7} *G_4 = N, \tag{3.18}$$

where N is an integer corresponding to the number of membranes. We can easily solve this as

$$R^6_{\mathbf{S}^7} = \frac{(2\pi l_p)^6}{2\pi^4} N. \tag{3.19}$$

We can show that this solution has 16 supersymmetries. In other words, this solution is a half BPS membrane solution.

Let us consider the near region to the membranes. Then the metric (3.14) goes to AdS_4 geometry

$$(3.14) \to \left(\frac{R_{\mathbf{S}^7}}{2}\right)^2 \frac{\eta_{\mu\nu} dx^\mu dx^\nu + dz^2}{z^2} + R^2_{\mathbf{S}^7} d\vec{\Omega}^2_{\mathbf{S}^7}, \tag{3.20}$$

and four-form flux

$$(3.15) \to -\frac{3R^3_{\mathbf{S}^7}}{8z^4} dt \wedge dx \wedge dy \wedge dz. \tag{3.21}$$

Here we do a coordinate transformation such that $r^2 = \frac{R^3_{\mathbf{S}^7}}{2z}$. We find that the space-time symmetry is changed into $SO(2,3) \times SO(8)$. Furthermore, we can show that this background has 32 supersymmetries. To see this, we solve the BPS equation $\delta\Psi_A = 0$ in this background. Plugging (3.20) and (3.21) into (3.12), we obtain the Killing spinor equations

$$\nabla_a \epsilon = \frac{1}{2R_{AdS_4}} \Gamma_a \gamma \epsilon, \tag{3.22}$$

$$\nabla_m \epsilon = -\frac{1}{2R_{\mathbf{S}^7}} \Gamma_m \gamma \epsilon. \tag{3.23}$$

Here the indices a and m describe coordinates of AdS_4 and $\mathbf{S}^7$, respectively. We also set $\gamma = -\Gamma_{\hat{t}\hat{r}\hat{\theta}\hat{\phi}}$, where r, θ, ϕ are the radial coordinate of AdS_4, and

$$R_{AdS_4} = \frac{R_{\mathbf{S}^7}}{2}. \tag{3.24}$$

We can solve the Killing spinor equations by using a constant Majorana spinor ϵ_0 with a fixed coordinate by

$$\epsilon = \left(e^{\frac{\rho}{2}\Gamma_{\hat{\rho}}\gamma} e^{\frac{t}{2}\Gamma_{\hat{t}}\gamma} e^{-\frac{\theta}{2}\Gamma_{\hat{\theta}}{}^{\hat{\rho}}} e^{-\frac{\phi}{2}\Gamma_{\hat{\rho}}{}^{\hat{\theta}}} \right) \left(e^{-\frac{\alpha_1}{2}\Gamma_1\gamma} e^{-\frac{\alpha_2}{2}\Gamma_2\gamma} e^{-\frac{\alpha_3}{2}\Gamma_3\gamma} \right) \left(e^{\frac{\beta_1}{2}\Gamma_1{}^4 + \frac{\beta_2}{2}\Gamma_2{}^5 + \frac{\beta_3}{2}\Gamma_3{}^6 - \frac{\beta_4}{2}\Gamma_M\gamma} \right) \epsilon_0. \tag{3.25}$$

See [94, 95] for details. Therefore, $\mathrm{AdS}_4 \times \mathbf{S}^7$ geometry have 32 supersymmetries. These supersymmetries are mixed with the above global symmetries, as mentioned in §2.2, and enhanced to $OSp(8|4)$. We give the algebra of $OSp(\mathcal{N}|4)$ in Appendix C.

This discussion is easily generalized to membranes at the tip of an eight-dimensional cone C^8 in Table 1.1. In this case, if we denote its fiber by X^7, the near-brane metric of membranes at such a background is given by

$$R^2_{AdS_4} ds^2_{AdS_4} + R^2_{X^7} ds^2_{X^7}. \tag{3.26}$$

The X^7 radius is given by

$$R^6_{X^7} = \frac{(2\pi l_p)^6}{6\mathrm{Vol}(X^7)} N, \tag{3.27}$$

where $\mathrm{Vol}(X^7)$ is the volume of X^7. The AdS_4 radius is given by (3.24) replacing $\mathbf{S}^7$ with X^7.

3.3 Dual of $\mathcal{N} = 4$ Chern-Simons theories

In this section, we consider the gravity dual of $\mathcal{N} = 4$ Chern-Simons theories. From the analysis of the moduli space of $\mathcal{N} = 4$ Chern-Simons theories, we read off the background of membranes described by them. An eight-dimensional cone is given by $\mathbf{C}^4_{p,q,k}$ and the eleven-dimensional space-time is given by

$$\mathbf{R}^{1,2} \times \mathbf{C}^4_{p,q,k}. \tag{3.28}$$

$\mathcal{N} = 4$ Chern-Simons theory with fixed p, q, k describes a pile of membranes at the tip of the orbifold $\mathbf{C}^4_{p,q,k}$ and extending in $\mathbf{R}^{1,2}$.

We apply $\mathrm{AdS}_4/\mathrm{CFT}_3$ duality via membranes to this system by taking the limits (3.1), (3.2) discussed in the previous section. By lifting superstring theory to M-theory, the string coupling constant g_s describes the size of M-circle, which is proportional to k^{-1}. The parameter k is quantized by the gauge symmetry, so we can fix it.[1] As N goes to large, the number of membranes is blowing up. In this limit, we cannot ignore the gravitational effect of membranes any more, and the space-time (3.28) gets curved according to the mass of membranes. With taking into account the back-reaction of membranes, the dual geometry of this system emerges as the near-brane limit of the black-membranes solution given by

$$AdS_4 \times \mathbf{S}^7_{p,q,k}, \tag{3.33}$$

$\mathbf{S}^7_{p,q,k}$ is the seven-sphere divided by the discrete subgroups defined by (2.52).

$$\mathbf{S}^7_{p,q,k} = (\mathbf{S}^7/(\mathbf{Z}_p \times \mathbf{Z}_q))/\mathbf{Z}_k. \tag{3.34}$$

The metric is given by (3.26). Here the radius of $\mathbf{S}^7_{p,q,k}$ is given by

$$R^6_{\mathbf{S}^7_{p,q,k}} = kpqR^6_{\mathbf{S}^7}. \tag{3.35}$$

Let us move on to global symmetry of this dual geometry. For this purpose, it is convenient to start with the un-orbifolded background, $AdS_4 \times \mathbf{S}^7$. Isometry of this geometry is

$$Sp(4,\mathbf{R}) \times SO(8), \tag{3.36}$$

which should be identified with the three-dimensional conformal group and R-symmtry in the field theory side. This geometry possesses the maximal supersymmetry.[2]

To obtain the M-theory dual of an $\mathcal{N}=4$ Chern-Simons theory, we divide the space by the discrete action (2.54), (2.55). Correspondingly, we define the subgroup $SO(4) \times SO(4)' \subset SO(8)$, which rotate two $\mathbf{C}^2$ spanned by z_1, z_2 or z_3, z_4, respectively. Because each $SO(4)$ is a product of two $SU(2)$ factors, we have in total four $SU(2)$ factors in this subgroup. We denote these four factors as follows.

$$SO(4) = SU(2)_R \times SU(2)_F, \quad SO(4)' = SU(2)'_R \times SU(2)'_F. \tag{3.37}$$

[1] Of course, we can take the zero size limit of M-circle, $k \to \infty$. In this limit, the dual theory goes to type IIA theory on $AdS_4 \times \mathbf{CP}^3_{p,q}$, where $\mathbf{CP}^3_{p,q}$ is given by the orbifolded $\mathbf{CP}^3$:

$$|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1, \tag{3.29}$$
$$(z_1, z_2, z_3, z_4) \sim (e^{-i\theta}z_1, e^{i\theta}z_2, e^{i\theta}z_3, e^{-i\theta}z_4), \tag{3.30}$$

where the coordinates z_l are identified as follows.

$$(z_1, z_2, z_3, z_4) \sim (e^{\frac{2\pi i}{p}}z_1, e^{-\frac{2\pi i}{p}}z_2, z_3, z_4), \tag{3.31}$$
$$(z_1, z_2, z_3, z_4) \sim (z_1, z_2, e^{\frac{2\pi i}{q}}z_3, e^{-\frac{2\pi i}{q}}z_4), \tag{3.32}$$

[2] Actually, in the case of $k=1$ and 2, ABJM model is expected to have $\mathcal{N}=8$ supersymmetry more than $\mathcal{N}=6$.

The supercharges of the $\mathcal{N}=8$ theory belong to the spinor representation $\mathbf{8}_s$ of $SO(8)$, and its branching in the subgroup $SU(2)_R \times SU(2)_F \times SU(2)'_R \times SU(2)'_F$ is

$$\mathbf{8}_s \rightarrow (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2}). \tag{3.38}$$

If we perform the orbifold projection by using an appropriate abelian discrete subgroup of $SU(2)_F \times SU(2)'_F$, the latter half on the right hand side in (3.38) is projected out, and we are left with $\mathcal{N}=4$ supersymmetry. The remaining four supercharges are transformed as a vector of

$$SO(4)_R = SU(2)_R \times SU(2)'_R. \tag{3.39}$$

This group is nothing but the R-symmetry (2.21) in the field theory side. Note that $SO(4)_R$ is not any of two $SO(4)$ groups in (3.37). The orbifolding breaks $SU(2)_F$ and $SU(2)'_F$ into abelian subgroups which commute with the orbifold groups. We denote them by $U(1)_P \subset SU(2)_F$ and $U(1)'_P \subset SU(2)'_F$. We identify these symmetries with the global symmetry (2.23) in $\mathcal{N}=4$ Chern-Simons theories.

An important feature of this dual geometry is that it includes orbifold singularities. The $\mathbf{Z}_p$ orbifolding in (3.34) generates A_{p-1} type singularities. The continuous set of the fixed points forms the fixed locus with topology $AdS_4 \times \mathbf{S}^3/\mathbf{Z}_{qk}$ in the total space. Similarly, the fixed locus associated with $\mathbf{Z}_q$ is $AdS_4 \times \mathbf{S}^3/\mathbf{Z}_{pk}$. We refer to these two fixed loci as $\mathcal{S}$ and $\mathcal{S}'$, respectively. We also use the notation $\mathcal{S}_U(\mathcal{S}_T)$ instead of $\mathcal{S}(\mathcal{S}')$, whose singularity is originated in untwisted (twisted) hyper-multiplets.

On $\mathcal{S}$ there exists a seven-dimensional $\mathcal{N}=2$ SU(p) vector-multiplet. Its Cartan part arises from the localized zero modes of the supergravity fields, while the non-Cartan part arises from M2-branes wrapping on vanishing 2-cycles at the singular locus. This fact can be also seen through duality chain. As discussed in §2.4.1, the orbifold singularity are produced by KK monopoles. By taking M/IIA duality, KK monopoles are mapped to D6-branes and M2-branes wrapping on the 2-cycles are transformed to open strings which are ending on D6-branes. Such open strings account for the non-abelian gauge group and the vector-multiplet mentioned above. We also have an SU(q) vector-multiplet localized at the other singular locus $\mathcal{S}'$. These two loci can be treated in a similar way, and we mainly focus on the vector-multiplet in $\mathcal{S}$. We define the gauge groups

$$G_{\mathcal{S}\times\mathcal{S}'} = G_{\mathcal{S}} \times G_{\mathcal{S}'} = SU(p) \times SU(q) \tag{3.40}$$

and their Cartan parts

$$H_{\mathcal{S}\times\mathcal{S}'} = H_{\mathcal{S}} \times H_{\mathcal{S}'} = U(1)^{p-1} \times U(1)^{q-1} \tag{3.41}$$

for later convenience.

As we mentioned above, $\mathcal{S}$ is $\mathbf{Z}_{qk}$ orbifold of its covering space $\widetilde{\mathcal{S}} = AdS_4 \times \mathbf{S}^3$. Once we obtain the Kaluza-Klein spectrum in $\widetilde{\mathcal{S}}$, we can derive the spectrum in $\mathcal{S}$ by an appropriate projection. Among global symmetries in (3.37), only $SU(2)'_R$ and $SU(2)'_F$ act on $\widetilde{\mathcal{S}}$ transitively, while $SU(2)_R$ and $SU(2)_F$ do not move the locus. This means that from the viewpoint of the seven-dimensional theory on the locus, $SU(2)_R$ and $SU(2)_F$ are internal symmetries while the other two are parts of the isometry group

$$Sp(4,\mathbf{R}) \times SU(2)'_R \times SU(2)'_F, \tag{3.42}$$

of the seven-dimensional spacetime. The last factor in this group, $SU(2)'_F$, is broken to $U(1)'_P$ by the orbifolding, and the other part, $Sp(4,\mathbf{R}) \times SU(2)'_R$, is a part of the bosonic subgroup $Sp(4,\mathbf{R}) \times SU(2)_R \times SU(2)'_R$ of the superconformal group $OSp(4|4)$.

3.4 Homology

We investigated the homology $H_*(\mathbf{S}^7_{p,q,k},\mathbf{Z})$ in [46]. Our motivation of this study was to classify M-branes wrapped on non-trivial cycles in the internal space, which is expected to have their counterparts in the field theory side. The results are given by

$$\begin{aligned} &H_0 = \mathbf{Z}, \quad H_1 = \mathbf{Z}_k, \quad H_2 = \mathbf{Z}^{p+q-2}, \quad H_3 = (\mathbf{Z}^{q-1}_{kp} \oplus \mathbf{Z}^{p-1}_{kq} \oplus \mathbf{Z}_{kpq})/(\mathbf{Z}_p \oplus \mathbf{Z}_q), \\ &H_4 = 0, \quad H_5 = \mathbf{Z}^{p+q-2} \oplus \mathbf{Z}_k, \quad H_6 = 0, \quad H_7 = \mathbf{Z}. \end{aligned} \tag{3.43}$$

Let us explain how these homologies can be obtained. First, let us focus on the free part of this homology. H_0 and H_7 are easily obtained by the fact that $\mathbf{S}^7_{p,q,k}$ is a connected manifold. The free parts of H_2 and H_5 originate in the $p+q-2$ 2-cycles coming from the $A_p \times A_q$ type singularity of the manifold $\mathbf{S}^7_{p,q,k}$. To justify this fact mathematically, homology algebra is a useful tool. First, we consider the $k=1$ case. Actually, the free part of the homology is independent of k. We use the Mayer-Vietoris exact sequence given by

$$\cdots \xrightarrow{\partial_*} H_*(X_1 \cap X_2) \xrightarrow{(\rho_{1*},-\rho_{2*})} H_*(X_1) \oplus H_*(X_2) \xrightarrow{j^1_*+j^2_*} H_*(X_1 \cup X_2) \xrightarrow{\partial_*} \cdots . \tag{3.44}$$

As usual, we divide $\mathbf{S}^7_{p,q,k}$ into two parts: one is the hemisphere including the north pole, the other is including the south pole. More concretely, we introduce a real coordinate $0 \leq t \leq 1$ in $\mathbf{S}^7_{p,q,k}$ as

$$|z_1|^2 + |z_2|^2 = t, \quad |z_3|^2 + |z_4|^2 = 1 - t. \tag{3.45}$$

At a generic value of t, this defines two 3-spheres, and the orbifold actions (2.54), (2.55) makes them Lens spaces L_p and L_q, respectively. Then the former hemisphere is described by $t \leq \frac{1}{2}$, the latter is $t \geq \frac{1}{2}$. We choose X_1 and X_2 in (3.44) as the two hemisphere, in which case $\mathbf{S}^7_{p,q,k}$ is realized by $X_1 \cup X_2$. Hence, the problem to obtain $H_*(\mathbf{S}^7_{p,q,k})$ reduces to that of the homologies of X_1, X_2 and $X_1 \cap X_2$. These manifolds have rather simpler homologies because they are topologically equivalent to the product spaces of lower-dimensional manifolds:

$$X_1 \sim \mathbf{D}^4_p \times L_q, \quad X_2 \sim \mathbf{D}^4_q \times L_p, \quad X_1 \cap X_2 \sim L_p \times L_q, \tag{3.46}$$

where $\mathbf{D}^4_p$ is a 4d compact disk with the A_{p-1} type singularity, and L_p is a lens space. The homologies of these manifolds are given in Table 3.1. To derive the homology of a product manifold, the universal coefficient theorem is useful, which is given by the following splitting exact sequence

$$0 \longrightarrow (H(A) \otimes H(B))_* \longrightarrow H_*(A \times B) \longrightarrow \bigoplus_{a+b=*-1} \mathrm{Tor}_1(H_a(A), H_b(B)) \longrightarrow 0. \tag{3.47}$$

As a result, we obtain the homologies of $\mathbf{D}^4_p \times L_q$ and $L_p \times L_q$, which are also listed in Table 3.1. In the table, (p, q) means the greatest common devisor of p and q.[3] Combining these data and the exact sequence (3.44), the free part of the homology (3.43) is easily reproduced. This result is consistent with the Poincare duality

$$b_q = b_{d-q}, \quad T_{q-1} = T_{d-q}, \tag{3.49}$$

where d is the dimension of the manifold, b_q is the rank of the q-th homology, called the q-th betti number, and T_q is the torsion part of the q-th homology. Of course such non-trivial cycles can be constructed concretely by using a coordinate system such as $\{z^l, t\}$ constrained by (3.45). In fact, we will construct 5-cycles in $H_5(\mathbf{S}^7_{p,q,k})$ by using the coordinate system in §6, to determine the volume of them.

[3]For derivation, we use the following formulas.

$$\mathbf{Z}_p \otimes \mathbf{Z}_q = \mathbf{Z}_{(p,q)}, \quad \mathbf{Z} \otimes \mathbf{Z}_q = \mathbf{Z}_q, \quad \mathrm{Tor}_1(\mathbf{Z}_p, \mathbf{Z}_q) = \mathbf{Z}_{(p,q)}, \quad \mathrm{Tor}_1(\mathbf{Z}, \mathbf{Z}_q) = 0. \tag{3.48}$$

Table 3.1: The homologies of a 4d compact disk with the A_{p-1} type singularity, a lens space, and some product spaces.

	$\mathbf{D}^4_p$	L_q	$\mathbf{D}^4_p \times L_q$	$L_p \times L_q$
H_0	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$
H_1	0	$\mathbf{Z}_q$	$\mathbf{Z}_q$	$\mathbf{Z}_p \oplus \mathbf{Z}_q$
H_2	$\mathbf{Z}^{p-1}$	0	$\mathbf{Z}^{p-1}$	$\mathbf{Z}_{(p,q)}$
H_3	0	$\mathbf{Z}$	$\mathbf{Z} \oplus \mathbf{Z}_q^{p-1}$	$\mathbf{Z}^2 \oplus \mathbf{Z}_{(p,q)}$
H_4	0	0	0	$\mathbf{Z}_p \oplus \mathbf{Z}_q$
H_5	0	0	$\mathbf{Z}^{p-1}$	0
H_6	0	0	0	$\mathbf{Z}$
H_7	0	0	0	0

As for the torsion part, we use the Poincare duality. For general k, we can obtain some of the torsion parts of $\mathbf{S}^7_{p,q,k}$ by hand as

$$T_0 = 0, \quad T_1 = \mathbf{Z}_k, \quad T_2 = 0. \tag{3.50}$$

Combining this fact and the Poincare duality for the torsion (3.49), we finally get the homology of $\mathbf{S}^7_{p,q,k}$ such as (3.43), except for $H_3(\mathbf{S}^7_{p,q,k})$. It seems a bit too complicated to determine $H_3(\mathbf{S}^7_{p,q,k})$ just by diagram chasing of the above exact sequences, so we will derive it by careful search of the manifold $\mathbf{S}^7_{p,q,k}$ in §5.2.

Chapter 4

Wrapped M2-branes and monopole operators

The strategy to confirm the proposal is to use indices.

4.1 Clues

Before studying indices, let us have a comment on some clues on the correspondence in the title of this chapter.

One of evidences of this correspondence is the agreement of the independent numbers in both sides [46]. The independent number of wrapped M2-branes can be obtained by counting the independent 2-cycles in the internal space. The number is given by the second Betti number $b_2(\mathbf{S}^7_{p,q,k}) := \mathrm{rank}\, H_2(\mathbf{S}^7_{p,q,k}) = r-2$. It is evident that this number is identical to that of the independent non-diagonal monopole operators.

Another evidence is the agreement of BPS spectra in both sides. As mentioned in §3.1, the reason why we restrict BPS spectrum is because it is independent of coupling constants and thus we can expect the exact agreement on AdS/CFT correspondence, which is the strong/weak duality.

One approach is to check the agreement between Kaluza-Klein modes of massless fields in the bulk geometry and chiral operators in the corresponding boundary CFT. Such analysis in the ABJM model was performed in [27, 49, 39]. For abelian $\mathcal{N} = 4$ Chern-Simons theories, the agreement of the both spectra of the twisted monopole operators and the wrapped M2-branes was confirmed by constructing a basis of chiral twisted monopole operators explicitly [80].

We generalized the result in [80] to the non-abelian case in another way. That is, we confirmed the agreement of indices calculated in both sides independently

[47, 48]. An index is a kind of a character with respect to global symmetries in a supersymmetric theory and free from quantum correction. By using this property, the BPS spectrum in a supersymmetric theory can be encoded into an index. Therefore the agreement of both indices in two theories leads to that of BPS spectra in both theories. Note that our generalization is necessary to use AdS/CFT correspondence, since it requires the large N limit as discussed in the previous chapter.

4.2 Gauge theory index

4.2.1 Overview

Let us overview works on indices for three-dimensional superconformal Chern-Simons-matter theories. Our analysis of indices highly depends on their works.

The study of indices of superconformal Chern-Simons theories was triggered by [102] for the ABJM model. Let (h_1, h_2, h_3) be the $SU(4)_R$ weight vector and h_4 be the $U(1)_B$ charge. Notice that h_4 is nothing but the M-momentum on the gravity side. The superconformal indices investigated in [102] are defined by[1]

$$I^{\text{gauge}}(x, y_2, y_3) = \text{Tr}\left[(-1)^F e^{-\beta'\{Q,S\}} x^{2(D+j)} y_2^{h_2} y_3^{h_3}\right] \tag{4.1}$$

where Q is one component of supercharge with R-charge $(h_1, h_2, h_3) = (1, 0, 0)$, and S is its Hermitian conjugate, D is the dilation, j is the spin component. On the gauge theory side, the trace in (4.1) is regarded as the summation over gauge invariant operators.

To be curious, the indices defined by (4.1) are independent of β' as in the four-dimensional case [103]. In other words, only BPS states, which is saturating the BPS bound

$$\{Q, S\} = \Delta - j - h_1 \geq 0, \tag{4.2}$$

contribute to the indices. In calculation, they took the large N and large k limit with the 't Hooft coupling $\lambda = N/k$ fixed. As a result, they succeeded to take the weak coupling limit $\lambda \to 0$ and perform the path integral. The explicit form is

$$I^{\text{gauge}}(x, y_2, y_3) = \prod_{n=1}^{\infty} \frac{(1 - x^{4n})^2}{(1 - x^{2n}y_2^n)(1 - \frac{x^{2n}}{y_2^n})(1 - \frac{x^{2n}}{y_3^n})(1 - x^{2n}y_3^n)}. \tag{4.3}$$

In exchange for success of path integral calculation, however, all the monopole contribution decouple in the large k limit. On the gravity side, this corresponds to

[1] We replace the variable x commonly used in the literature by x^2 to avoid fractional power. And we denote the variable for h_l as y_l.

the decoupling of Kaluza-Klein modes with non-vanishing M-momentum h_4. The corresponding graviton index is obtained by a projection of the graviton index for $\mathrm{AdS}_4 \times \mathbf{S}^7$, which restricts to the sector without M-momentum contribution. The single particle index is given as

$$I_{G_{\mathbf{CP}^3}} = \frac{x^2}{y_2 - x^2} + \frac{1}{1 - x^2 y_2} + \frac{x^2}{y_3 - x^2} + \frac{1}{1 - x^2 y_3} - \frac{2}{1 - x^4}. \tag{4.4}$$

This index is derived in §4.3.1. By deriving the multi-particle index from (4.4), they confirmed the agreement of both indices.

This result can be generalized in two ways. One way is to apply for more generic superconformal Chern-Simons-matter theories. Such a generalization is performed in [104]. They compute a superconformal index for $\mathcal{N} = 4$ theories obtained as $\mathbf{Z}_M$ orbifold of ABJM model. Due to less supersymmtries, the definition of the index is changed from (4.3) by setting $y_2 = 1$:

$$I^{\text{gauge}}(x, y_3) = \mathrm{Tr}\left[(-1)^F e^{-\beta'\{Q,S\}} x^{2(D+j)} y_3^{h_3}\right]. \tag{4.5}$$

The result is given by

$$I^{\text{gauge}}(x, y_3) = \prod_{n=1}^{\infty} \left[\frac{(1 - x^{4n})^2}{(1 - x^{2nM})^2 (1 - \frac{x^{2n}}{y_3^n})(1 - x^{2n} y_3^n)}\right]^M. \tag{4.6}$$

This result is consistent with the index of the ABJM model (4.3) by setting $y_2 = M = 1$.

From this result, the corresponding single particle index can be read off as follows.

$$\begin{aligned} I^{\text{sp}}(x, y_3) \quad &= \quad \frac{1}{1 - x^2 y_3} + \frac{1}{1 - x^2/y_3} - \frac{2}{1 - x^4} + \frac{2x^{2M}}{1 - x^{2M}} \\ &\quad + (M-1)\left(\frac{1}{1 - x^2 y_3} + \frac{1}{1 - x^2/y_3} - \frac{2}{1 - x^4}\right). \end{aligned} \tag{4.7}$$

An interesting feature of this result is that this index consists of two parts of different origins. The first line in (4.7) is obtained from the single-particle index (4.4) by the projection which leaves only terms invariant under the $\mathbf{Z}_M$ rotation $y_2 \to e^{2\pi i/M} y_2$:

$$\text{The first line in (4.7)} = \frac{1}{M} \sum_{m=1}^{M} I_{G_{\mathbf{CP}^3}}(x, e^{2\pi i m/M} y_2, y_3)|_{y_2 \to 1}. \tag{4.8}$$

Thus, the first line is regarded as the bulk contribution. On the other hand, the second line is interpreted as the contribution of twisted sectors:

$$\Delta I^{twist} = (M-1)\left(\frac{1}{1 - x^2 y_3} + \frac{1}{1 - x^2/y_3} - \frac{2}{1 - x^4}\right). \tag{4.9}$$

They discuss that the contribution of the twisted sectors comes from the $H_{\mathcal{S}\times\mathcal{S}'}$ vector-multiplets, which arises as localized mode at the singular loci. They did not derive the contribution in the gravity side. We derived the contribution in [48] including that of $H_{\mathcal{S}\times\mathcal{S}'}$ charged part. We discuss it in §4.3.4.

Another way to generalize the result (4.3) is to include the contribution of monopole operators. Such a generalization is performed in [105] for the ABJM model. The index is defined by

$$I^{\text{gauge}}(x, y_2, y_3, y_4) = \text{Tr}\left[(-1)^F e^{-\beta'\{Q,S\}} x^{2(D+j)} y_2^{h_2} y_3^{h_3} y_4^{h_4}\right], \tag{4.10}$$

where y_4 is introduced as the chemical potential for the charge h_4, which is related to the monopole charge m by $h_4 = km$. To include the contribution of monopole operators, the Chern-Simons level k needs to be finite. The trick to compute the index (4.10) without taking the large k limit is to use the localization method by deforming the action by adding Q-exact terms (Q-exact deformation) [105]. As a result, the index can be computed in the weak coupling limit. It is confirmed that this agrees with the multi-particle index.

Combining the techniques developed by them, we computed an index in generic $\mathcal{N} = 4$ Chern-Simons theories with taking account of the monopole contribution in [47]. The gauge theory index I^{gauge} is defined by

$$I^{\text{gauge}}(x, z, z'; \vec{\tau}) = \text{Tr}\left[(-)^F e^{-\beta'\{Q,S\}} x^{2(D+j)} z^P z'^{P'} \prod_{a=1}^{r} \tau_a^{m_a}\right], \tag{4.11}$$

where we introduce chemical potentials τ_a for the magnetic charges m_a defined by (2.24). Q is a certain component of the supercharges and S is its Hermitian conjugate. This choice of the two supercharges breaks the R-symmetry $SO(4)_R$ down to $SO(2) \times SO(2)$: $H_1 = T_3 + T_3'$ is the R-charge rotating Q. The global symmetries commuting with this R-charge is generated by P, P', and $H_2 = -T_3 + T_3'$. Among these three $U(1)$ symmetries, the last one is broken when we deform the theory. This is the reason why we insert chemical potentials only for the charges P and P'.

4.2.2 Localization

In this subsection, we discuss the localization method in a general $\mathcal{N} = 2$ superconformal field theory [106]. By using this technique, we can calculate a gauge theory index in the free theory limit.

A basic fact to localization is that a function of β' in the following form

$$I_R = \mathrm{Tr}_R\Big((-)^F e^{-\beta'\{Q,S\}}\, C\Big) \tag{4.12}$$

is independent of β'. Here Tr_R means a trace of a unitary representation R of the superconformal group, F is the fermion number operator, Q is a certain supersymmetry generator, S is its hermitian conjugate, and C is a hermitian operator which commutes with Q. To see this, we first consider the case $C = 1$. It is sufficient to see that the states which contribute to (4.12) are only the Q-invariant ones. In other words, the contribution from the other states is totally cancelled. Since Q and S are nilpotent,[2] $\Delta = \{Q, S\}$ commutes with Q and S. This means that any eigenspace of Δ, except for the space of the ground states, splits into two. One can be written as the image of Q, or Q-exact form, the other is that of S, or S-exact form. The two spaces are transformed to each other by a linear map S or Q (with a certain normalization). Therefore, there is one-to-one correspondence between the two spaces. In particular, a pair of states transformed to each other have different statistics. Therefore, the contributions from these two spaces are totally cancelled due to insertion $(-)^F$. This means that only Δ-invariant states contribute to (4.12). A Δ-invariant state is nothing but a Q-invariant one in a unitary representation. Next, we consider $C \neq 1$ case. Since C is hermitian and commutes with Q, it also commutes with S and thus with Δ. Therefore, by making the states diagonal with respect to Δ and C simultaneously, the above discussion can be applied as it is.

By using this fact, it is not difficult to see that the function (4.12) such as (4.11) is invariant under a Q-exact deformation, or adding a Q-exact term into the Lagrangian. To see this, it is useful to rewrite the trace in (4.12) as the path integral form on $\mathbf{S}^2\times\mathbf{S}^1$. For this purpose, we first map the conformal field theory on $\mathbf{R}^3$ to that on $\mathbf{S}^2 \times \mathbf{R}^1$ in the usual way. In particular, the radial direction of $\mathbf{R}^3$, denoted by r, is related to that of $\mathbf{R}^1$, τ, by

$$r = e^\tau \quad \leftrightarrow \quad \tau = \log r. \tag{4.13}$$

A local operator in $\mathbf{R}^3$ is mapped to a state in the Fock space in $\mathbf{S}^2 \times \mathbf{R}$. To read off the charges of the fields and calculate the trace in (4.12), we compactify the "time" direction τ with the period β. Then, (4.12) can be written as a path integral form as

$$I_R = \int [d\Psi] e^{-S[\Psi]}, \tag{4.14}$$

[2] Nilpotency of Q can be easily checked by (B.9) or (C.21).

where Ψ is a collective set of the fields in the theory on $\mathbf{S}^2 \times \mathbf{S}^1$ with a suitable boundary condition. Here let us deform the action $S[\Psi]$ by adding a Q-exact term

$$S[\Psi] \to S[\Psi, t] = S[\Psi] + t \int QV[\Psi], \tag{4.15}$$

where t is a deformation parameter and Q means the supersymmetry transformation acting on the fields. Then (4.14) is changed as

$$I_R \to I_R(t) = \int [d\Psi] e^{-S[\Psi,t]}. \tag{4.16}$$

Let us rewrite $I_R(t)$ by the operator formalism.

$$\begin{aligned} I_R(t) &= \mathrm{Tr}\Big((-)^F e^{-\beta'\{Q,S\}} e^{-t[Q,\int \hat{V}[\Psi]]}\, C \Big) \\ &= \sum_{k:\Delta=0} \langle k| \Big((-)^F e^{-\beta'\{Q,S\}} e^{-t[Q,\int V[\hat{\Psi}]]}\, C \Big) |k\rangle. \end{aligned} \tag{4.17}$$

Here the state $|k\rangle$ is annihilated by Δ or Q. Since C commutes with Q, the insertion of $e^{-t[Q,\int V[\hat{\Psi}]]}$ becomes trivial and the right hand side is identical to I_R.

We can realize kinetic terms of vector and chiral multiplets as Q-exact terms in a conformally flat background [107]. Indeed, the kinetic term of a vector-multiplet (B.48) on $\mathbf{R}^3$ can be written as

$$-\mathcal{L}_v^{YM} = \frac{1}{4\bar{\epsilon}^2} \Delta^2_{\epsilon=0,\bar{\epsilon}} \bar{\lambda}^2. \tag{4.18}$$

Here we set the supersymmetry transformation Δ^{WZ}_ϵ in (B.27) to $\Delta_{\epsilon,\bar{\epsilon}}$. For a chiral multiplet on $\mathbf{R}^3$ such as (B.53), (B.55), (B.57),

$$-\mathcal{L}_Q = -\frac{1}{4\bar{\epsilon}^2} \Delta^2_{\epsilon=0,\bar{\epsilon}} (F q^\dagger). \tag{4.19}$$

It is evident that they are written as Q-exact forms.

These facts are also the case with a conformally flat background. In such a background, the supersymmetry transformation is corrected from that in the flat background since it becomes covariant under the Weyl transformation. The explicit form of a vector-multiplet is given by

$$\Delta_{\epsilon,\bar{\epsilon}} A_{\alpha\beta} = (\Delta_{\epsilon,\bar{\epsilon}} A_{\alpha\beta})^{\mathbf{R}^3}, \tag{4.20}$$

$$\Delta_{\epsilon,\bar{\epsilon}} \sigma = (\Delta_{\epsilon,\bar{\epsilon}} \sigma)^{\mathbf{R}^3}, \tag{4.21}$$

$$\Delta_{\epsilon,\bar{\epsilon}} \lambda_\alpha = (\Delta_{\epsilon,\bar{\epsilon}} \lambda_\alpha)^{\mathbf{R}^3} + \frac{2}{3} (\nabla_\mu \epsilon) \gamma^\mu_\alpha \sigma, \tag{4.22}$$

$$\Delta_{\epsilon,\bar{\epsilon}} \bar{\lambda}_\alpha = (\Delta_{\epsilon,\bar{\epsilon}} \bar{\lambda}_\alpha)^{\mathbf{R}^3} + \frac{2}{3} (\nabla_\mu \bar{\epsilon}) \gamma^\mu_\alpha \sigma, \tag{4.23}$$

$$\Delta_{\epsilon,\bar{\epsilon}} D = (\Delta_{\epsilon,\bar{\epsilon}} D)^{\mathbf{R}^3} + \frac{1}{3} (\nabla_\mu \bar{\epsilon}) \gamma^\mu \lambda + \frac{1}{3} (\nabla_\mu \epsilon) \gamma^\mu \bar{\lambda}, \tag{4.24}$$

where $(\Delta\Psi)^{\mathbf{R}^3}$ is the susy transformation in the flat euclidean space given by (B.30), (B.31), (B.32), (B.33) and (B.33), respectively. ϵ is a killing vector on the conformally flat background. For a chiral multiplet,

$$\begin{aligned}
\Delta_{\epsilon,\bar{\epsilon}}q &= (\Delta_{\epsilon,\bar{\epsilon}}q)^{\mathbf{R}^3}, &(4.25)\\
\Delta_{\epsilon,\bar{\epsilon}}\psi_\alpha &= (\Delta_{\epsilon,\bar{\epsilon}}\psi_\alpha)^{\mathbf{R}^3} + \frac{4}{3}(\nabla_\mu\bar{\epsilon})\gamma^\mu_\alpha q, &(4.26)\\
\Delta_{\epsilon,\bar{\epsilon}}F &= (\Delta_{\epsilon,\bar{\epsilon}}F)^{\mathbf{R}^3} + \frac{2}{3}(\Delta_q - \frac{1}{2})(\nabla_\mu\bar{\epsilon})\gamma^\mu\psi, &(4.27)
\end{aligned}$$

where Δ_q is the conformal dimension of q. The first term of the right hand side in each equation is the susy transformation in the flat background, which is given by (B.39), (B.40) and (B.41), respectively. For an anti-chiral multiplet,

$$\begin{aligned}
\Delta_{\epsilon,\bar{\epsilon}}\bar{q} &= (\Delta_{\epsilon,\bar{\epsilon}}\bar{q})^{\mathbf{R}^3}, &(4.28)\\
\Delta_{\epsilon,\bar{\epsilon}}\bar{\psi}_\alpha &= (\Delta_{\epsilon,\bar{\epsilon}}\bar{\psi}_\alpha)^{\mathbf{R}^3} + \frac{4}{3}(\nabla_\mu\epsilon)\gamma^\mu_\alpha\bar{q}, &(4.29)\\
\Delta_{\epsilon,\bar{\epsilon}}\bar{F} &= (\Delta_{\epsilon,\bar{\epsilon}}\bar{F})^{\mathbf{R}^3} + \frac{2}{3}(\Delta_q - \frac{1}{2})(\nabla_\mu\epsilon)\gamma^\mu\bar{\psi}. &(4.30)
\end{aligned}$$

See (B.42), (B.43) and (B.44) for the susy transformation in the flat background. Notice that they are consistent with the case of the flat background, where the killing spinors are constant.

The supersymmetric Lagrangians for a vector-multiplet and a chiral multiplet are the same forms as (4.18) and (4.19), respectively. The killing spinor equation on $\mathbf{S}^2 \times \mathbf{S}^1$ is given by

$$\nabla_\mu\bar{\epsilon} = \frac{1}{2r}\gamma_\mu\gamma_3\bar{\epsilon}, \tag{4.31}$$

where r is the $\mathbf{S^2}$ radius. The other two ϵ satisfy the equation with opposite sign. By using (4.31), we can compute the supersymmetric Lagrangians for a vector-multiplet and a chiral multiplet as follows.

$$\begin{aligned}
-\mathcal{L}_v^{YM} &= (-\mathcal{L}_v^{YM})^{\mathbf{R}^3} + \frac{1}{r}F_{12}\sigma - \frac{1}{2r^2}\sigma^2 - \frac{1}{2r}\bar{\lambda}\gamma^3\lambda, &(4.32)\\
-\mathcal{L}_Q &= (-\mathcal{L}_Q)^{\mathbf{R}^3} + \frac{\Delta_q - \frac{1}{2}}{r}\left(2(D^3q)^\dagger q + \frac{1}{2}\bar{\psi}\gamma^3\psi\right) + \frac{\Delta_q(\Delta_q - 1)}{r^2}\bar{q}q. &(4.33)
\end{aligned}$$

The Lagrangians in the flat background are given by (B.48), (B.53), respectively. Again, we can add these terms into the original Lagrangian without changing the index.

Let us find Lorentz invariant vacua or saddle points in the deformed action (4.15). Lorentz invariance requires the fermions to vanish. The potential minimum of scalar fields q, D, F is also trivial. Other component fields in a vector-multiplet get non-trivial vevs. The equation of motion with respect to σ is

$$F_{12} - \frac{i}{r}\sigma = 0. \tag{4.34}$$

This equation is a monopole equation on $\mathbf{S}^2 \times \mathbf{S}^1$. Solutions are given by a superposition of Dirac monopoles [108]

$$A_1 = 0, \quad A_2 = \frac{-M}{2r} \cot\theta, \quad \sigma = \frac{M}{2r} = \frac{s}{r}. \tag{4.35}$$

where $M = 2s$ is a magnetic charge, which takes values in the Cartan algebra of the gauge group. It is quantized by the Dirac quantization condition. For A_3, we can turn on zero-mode (holonomy) in the $\mathbf{S}^1$ direction:

$$\int_{\mathbf{S}^1} A_3 d\tau = -\alpha \quad \rightarrow \quad A_3 = \frac{-\alpha}{\beta}, \tag{4.36}$$

where β is the $\mathbf{S}^1$ circumference. With taking account of the equation of motion of the gauge field, the gauge field configuration gets correction at the order $\mathcal{O}(t^{-1})$. Actually, such a correction vanishes under the limit of the coupling constant t to infinity.

We take into account GNO monopoles as saddle points. After an appropriate gauge fixing, α and M take values in the Cartan part of the Lie algebra of the gauge group G. When we perform the Gaussian integral, all fields in this background is decomposed into vacuum expectation values $\Psi^{(0)}$ and the fluctuations Ψ' as

$$\Psi = \Psi^{(0)} + \frac{1}{\sqrt{t}}\Psi'. \tag{4.37}$$

Substituting this into the total action $S[\Psi, t]$, all interaction terms including more than two fluctuations vanish in $t \to \infty$ limit. After taking the limit, we are left with

$$I_R = \sum_M \left[\int [d\Psi'] e^{-S^{(0)}} e^{-S'[\Psi']} \right]. \tag{4.38}$$

Here $S^{(0)}$ is the expectation value of the original action. Almost all terms in the original action vanish when the expectation values are substituted. If the action includes Chern-Simons terms, it gives the non-vanishing contribution

$$S^{(0)}_{\mathrm{CS}} = \frac{i}{4\pi} \int \mathrm{Tr}' \left(A^{(0)} dA^{(0)} - \frac{2i}{3} A^{(0)} A^{(0)} A^{(0)} \right) = 2i \mathrm{Tr}'(\alpha s). \tag{4.39}$$

The definition of the trace "Tr′" here includes Chern-Simons levels and it does not have to be positive definite. $S'[\Psi']$ is the quadratic action of the fluctuation fields with the monopole background. For a vector-multiplet, the Lagrangian is given by

$$-\mathcal{L}'_v \;=\; \mathrm{Tr}\left[-\frac{1}{2} V_\mu V^\mu + \frac{1}{2} D^2 + \bar{\lambda}[\not{D} + \frac{s}{r}, \lambda] - \frac{1}{2r} \bar{\lambda}\gamma^3 \lambda \right]. \tag{4.40}$$

where V_μ is set to

$$V_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho}F^{\nu\rho} + [D_\mu, \sigma] + [A_\mu, \frac{s}{r}] - \varepsilon_{\mu 12}\frac{\sigma}{r}. \tag{4.41}$$

Here $\varepsilon_{123} = 1$ and $D_i\,(i = 1, 2)$ is the covariant derivative in the monopole background

$$D_i = \partial_i - iS_{\text{eff}}\omega_i, \tag{4.42}$$

where ω_i is the spin connection in $\mathbf{S}^2$, which coincides with A_i in (4.35) with the unit magnetic flux and S_{eff} is given by sum of spin S and background flux s.

$$S_{\text{eff}} = S + \frac{M}{2} = S + s. \tag{4.43}$$

For a chiral multiplet,

$$\begin{aligned} -\mathcal{L}'_Q &= -\bar{q}D_\mu D^\mu q + \frac{1}{r^2}\bar{q}ssq + \frac{\Delta_q(1-\Delta_q)}{r^2}\bar{q}q + \frac{1-2\Delta_q}{r}\bar{q}D_3 q + \bar{F}F \\ &\quad + \frac{1}{r}(\overline{\psi}s\psi) + (\overline{\psi}\gamma^\mu D_\mu\psi) + \frac{2\Delta_q - 1}{2r}(\overline{\psi}\gamma_3\psi). \end{aligned} \tag{4.44}$$

Here we denote the field Ψ' as Ψ. Therefore, the final result of I_R can be decomposed into each contribution of the fields.

$$I_R = \sum_M \left[\prod_v I^v e^{-S^{(0)}} \prod_Q I^Q\right], \tag{4.45}$$

where I^v, I^Q is given by

$$I^v = \int [dv] e^{-\int d^3x\sqrt{g}\mathcal{L}'_v}, \quad I^Q = \int [dQ] e^{-\int d^3x\sqrt{g}\mathcal{L}'_Q}. \tag{4.46}$$

This is the localization method on $\mathbf{S}^2 \times \mathbf{S}^1$. Since the Lagrangians $\mathcal{L}_v, \mathcal{L}_Q$ are given by the quadratic expression of the fields, the path integral reduces to the products of gaussian integrals.

Let us apply the localization method to an index of an $\mathcal{N} = 4$ Chern-Simons theory (4.11). In this case, the magnetic charge M and holonomy α are given for each node:

$$M \to \text{diag}(m_1^a, \cdots, m_N^a), \quad \alpha \to \text{diag}(\alpha_1^a, \cdots, \alpha_N^a). \tag{4.47}$$

The index (4.11) is given by

$$I^{\text{gauge}}(x, z, z'; \vec{\tau}) = \sum_M \left[\prod_a I^{V_a}(x, z, z') e^{ik_a m_s^a \alpha_s^a} \prod_I I^{H_I}(x, z, z') \prod_{a=1}^r \tau_a^{m_a}\right], \tag{4.48}$$

where we use s, t as suffixes of $U(N)_a$ gauge group. Here $I^{V_a}(x, z, z')$ and $I^{H_I}(x, z, z')$ are the contributions from a-th vector-multiplet or I-th matter-multiplet, respectively.

$$I^{V_a}(x, z, z') = \text{Tr}_{V_a}\left[(-)^F e^{-\beta'\{Q,S\}} x^{2(D+j)} z^P z'^{P'}\right], \tag{4.49}$$

$$I^{H_I}(x, z, z') = \text{Tr}_{H_I}\left[(-)^F e^{-\beta'\{Q,S\}} x^{2(D+j)} z^P z'^{P'}\right]. \tag{4.50}$$

4.2.3 Radial quantization

In this subsection, we perform the path integral (4.46) up to the over all constant. Since the path integral reduces to gaussian integral, we can perform the path integral exactly.

For this aim, let us fix the boundary condition of the $\mathbf{S}^1$ direction. The Killing equation (4.31) implies $\bar{\epsilon}_1 \propto e^{\tau/(2r)}$, and we cannot impose the periodic boundary condition on $\bar{\epsilon}_1$. Instead, it satisfies

$$\bar{\epsilon}(\tau + \beta) = e^{\beta/(2r)} \bar{\epsilon}(\tau). \tag{4.51}$$

We interpret the extra factor $e^{\beta/(2r)}$ on the right hand side as an insertion of a twist operator. Namely, by using the quantum numbers

$$R(\bar{\epsilon}_1) = -1, \quad j_3(\bar{\epsilon}_1) = \frac{1}{2}, \quad F_i(\bar{\epsilon}_1) = 0, \tag{4.52}$$

we can rewrite (4.51) as

$$\bar{\epsilon}_1(\tau + \beta) = e^{(-R-j_3)\beta_1 + j_3\beta_2 + F_i\gamma_i} \bar{\epsilon}_1(\tau), \tag{4.53}$$

where β_1, β_2, and γ_i are real parameters satisfying $\beta/r = \beta_1 + \beta_2$. For the consistency, the same boundary condition should be imposed on all fields in the theory. Namely, we impose

$$\Psi(\tau + \beta r) = e^{(-R-j_3)\beta_1 + j_3\beta_2 + F_i\gamma_i} \Psi(\tau) \tag{4.54}$$

for an arbitrary field Ψ. The path integral over $\mathbf{S}^2 \times \mathbf{S}^1$ with this twisted boundary condition gives the index I_R defined in (4.12) with $C = x^{(D+j)} z_i^{F_i}$. The variables x', x, and z_i are related to β_1, β_2, and γ_i by

$$x' = e^{-\beta_1}, \quad x = e^{-\beta_2}, \quad z_i = e^{-\gamma_i}. \tag{4.55}$$

First, we perform the path integral for a scalar field q. The integration of the auxiliary field F gives a constant factor and we can simply drop it. The path integral of the complex scalar field q gives the determinant factor $(\det \Box_q)^{-1}$ with the differential operator

$$\Box_q = -D_3 D_3 - D_i D_i + \frac{1}{r^2} s^2 + \frac{\Delta_q(1-\Delta_q)}{r^2} + \frac{1-2\Delta_q}{r} D_3. \tag{4.56}$$

Let us focus on a component of the scalar field with weight $\rho \in R_q$. Although the spin of scalar field is $S = 0$, the coupling to the background flux shifts the effective spin to $S_{\rm eff}=\rho(s)$. We can expand a field with spin S and weight ρ in the background flux s by spin $S_{\rm eff}$ spherical harmonics $Y^{S_{\rm eff}}_{j,j_3}$:

$$Y^{S_{\rm eff}}_{j,j_3},\quad j\geq|S_{\rm eff}|,\quad -j\leq j_3\leq j. \tag{4.57}$$

The eigenvalue of the Laplacian D_iD_i corresponding to $Y^{S_{\rm eff}}_{j,j_3}$ is

$$D_iD_iY^{S_{\rm eff}}_{j,j_3}=-\frac{1}{r^2}[j(j+1)-S_{\rm eff}^2]Y^{S_{\rm eff}}_{j,j_3}. \tag{4.58}$$

Substituting (4.58) into (4.56), we obtain the eigenvalue

$$\square_q=\frac{1}{r^2}(j+\Delta_q+rD_3)(j+1-\Delta_q-rD_3). \tag{4.59}$$

In this expression D_3 should be understood to be its eigenvalue. By taking the twisted boundary condition (4.54) into account, the eigenvalues of D_3 are given by

$$D_3=\frac{1}{\beta r}\Big[2\pi in-i\rho(\alpha)+(-R-j_3)\beta_1+j_3\beta_2+F_i\gamma_i\Big],\quad n\in\mathbf{Z} \tag{4.60}$$

For the scalar field q the charge R in (4.60) is replaced by Δ_q. Taking the product of all the eigenvalues, we obtain the scalar field contribution to the Gaussian integral.

$$I^q=\left[\prod_{\rho\in R_q}\prod_{j=|\rho(s)|}^{\infty}\prod_{j_3=-j}^{j}\prod_{n=-\infty}^{\infty}(j+\Delta_q+rD_3)(j+1-\Delta_q-rD_3)\right]^{-1}. \tag{4.61}$$

Next, let us consider Gaussian integral of the fermion field ψ. The differential operator acting on ψ in the action (4.44) is

$$D_{\rm fer}=\gamma^\mu D_\mu-\frac{1-2\Delta_q}{2r}\gamma_3+\frac{s}{r}. \tag{4.62}$$

We focus on a component with weight $\rho\in R_q$. Including the shift due to the background flux, the upper and lower components of the spinor ψ have the effective spins $S_{\rm eff}=\rho(s)-1/2$ and $S_{\rm eff}=\rho(s)+1/2$, respectively. They are expanded by spherical harmonics $Y^{\rho(s)-1/2}_{j,j_3}$ and $Y^{\rho(s)+1/2}_{j,j_3}$. Let us focus on a mode with spin j.

In the case with $j\geq|\rho(s)|+1/2$, both $Y^{\rho(s)-1/2}_{j,j_3}$ and $Y^{\rho(s)+1/2}_{j,j_3}$ exist. The differential operator $D_{\rm fer}$ acting on ψ takes the matrix form

$$D_{\rm fer}=\begin{pmatrix} D_3-\frac{1-2\Delta_q}{2r}+\frac{\rho(s)}{r} & -iD_- \\ iD_+ & -D_3+\frac{1-2\Delta_q}{2r}+\frac{\rho(s)}{r}\end{pmatrix}, \tag{4.63}$$

where $D_\pm = D_1 \pm iD_2$. The determinant of the matrix (4.63) is

$$\det{}' D_{\rm fer} = \frac{\rho(s)^2}{r^2} - \left(D_3 - \frac{1-2\Delta_q}{2r}\right)^2 - D_+D_-. \tag{4.64}$$

$\det'$ in (4.64) represents the determinant of the 2×2 matrix. Note that D_+ and D_- do not commute with each other and D_+D_- and D_-D_+ are different operators. If we adopt D_+D_- as in (4.64) we should regard it as an operator acting on the upper component of ψ, which has the effective spin $\rho(s)-1/2$. The eigenvalue is

$$D_+D_-Y^{\rho(s)-1/2}_{j,j_3} = -\frac{1}{r^2}\left[\left(j+\frac{1}{2}\right)^2 - \rho(s)^2\right]Y^{\rho(s)-1/2}_{j,j_3}. \tag{4.65}$$

We can also use D_-D_+ acting on $Y^{\rho(s)+1/2}_{j,j_3}$, and obtain the same eigenvalue as (4.65). By substituting this eigenvalue into (4.64) we obtain

$$\det D_{\rm fer} = \frac{1}{r^2}(j+\Delta_q + rD_3)(j+1-\Delta_q - rD_3). \tag{4.66}$$

In the case with $j = |\rho(s)| - 1/2$, on the other hand, only one of $Y^{\rho(s)-1/2}_{j,j_3}$ or $Y^{\rho(s)+1/2}_{j,j_3}$ exists. Therefore, only top-left or bottom-right component in the matrix (4.63) exists. The eigenvalue in this case is

$$D_{\rm fer} = \frac{1}{r}(j+\Delta_q + rD_3). \tag{4.67}$$

Here we ignore the sign factor. It is not relevant if the matter fields are bifundamental for the gauge group. Combining (4.66) and (4.67), we obtain

$$\begin{aligned} I^\psi = \det D_{\rm fer} &= \prod_{\rho\in R_q}\prod_{j=|\rho(s)|-1/2}^{\infty}\prod_{j_3=-j}^{j}\prod_{n=-\infty}^{\infty}(j+\Delta_q+rD_3) \\ &\times \prod_{\rho\in R_q}\prod_{j=|\rho(s)|+1/2}^{\infty}\prod_{j_3=-j}^{j}\prod_{n=-\infty}^{\infty}(j+1-\Delta_q-rD_3). \end{aligned} \tag{4.68}$$

Recall that the differential operator D_3 should be understood as its eigenvalue given in (4.60). For the fermion field ψ R in (4.60) is replaced by $\Delta_q - 1$.

As a result, we obtain an index for a chiral superfield $I^Q = I^q I^\psi$ as follows.

$$I^Q = \prod_{\rho\in R_q}\left[\frac{\prod_{\mathcal{F}_1}\prod_{n=-\infty}^{\infty}(j+\Delta_q+rD_3)\prod_{\mathcal{F}_2}\prod_{n=-\infty}^{\infty}(j+1-\Delta_q-rD_3)}{\prod_{\mathcal{B}}\prod_{n=-\infty}^{\infty}(j+\Delta_q+rD_3)(j+1-\Delta_q-rD_3)}\right]. \tag{4.69}$$

Here $\mathcal{F}_1, \mathcal{F}_2, B$ mean

$$\mathcal{F}_1 \;:\; j \ge |\rho(s)| - 1/2, \quad |j_3| \le j, \tag{4.70}$$

$$\mathcal{F}_2 \;:\; j \ge |\rho(s)| + 1/2, \quad |j_3| \le j, \tag{4.71}$$

$$\mathcal{B} \;:\; j \ge |\rho(s)|, \quad |j_3| \le j. \tag{4.72}$$

Let us regularize the infinite product of the eigenvalues in a standard way. One of the eigenvalues is of the following form

$$j+\Delta_q+rD_3=\frac{r}{\beta}\Big(2\pi in-i\rho(\alpha)+(-R+\Delta_q+j-j_3)\beta_1+(j+\Delta_q+j_3)\beta_2+F_i\gamma_i\Big). \tag{4.73}$$

We set z_1 to be the right hand side with $2\pi in$ removed. The other is

$$j-\Delta_q+rD_3+1=\frac{r}{\beta}\Big(2\pi in+i\rho(\alpha)+(R-\Delta_q+j+j_3+1)\beta_1+(j-\Delta_q-j_3+1)\beta_2-F_i\gamma_i\Big). \tag{4.74}$$

We define z_2 to be the right hand side with $2\pi in$ removed. We first carry out the product over the integer n by using the formula

$$\prod_{n=-\infty}^{\infty}(2\pi in+z)=2\sinh\frac{z}{2}=e^{\frac{z}{2}}(1-e^{-z})=e^{\frac{z}{2}}\exp\left[-\sum_{m=1}^{\infty}\frac{1}{m}e^{-mz}\right]. \tag{4.75}$$

At the first equality we neglect a divergent constant. With this formula all the products in the definition of I^Q other than $\prod_{n=-\infty}^{\infty}$ can be rewritten by the summation.

$$\prod_{\mathcal{X}}\prod_{n=-\infty}^{\infty}(2\pi in+z)^{-(-)^F}=e^{-\sum_{\mathcal{X}}(-)^F\frac{z}{2}}\exp\left[\sum_{m=1}^{\infty}\frac{1}{m}\sum_{\mathcal{X}}(-)^Fe^{-mz}\right]. \tag{4.76}$$

F is the fermion number of the corresponding field. We define a function f by

$$f(e^{i\alpha},x',x,z_i)=\sum_{\mathcal{X}}(-)^Fe^{-z}. \tag{4.77}$$

We call this a letter index because this can be regarded as an index for elementary excitations, which are often called letters. For the eigenvalue (4.73) for a scalar field, e^{-z} is given by

$$e^{-z_1}=e^{i\rho(\alpha)-(j-j_3)\beta_1-(j+\Delta_q+j_3)\beta_2-F_i\gamma_i}=e^{i\rho(\alpha)}x'^{j-j_3}x^{j+\Delta_q+j_3}z_i^{F_i}, \tag{4.78}$$

and the corresponding letter index is

$$f=\sum_{\mathcal{B}}e^{-z_1}=e^{i\rho(\alpha)}x^{\Delta_q}z_i^{F_i}\sum_{j=|\rho(s)|}^{\infty}\sum_{j=-j_3}^{j_3}(x'x)^j\left(\frac{x}{x'}\right)^{j_3}. \tag{4.79}$$

We compute the letter index for other series of eigenvalues in the integrand in (4.69) in the same way. We give only the results. From the other factor $(j+1-\Delta_q-rD_3)$ in I^q we obtain

$$f=e^{-i\rho(\alpha)}x^{-\Delta_q}z_i^{-F_i}\sum_{j=|\rho(s)|}^{\infty}\sum_{j_3=-j}^{j}(x'x)^{j+1}\left(\frac{x'}{x}\right)^{j_3}. \tag{4.80}$$

The factor $(j+\Delta_q+rD_3)$ in I^ψ gives

$$f=-e^{i\rho(\alpha)}x^{\Delta_q}z_i^{F_i}\sum_{k=|\rho(s)|}^{\infty}\sum_{l=-k}^{k-1}(x'x)^k\left(\frac{x}{x'}\right)^l, \tag{4.81}$$

and the other factor $(j+1-\Delta_q-rD_3)$ in I^ψ gives

$$f=-e^{-i\rho(\alpha)}x^{-\Delta_q}z_i^{-F_i}\sum_{k=|\rho(s)|}^{\infty}\sum_{l=-k-1}^{k}(x'x)^{k+1}\left(\frac{x'}{x}\right)^l. \tag{4.82}$$

By summing up (4.79), (4.80), (4.81), and (4.82) we obtain the letter index for a chiral multiplet Q. Finally we sum up the letter index for all chiral multiplets, and obtain

$$f^Q(e^{i\alpha},x,z_i)=e^{i\rho(\alpha)}z_i^{F_i}\frac{x^{2|\rho(s)|+\Delta_q}}{1-x^2}-e^{-i\rho(\alpha)}z_i^{-F_i}\frac{x^{2|\rho(s)|+2-\Delta_q}}{1-x^2}. \tag{4.83}$$

This does not depend on the variable x'. This is consistent with the fact that only BPS states contribute to the index I_R. When $\Delta_q=1/2$ (4.83) agrees with the corresponding function in [105].

We also need to evaluate the first factor in (4.76). It is a monomial of the variables $e^{i\alpha}$, x, and z_i. We define b_0, ϵ_0, and q_{0i} by

$$\exp\left(-\sum_{\mathcal{X}}(-)^F\frac{z}{2}\right)=e^{ib_0(a)}x^{\epsilon_0}z_i^{q_{0i}}. \tag{4.84}$$

ϵ_0 and q_{0i} are zero-point contribution to the energy and the flavor charges. $b_0(a)$ is a linear function of a which represents the zero-point charge coupling to the gauge fields. The zero-point energy is given by

$$\epsilon_0=\frac{1}{2}\frac{\partial f^Q}{\partial x}\bigg|_{e^{i\alpha}=x=z_i=1}=(1-\Delta_q)|\rho(s)|. \tag{4.85}$$

The zero-point flavor charges q_{0i} are also obtained in the same way.

$$q_{0i}=\frac{1}{2}\frac{\partial f^Q}{\partial z_i}\bigg|_{e^{i\alpha}=z_i=1}=-F_i\left[\frac{1}{2(x-1)}+\left(\frac{1}{4}+|\rho(s)|\right)+\mathcal{O}(x-1)\right]. \tag{4.86}$$

Since this is divergent, we need to regularize and normalize (renormalize) it suitably. Because (4.86) does not depend on Δ_q, it is plausible that after an appropriate regularization q_{0i} does not depend on Δ_q. Thus we take zero-point charges for canonical fields,

$$q_{0i}=-|\rho(s)|F_i. \tag{4.87}$$

Similarly, $b_0(a)$ is given by

$$b_0(\alpha)=-|\rho(s)|\rho(\alpha). \tag{4.88}$$

(4.88) can be regarded as the 1-loop correction to Chern-Simons levels. This vanishes when the matter representation is vector-like.

Finally, the index of a chiral superfield (4.69) is written as follows.

$$I^Q = \prod_{\rho\in R_q}\left[e^{-i|\rho(s)|\rho(\alpha)}x^{(1-\Delta_q)|\rho(s)|}z_i^{-|\rho(s)|F_i}\exp\left[\sum_{m=1}^{\infty}\frac{1}{m}f^Q(x^m, z_i^n, e^{im\rho(\alpha)})\right]\right]. \tag{4.89}$$

The letter index of vector multiplets is also obtained in a similar way. The contribution of the bosonic part of a vector-multiplet $I^{A,\sigma,D}$ is given by

$$I^{A,\sigma,D} = \left(\prod_{\rho(\alpha)=0}\int d\alpha\right)(\mathrm{FPD})\prod_{\rho\in G}\left[\frac{1}{\prod_{\mathcal{B}_1}\prod_{n=-\infty}^{\infty}(j+1-rD_3)\prod_{\mathcal{B}_2}\prod_{n=-\infty}^{\infty}(j+rD_3)}\right], \tag{4.90}$$

where B_1, B_2 are

$$\mathcal{B}_1 \ : \ j \geq |\rho(s)| - 1, \quad |j_3| \leq j, \tag{4.91}$$

$$\mathcal{B}_2 \ : \ j \geq |\rho(s)| + 1, \quad |j_3| \leq j. \tag{4.92}$$

$\prod_{\rho\in G}$ represents the product over all roots. The factor (FPD) is the Faddev-Popov determinant for a certain gauge fixing. When the gauge group is unitary group, (FPD) is given by

$$(\mathrm{FPD}) = \prod_{\substack{\rho(\alpha)\neq 0\\ \rho(s)=0}} 2i\sin\left(\frac{\rho(\alpha)}{2}\right). \tag{4.93}$$

The contribution of the gaugino I^λ is given by

$$I^\lambda = \prod_{\rho\in G}\left[\prod_{\mathcal{F}_1}\prod_{n=-\infty}^{\infty}(j+1+rD_3)\prod_{\mathcal{F}_2}\prod_{n=-\infty}^{\infty}(j-rD_3)\right]. \tag{4.94}$$

$\mathcal{F}_1, \mathcal{F}_2$ are given by (4.70), (4.71), respectively.

Combining these results, we obtain the index for a vector-multiplet $I^v = I^{A,\sigma,D}I^\psi$.

$$I^v = \left(\prod_{\rho(\alpha)=0}\int d\alpha\right)(\mathrm{FPD})\prod_{\rho\in G}\left[\frac{\prod_{\mathcal{F}_1}\prod_{n=-\infty}^{\infty}(j+1+rD_3)\prod_{\mathcal{F}_2}\prod_{n=-\infty}^{\infty}(j-rD_3)}{\prod_{\mathcal{B}_1}\prod_{n=-\infty}^{\infty}(j+1-rD_3)\prod_{\mathcal{B}_2}\prod_{n=-\infty}^{\infty}(j+rD_3)}\right]. \tag{4.95}$$

We can regularize the infinite product in the same way before. Because any vector multiplet carries no flavor charges, it is a function of only s, $e^{i\alpha}$ and x. The result is

$$I^v = \left(\prod_{\rho(\alpha)=0}\int d\alpha\right)(\mathrm{FPD})\prod_{\rho\in G}\left[x^{-|\rho(s)|}\exp\left[\sum_{m=1}^{\infty}\frac{1}{m}f^v(x^m, e^{im\rho(\alpha)})\right]\right]. \tag{4.96}$$

where the letter index f^v is given by

$$f^v(x, e^{i\alpha}) = -e^{i\rho(\alpha)}x^{2|\rho(s)|}, \tag{4.97}$$

and the factor $x^{-|\rho(s)|}$ is the zero-point energy.

Let us apply this result to an indices of an $\mathcal{N} = 4$ Chern-Simons theory (4.49) and (4.50). Remember that the definition of x is x^2 in (4.49) and (4.50). First, we give the contribution of an $\mathcal{N} = 4$ vector-multiplets V^a. The conformal dimension of the chiral superfield ϕ is 1. The index is given by

$$I^{V_a} = I^{v^a} I^{\phi^a} = \left(\prod_{s=1}^{N} \int d\alpha_s^a \right) \prod_{s,t=1}^{N} \left[x^{-|m_s^a - m_t^a|} \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} f^{V_a}(x^n, e^{in\beta_{st}^a}) \right] \right]. \tag{4.98}$$

Here we set $\beta_{st}^a = \alpha_s^a - \alpha_t^a$ ($\alpha_s^a := \alpha_{s,s}^a$) and

$$f^{V_a}(x, e^{i\beta_{st}^a}) \;\; = \;\; -(1 - \delta_{st})x^{2|m_s^a - m_t^a|} e^{i\beta_{st}^a}. \tag{4.99}$$

Next, we consider the contribution of the bi-fundamental hyper-multiple H^I. The conformal dimension of the chiral superfields is $\Delta_q = 1/2$. We obtain the index as

$$I^{H_I} = I^{Q_I} I^{\widetilde{Q}_I} = \prod_{s,t=1}^{N} \left[x^{|m_s^{L(I)} - m_t^{R(I)}|} \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} f^{H_I}(x^n, z_I^n, e^{in\beta_{st}^I}) \right] \right], \tag{4.100}$$

where we set $\beta_{st}^I = \alpha_s^{L(I)} - \alpha_t^{R(I)}$, $z_I = z\,(z')$ for $s_I = 0\,(1)$, and

$$f^{H_I}(x, z_I, e^{i\beta_{st}^I}) \;\; = \;\; \frac{x(x^{2|m_s^{L(I)} - m_t^{R(I)}|})}{1 + x^2} \left(e^{i\beta_{st}^I} z_I + \frac{1}{e^{i\beta_{st}^I} z_I} \right). \tag{4.101}$$

This result correctly encodes the spectrum obtained from the radial quantization method [109, 110].

4.2.4 Large N computation

Let us perform the holonomy integrals of the index under the large N limit.

$$\begin{aligned} I_{\{m_s^a\}}(x, z, z') &= \prod_a I^{V_a}(x, z, z') e^{-S^{(0)}} \prod_I I^{H_I}(x, z, z') && (4.102) \\ &= \left(\prod_{a=1}^{r} \prod_{s=1}^{N} \int \frac{d\alpha_s^a}{2\pi} \right) I^{\text{gauge}}(x, z, z'). && (4.103) \end{aligned}$$

where $I^{\rm gauge}(x,z,z')$ is given by

$$
\begin{aligned}
I^{\rm gauge}(x,z,z') &= x^{2\epsilon_0(\{m_s^a\})}\exp\left(i\sum_{a=1}^{r}\sum_{s=1}^{N}k_a m_s^a\alpha_s^a\right) \\
&\times\prod_{a=1}^{r}\prod_{s,t=1}^{N}\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f^{V_a}(x^n,e^{in\beta_{st}^a})\right] \\
&\times\prod_{I=1}^{r}\prod_{s,t=1}^{N}\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f^{H_I}(x^n,z_I^n,e^{in\beta_{st}^I})\right]. \qquad (4.104)
\end{aligned}
$$

Here we use (4.98) and (4.100). $\epsilon_0(\{m_s^a\})$ is the zero point energy due to the vacuum polarization in $\mathbf{S}^2$, which is given by

$$
\epsilon_0(\{m_s^a\}) = -\frac{1}{2}\sum_{a=1}^{r}\sum_{s,t=1}^{N}|m_s^a-m_t^a| + \frac{1}{2}\sum_{I=1}^{r}\sum_{s,t=1}^{N}|m_s^{L(I)}-m_t^{R(I)}|. \qquad (4.105)
$$

To obtain the gauge theory index which can be compared with the graviton index, we should take the large N limit. This limit is taken by adding vanishing entries to the monopole charges $\{m_s^a\}$. For each a, the monopole charge is described by N integers m_s^a $(s=1,\ldots,N)$. Let M_a be the number of non-vanishing components among them. When we take the large N limit, we keep M_a at $\mathcal{O}(1)$. For this limit to be well defined, the zero-point energy should not diverge in the limit. This is indeed easily confirmed by rewriting (4.105) as

$$
\begin{aligned}
\epsilon_0(\{m_s^a\}_*) &= -\frac{1}{2}\sum_{a=1}^{r}\sum_{s\in M_a}\sum_{t\in M_a}|m_s^a-m_t^a| + \frac{1}{2}\sum_{a=1}^{r}\sum_{s\in M_{L(I)}}\sum_{t\in M_{R(I)}}|m_s^{L(I)}-m_t^{R(I)}| \\
&+\frac{1}{2}\sum_{a=1}^{r}(2M_a-M_{a+1}-M_{a-1})\sum_{s\in M_a}|m_s^a|, \qquad (4.106)
\end{aligned}
$$

where $\{m_s^a\}_*$ is the collection of non-vanishing components in $\{m_s^a\}$, and $\sum_{s\in M_a}$ represents the summation over M_a non-vanishing components in the magnetic charges. This expression is manifestly independent of N, and well behaves in the large N limit.

The integration with respect to angular variables α_s^a associated with vanishing magnetic charges m_s^a can be carried out by introducing the variables λ_{an} by

$$
\lambda_{an} = \frac{1}{N-M_a}\sum_{s=M_a+1}^{N}e^{in\alpha_s^a}, \quad n=\pm1,\pm2,\ldots. \qquad (4.107)
$$

The exponential factors in the second and the third lines in (4.104) can be rewritten as a Gaussian factor including

$$
\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\sum_{a,b=1}^{r}\lambda_{an}M_{ab}(x^n,z^n,z'^n)\lambda_{b-n}+\cdots\right) \qquad (4.108)
$$

where $\cdots$ include the first and the zeroth order terms of λ_{an}, and the matrix M is

$$M(x,z,z') = \begin{pmatrix} \ddots & & & & & -\frac{xz_r^{-1}}{1+x^2} \\ & 1 & -\frac{xz_{I-1}}{1+x^2} & & & \\ & -\frac{xz_{I-1}^{-1}}{1+x^2} & 1 & -\frac{xz_I}{1+x^2} & & \\ & & -\frac{xz_I^{-1}}{1+x^2} & 1 & -\frac{xz_{I+1}}{1+x^2} & \\ & & & -\frac{xz_{I+1}^{-1}}{1+x^2} & 1 & \\ -\frac{xz_r}{1+x^2} & & & & & \ddots \end{pmatrix}. \tag{4.109}$$

After the Gaussian integral with respect to λ_{an}, we are left with the following expression including the finite number of integrals.

$$I_{\{m_s^a\}}(x,z,z') = I^{(0)}(x,z,z')I^{(*)}_{\{m_s^a\}_*}(x,z,z'), \tag{4.110}$$

where $I^{(0)}$ is the determinant factor associated with the Gaussian integral of λ_{an}:

$$I^{(0)}(x,z,z') = \prod_{n=1}^{\infty}\frac{1}{\det M(x^n,z^n,z'^n)}, \tag{4.111}$$

and $I^{(*)}_{\{m_s^a\}_*}$ is given by

$$\begin{aligned} I^{(*)}_{\{m_s^a\}_*}(x,z,z') \;=\; & \frac{x^{2\epsilon_0(\{m_s^a\}_*)}}{(\text{symmetry})}\left(\prod_{a=1}^{r}\prod_{s=1}^{M_a}\int\frac{d\alpha_s^a}{2\pi}\right)\exp\left(i\sum_{a=1}^{r}\sum_{s=1}^{M_a}k_a m_s^a\alpha_s^a\right) \\ & \times\prod_a\prod_{s,t}\left[\exp\sum_{n=1}^{\infty}\frac{1}{n}\mathbf{f}^{V_a}(x^n,e^{in\beta_{st}^a})\right] \\ & \times\prod_I\prod_{s,t}\left[\exp\sum_{n=1}^{\infty}\frac{1}{n}\mathbf{f}^{H_I}(x^n,z_I^n,e^{in\beta_{st}^I})\right]. \end{aligned} \tag{4.112}$$

Here $\mathbf{f}^{\text{vec}}_{ast}$ and $\mathbf{f}^{\text{hyp}}_{Ist}$ are given by

$$\mathbf{f}^{V_a}(x,e^{i\beta_{st}^a}) \;=\; \left(-(1-\delta_{st})x^{2|m_s^a-m_t^a|}+x^{2(|m_s^a|+|m_t^a|)}\right)e^{i\beta_{st}^a}, \tag{4.113}$$

$$\mathbf{f}^{H_I}(x,z_I,e^{i\beta_{st}^I}) \;=\; \frac{x(x^{2|m_s^{L(I)}-m_t^{R(I)}|}-x^{2(|m_s^{L(I)}|+|m_t^{R(I)}|)})}{1+x^2}\left(z_Ie^{i\beta_{st}^I}+\frac{1}{z_Ie^{i\beta_{st}^I}}\right) \tag{4.114}$$

4.2.5 Factorization

We have already finished the calculation in the large N limit. The index (4.112) can be factorized three parts in neutral, positive and negative parts with respect to magnetic charges.

For later convenience we divide the magnetic charges $\{m_s^a\}_*$ into the collection of positive charges, denoted by $\{m_s^a\}_+$, and the collection of negative charges, $\{m_s^a\}_-$. Using the convention in §2.4, we write as

$$\{m_s^a\}_+\in\Gamma_M^{(+)},\quad \{m_s^a\}_-\in-\Gamma_M^{(+)}. \tag{4.115}$$

Each of $\{m_s^a\}_+$ and $\{m_s^a\}_-$ can be represented as a set of r Young diagrams by reordering the magnetic charges by decending order. For example,

$$\{\yng(2,1,1),\yng(4,2,2,1)\} = (\mathrm{diag}(2,1,1), \mathrm{diag}(4,2,2,1)) \tag{4.116}$$

We also introduce M_a^+ (M_a^-), the number of positive (negative) components in m_s^a for each a. The symmetry factor in (4.112) is the product of the symmetry factors of the $2r$ Young diagrams. The symmetry factor for a single Young diagram is defined as the product of the factorial of the number of the lines with the same length in the Young diagram. For example, the symmetry factor for $\yng(2,1,1)$ and $\yng(4,4,1,1,1)$ are 1!2! and 2!3!, respectively.

We can easily see that $I^{(*)}_{\{m_s^a\}_*}$ is further factorized into $I^{(+)}_{\{m_s^a\}_+}$ depending only on $\{m_s^a\}_+$, and $I^{(-)}_{\{m_s^a\}_-}$ depending only on $\{m_s^a\}_-$. To show the factorization of the zero-point energy contribution $x^{2\epsilon_0}$, we divide the range of all the summations of color indices s and t in (4.106) into two parts as $\sum_{s\in M_a} = \sum_{s\in M_a^+} + \sum_{s\in M_a^-}$. The first term in (4.106) is decomposed as

$$\begin{aligned} &-\sum_{a=1}^{r}\sum_{s\in M_a^+}\sum_{t\in M_a^+}|m_s^a - m_t^a| - \sum_{a=1}^{r}\sum_{s\in M_a^-}\sum_{t\in M_a^-}|m_s^a - m_t^a| \\ &-\sum_{a=1}^{r}\left(2M_a^-\sum_{s\in M_a^+}|m_s^a| + 2M_a^+\sum_{s\in M_a^-}|m_s^a|\right). \end{aligned} \tag{4.117}$$

In the first line, the contributions of $\{m_s^a\}_+$ and $\{m_s^a\}_-$ decouple from each other. The two terms in the second line depend on both $\{m_s^a\}_+$ and $\{m_s^a\}_-$, and for the factorization, these terms should be canceled by other terms. Actually, these terms are precisely canceled by the mixed terms arising from the $\sum 2M_a \sum |m_s^a|$ term in the second line in (4.106). In this way, all the mixed terms cancel, and the zero-point energy is represented as

$$\epsilon_0(\{m_s^a\}_*) = \epsilon_0(\{m_s^a\}_+) + \epsilon_0(\{m_s^a\}_-). \tag{4.118}$$

The factorization of the second and the third lines in (4.112) is shown by using the fact that the factors in the form $x^{2|m-m'|} - x^{2(|m|-|m'|)}$ appearing in (4.113) and (4.114) vanish when m and m' have opposite signatures.

Now we have shown that the gauge theory index factorizes into the three parts:

$$I_{\{m_s^a\}}(x,z,z') = I^{(0)}(x,z,z')I^{(+)}_{\{m_s^a\}_+}(x,z,z')I^{(-)}_{\{m_s^a\}_-}(x,z,z'). \tag{4.119}$$

Because the summations over $\{m_s^a\}_+$ and $\{m_s^a\}_-$ are independent in the large N limit, the total index also factorizes into three parts

$$I^{\text{guage}}(x, z, z'; \vec{\tau}) = I^{(0)}(x, z, z') I^{(+)}(x, z, z'; \vec{\tau}) I^{(-)}(x, z, z'; \vec{\tau}), \tag{4.120}$$

where $I^{(\pm)}\ (x, z, z'; \vec{\tau})$ is defined by

$$I^{(\pm)}(x, z, z'; \vec{\tau}) = \sum_{\{m_s^a\}_\pm} I^{(\pm)}_{\{m_s^a\}_\pm}(x, z, z') \tau_1^{m_1} \cdots \tau_r^{m_r}. \tag{4.121}$$

We also define the index for a specific gauge invariant monopole charges $\{m_a\}$ as the sum of contributions of all the monopole backgrounds with the same $\{m_a\}$. For example, the index for $\{m_a\} = \{2, 2\}$ is the sum of four contributions:

$$I^{(+)}_{\{2,2\}} = I^{(+)}_{\{\ydiagram{2},\ydiagram{2}\}} + I^{(+)}_{\{\ydiagram{2},\ydiagram{1,1}\}} + I^{(+)}_{\{\ydiagram{1,1},\ydiagram{2}\}} + I^{(+)}_{\{\ydiagram{1,1},\ydiagram{1,1}\}}, \tag{4.122}$$

where we used the Young diagrams to represent the charges $\{m_s^a\}_+$.

4.2.6 Rule of selection

The integration with respect to the angular variable α_{ia} leaves only terms whose P and P', the numbers of z and z' in the terms, satisfy certain selection rules, which correspond to conditions of gauge-invariance of operators. For operators which carries only the diagonal U(1) magnetic charge, the selection rules are expected to be

$$\frac{1}{p} P \in \mathbf{Z}, \quad \frac{1}{q} P' \in \mathbf{Z}, \tag{4.123}$$

which means that such operators are invariant under the residual gauge transformations (2.52). In this subsection, we derive selection rules for general magnetic charges, which will be compared to the spectrum of the Kaluza-Klein momenta derived on the gravity side.

Let us start from (4.104). For every vertex ($U(N)$ gauge group) a, we have N angular variables α_s^a ($s = 1, \ldots, N$). Instead of these, let us take α_1^a and β_{1s}^a ($s = 2, \ldots, N$) as N independent angular variables. We replace all α_s^a ($s \geq 2$) in (4.104) by $\alpha_1^a - \beta_{1s}^a$. By this replacement, the exponential factor including the levels k_a becomes

$$\exp\left(i \sum_{a=1}^{r} \sum_{s=1}^{N} k_a m_s^a \alpha_s^a\right) = \exp\left(i \sum_{a=1}^{r} k_a m_a \alpha_1^a\right) \times \exp\left(-i \sum_{a=1}^{r} \sum_{s=2}^{N} k_a m_s^a \beta_{1s}^a\right). \tag{4.124}$$

The variables β_{st}^I in $f_{Ist}^{\text{hyp}}(x, z_I e^{i\beta_{st}^I})$ become

$$\beta_{st}^I = \beta_{11}^I - \beta_{1s}^{L(I)} + \beta_{1t}^{R(I)}. \tag{4.125}$$

As a result, the parameter z_I is always accompanied by $e^{i\beta^I_{11}}$. After integrating out β^a_{1s} $(s \geq 2)$, we obtain

$$I_{\{m^a_s\}}(x,z,z') = \left(\prod_{a=1}^r \int d\alpha^a_1\right) \exp\left(i\sum_{a=1}^r k_a m_a \alpha^a_1\right) f(z_I e^{i\beta^I_{11}}), \tag{4.126}$$

where $f(z_I e^{i\beta^I_{11}})$ is a certain function of r variables $z_I e^{i\beta^I_{11}}$ $(I = 1, \ldots, r)$. This function also depends on x, but we do not take care about it here.

We now have r angular variables α^a_1 to be integrated. Instead of these, let us use $\alpha_{\bullet 1}$ and β^I_{11} as r independent variables, where $a = \bullet$ is the reference vertex, which is given in §2.4. By definition β^I_{11} satisfy

$$\sum_{I=1}^r \beta^I_{11} = 0. \tag{4.127}$$

To treat all β^I_{11} as independent variables, we insert the δ function

$$\delta(\sum_{I=1}^r \beta^I_{11}) = \sum_{d=-\infty}^{\infty} \exp(-id\sum_{I=1}^r \beta^I_{11}) \tag{4.128}$$

into (4.126). ($\delta(\theta)$ in this equation is the δ-function for an angular variable. Namely it has the periodic support $\theta = 2\pi n$.) We rewrite the exponential factor in (4.126) by using

$$\sum_{a=1}^r k_a m_a \alpha^a_1 = k\sum_{I=1}^r c_I \beta^I_{11} - k\alpha_{\bullet 1}\sum_{I=1}^r s_I \mu_I, \tag{4.129}$$

where μ_I is the relative magnetic charge defined by (2.30) and c_I is defined by

$$c_I = \sum_{\bullet < J < I} (s_I - s_J)\mu_J - m_\bullet s_I. \tag{4.130}$$

As a result, we obtain

$$\begin{aligned} I_{\{m^a_s\}}(x,z,z') &= \sum_{d=-\infty}^{\infty} \int d\alpha_{\bullet 1} \left(\prod_{I=1}^r \int d\beta^I_{11}\right) F(z_I e^{i\beta^I_{11}}) \\ &\quad \times \exp\left(i\sum_{I=1}^r (kc_I - d)\beta^I_{11} - k\alpha_{\bullet 1}\sum_{I=1}^r s_I\mu_I\right). \end{aligned} \tag{4.131}$$

The integration of $\alpha_{\bullet 1}$ gives the constraint

$$\sum_{I=1}^r s_I \mu_I = 0, \tag{4.132}$$

imposed on μ_I. This is equivalent to (2.28), which comes from the gauge-invariance of monopole operators.

For every I, the β^I_{11} integration picks up terms proportional to $z_I^{d-kc_I}$. Therefore, P and P', the total numbers of z and z', are given by

$$P = \sum_{i=1}^{p}(d-kc_i) = pd + \sum_{i'=1}^{q} l_{i'}\mu_{i'}, \tag{4.133}$$

$$P' = \sum_{i'=1}^{q}(d-kc_{i'}) = qd + kqm_\bullet - \sum_{i=1}^{p} l_i\mu_i, \tag{4.134}$$

where l_I are the linking numbers defined by (5.35). We number edges in the linear quiver diagram in ascending order from left to right. Here $n_I = N_{L(I)} - N_{R(I)}$ represents the number of D3-branes ending on the fivebrane I. In this subsection, we only consider the case of $n_I = 0$, and the linking numbers are multiples of k. From these equations, we can see that the selection rules in (4.123) are corrected when the relative magnetic charges μ_I are non-vanishing.

The charge P_M is calculated by using (2.53) as

$$P_M = m_\bullet - \frac{1}{kq}\sum_{i=1}^{p} l_i\mu_i - \frac{1}{kp}\sum_{i'=1}^{q} l_{i'}\mu_{i'}. \tag{4.135}$$

The right hand side of this equation is independent of d, and a function of the magnetic charges m_a. Although each of three terms in (4.135) separately depends on the choice of the reference point, the sum of them is independent of the choice. We can obtain the relation (4.135) in more direct way from (4.104). The reason why P_M is related to m_a is that the flavor rotation generated by P_M is gauge equivalent to the shift of the dual photon field $\tilde{a}$. The gauge invariance of an operator requires its charges associated with these two shifts to be the same. When the gauge group is $U(1)^r$, the gauge symmetry connecting these two shifts are parameterized by φ defined in (2.45). For $U(N)^r$ gauge group we can define such a parameter φ by

$$\partial_\varphi\beta^i_{st} = \frac{1}{p}, \quad \partial_\varphi\beta^{i'}_{st} = -\frac{1}{q}, \quad \partial_\varphi\beta^a_{st} = 0. \tag{4.136}$$

The action of ∂_φ on the parameters α^a_s is

$$\partial_\varphi\alpha^a_s = \gamma_a, \tag{4.137}$$

where γ_a are constants satisfying

$$\gamma_{L(i)} - \gamma_{R(i)} = \frac{1}{p}, \quad \gamma_{L(i')} - \gamma_{R(i')} = -\frac{1}{q}. \tag{4.138}$$

These conditions determine γ_a up to overall shift. Integration of φ leaves only the contribution of operators which are invariant under the φ gauge transformation, and reproduces the relation (4.135) as we see below. Let us perform the integration over φ orbit in (4.104). A term proportional to $z^P z^{P'}$ is accompanied by the factor $e^{-ikP_M\varphi}$. The other factor including φ is the exponential factor including the Chern-Simons levels. It includes

$$\exp\left(i\varphi\sum_{a=1}^{r}k_a m_a\gamma_a\right). \tag{4.139}$$

Therefore, for the term to survive after φ integration, the following relation must hold.

$$P_M=\frac{1}{k}\sum_{a=1}^{r}k_a m_a\gamma_a. \tag{4.140}$$

This is equivalent to (4.135). This expression is manifestly independent of the reference point. Due to the constraint (2.28), (4.140) is not changed by the overall shift of γ_a, and is determined unambiguously. We can easily show $(1/k)\sum_a k_a\gamma_a = 1$, and P_M is a weighted average of the magnetic charges.

4.3 AdS$_4$ particle index

The indices we consider in gravity side are defined by

$$I(x,z,z';\vec{t},\vec{t}')=\mathrm{Tr}\left[(-)^F e^{-\beta'\{Q,S\}}x^{2(D+j)}y_3^{h_3}z^P z'^{P'}\vec{t}^{\vec{\rho}}\vec{t}'^{\vec{\rho}'}\right], \tag{4.141}$$

We here introduced new chemical potentials $\vec{t}=(t_1,\ldots,t_p),\vec{t}'=(t'_1,\ldots,t'_q)$ for the $H_{\mathcal{S}},H_{\mathcal{S}'}$ charges $\vec{\rho},\vec{\rho}'$, respectively. The meaning of the last part in (**??**) is

$$\vec{t}^{\vec{\rho}}\vec{t}'^{\vec{\rho}'}=\prod_{i=1}^{p}t_i^{\rho_i}\prod_{i'=1}^{q}t_{i'}^{\rho_{i'}}. \tag{4.142}$$

In addition, we insert the chemical potential y_3 for the charge h_3, which is set to 1 in comparing the gauge theory index.

By this definition (4.141), we can define two indices: One is the single-particle index $I^{\rm sp}$, which is defined by taking the trace over all single-particle states, the other is the multi-particle index $I^{\rm mp}$, which is also defined by the same equation by summing up all the multi-particle states including single- and no-particle states. We compare the gauge theory indices obtained in the previous section with this corresponding multi-particle indices for M-theory in the dual geometry AdS$_4\times$ $\mathbf{S}^7_{p,q,k}$. In general, a single-particle index and the corresponding multi-particle index are related by

$$I^{\rm mp}(\cdot)=\exp\sum_{n=1}^{\infty}\frac{1}{n}I^{\rm sp}(\cdot^n), \tag{4.143}$$

where "$(\cdot)$" represents the sequence of the arguments corresponding to additive charges and "$(\cdot^n)$" on the right hand side is the sequence with every argument replaced by its n-th power. This index does not depend on β', and only operators saturating the BPS bound

$$\{Q,S\} = D - j - (T_3 + T_3') \geq 0 \tag{4.144}$$

contribute. We will compute a superconformal index of the Kaluza-Klein modes. We denote generators of these two $SO(2)$ symmetries by H_1 and H_2. They are related to T_3 and T_3' by

$$H_1 = T_3 + T_3', \quad H_2 = -T_3 + T_3'. \tag{4.145}$$

See (D.19) in Appendix D.

As we have already mentioned, the internal space $\mathbf{S}^7_{p,q,k}$ includes two fixed loci $\mathcal{S}_U$ and $\mathcal{S}_T$, and we should take account of the twisted sectors associated with these. The two sectors can be treated in parallel ways, and we first consider the contribution of the $\mathcal{S}_U$ sector in detail. Because $\mathcal{S}_U$ is the A_{p-1} type singularity, we expect that there exists an $SU(p)$ vector-multiplet localized on the locus. With the coordinates defined in (3.45), $\mathcal{S}_U$ is given by $t = 0$, or $z_1 = z_2 = 0$, and is spanned by two complex coordinates z_3 and z_4 constrained by

$$|z_3|^2 + |z_4|^2 = 1. \tag{4.146}$$

This equation together with the identification by the $\mathbf{Z}_{kq}$ generated by the third generator in (2.52) defines the Lens space $\mathbf{S}^3/\mathbf{Z}_{kq}$. Because this orbifold does not have fixed points, we can obtain the single-particle index for a vector-multiplet in this manifold by the $\mathbf{Z}_{kq}$ projection from the index for the covering space $\mathbf{S}^3$. The $H_{\mathcal{S}\times\mathcal{S}'}$ charges are represented as a vector in the $G_{\mathcal{S}\times\mathcal{S}'}$ root lattice, and we can regard them as wrapping numbers of M2-branes on the vanishing two-cycles at the singularities.

By summing these contribution coming from the twisted sectors and that of bulk particles, we can obtain the single-particle index as follow:

$$I^{\rm sp}(x,z,z';\vec{t},\vec{t}') = I^{\mathcal{B}}(x,z,z') + I^{\mathcal{S}_U}(x,z';\vec{t}) + I^{\mathcal{S}_T}(x,z;\vec{t}'). \tag{4.147}$$

For convenience, we give our result of these indices in advance. To do this, let us denote the coefficient of a function $I^*(x,z,z';\vec{t},\vec{t}')$ in the expansion $\vec{t},\vec{t}'$ and a new variable c, which is used to pick up the specific M-momentum, as $I^*_{(P_M,\vec{\rho},\vec{\rho}')}(x,z,z')$:

$$I^*(x,c^{-\frac{1}{kp}}z,c^{\frac{1}{kq}}z';\vec{t},\vec{t}') = \sum_{(P_M,\vec{\rho},\vec{\rho}')} I^*_{(P_M,\vec{\rho},\vec{\rho}')}(x,z,z')c^{P_M}\vec{t}^{\vec{\rho}}\vec{t}'^{\vec{\rho}'}. \tag{4.148}$$

In this notation, the single-particle index is given by

$$I^{\rm sp}_{(P_M,\vec{\rho},\vec{\rho}')}(x,z,z') = \delta_{\vec{\rho},\vec{0}}\delta_{\vec{\rho}',\vec{0}} I^{\mathcal{B}}_{P_M}(x,z,z') + \delta_{\vec{\rho}',\vec{0}} I^{\mathcal{S}_U}_{(P_M,\vec{\rho})}(x,z') + \delta_{\vec{\rho},\vec{0}} I^{\mathcal{S}_T}_{(P_M,\vec{\rho}')}(x,z), \tag{4.149}$$

where each contribution on the right hand side is

$$\begin{aligned} I^{\mathcal{B}}_{P_M}(x,z,z') &= \sum_{a=-\infty}^{\infty} I^{\rm grav}_{pa,q(a+kP_M)}(x) z^{pa} z'^{q(a+kP_M)}, && (4.150)\\ I^{\mathcal{S}_U}_{(P_M,\vec{\rho})}(x,z') &= \deg(\vec{\rho}) I^{\rm vec}_{kqP_M}(x) z'^{kqP_M}, && (4.151)\\ I^{\mathcal{S}_T}_{(P_M,\vec{\rho}')}(x,z) &= \deg(\vec{\rho}') I^{\rm vec}_{-kpP_M}(x) z^{-kpP_M}. && (4.152)\end{aligned}$$

Here $\deg(\vec{\rho})$ is the degeneracy for the adjoint representation at $\vec{\rho}$ in the $SU(p)$ root lattice. Namely,

$$\deg(\vec{\rho}) = \begin{cases} 1 & (|\vec{\rho}|^2 = 2) \\ p-1 & (\vec{\rho} = 0) \\ 0 & (\text{others}). \end{cases} \tag{4.153}$$

We derive this expression in this subsection.

As noted, once we obtained the single-particle index, the corresponding multi-particle index is straightforward by (4.143). In this case, the relation is given by

$$I^{\rm mp}(x,z,z';\vec{t},\vec{t}') = \exp \sum_{n=1}^{\infty} \frac{1}{n} I^{\rm sp}(x^n, z^n, z'^n; \vec{t}^n, \vec{t}'^n). \tag{4.154}$$

4.3.1 Bulk sector

First, we discuss the bulk sector. In general, the index for bulk particles in an orbifold can be obtained from the index for its covering space by the projection which leaves modes invariant under the orbifold action. The single-particle index for bulk gravitons in $AdS_4 \times \mathbf{S}^7$ was obtained by [111] as follows[3]:

$$I_{G_{\mathbf{S}^7}}(x,y_2,y_3,y_4) = \mathrm{Tr}\left[(-)^F e^{-\beta'\{Q,S\}} x^{2(\Delta+j)} y_2^{h_2} y_3^{h_3} y_4^{h_4}\right] = \frac{(\text{numerator})}{(\text{denominator})}, \tag{4.156}$$

[3]The single-particle index for the orbifold $AdS_4 \times \mathbf{S}^7/\mathbf{Z}_k$ in the large k limit is obtained from (4.156) by picking up y_4 independent terms as

$$I_{G_{\mathbf{CP}^3}} := I_{G_{\mathbf{S}^7}}|_{y_4=0}. \tag{4.155}$$

where the numerator and the denominator are given by

$$\begin{aligned}
(\text{numerator}) &= (y_3y_2+y_2y_4+y_4y_3+y_3y_2y_4(y_3+y_2+y_4))(x^6-x^2) \\
&\quad +\sqrt{y_3y_2y_4}(1+y_3y_2+y_2y_4+y_4y_3)x \\
&\quad -\sqrt{y_3y_2y_4}(y_3+y_2+y_4+y_3y_2y_4)x^7, \qquad (4.157)\\
(\text{denominator}) &= (1-x^4)(\sqrt{y_4}-x\sqrt{y_3y_2})(\sqrt{y_3}-x\sqrt{y_2y_4}) \\
&\quad \times(\sqrt{y_2}-x\sqrt{y_4y_3})(\sqrt{y_3y_2y_4}-x). \qquad (4.158)
\end{aligned}$$

For the aim to be self-contained, we derive this index in Appendix D.

To obtain the index for $\mathbf{S}^7_{p,q,k}$ given by (3.34), let us first rewrite the index (4.156) as a function of x, z, and z'. Because $z^P z'^{P'} = (zz')^{h_2}(z/z')^{h_4}$, we can change the variables by substituting

$$y_2 = zz', \quad y_3 = 1, \quad y_4 = \frac{z}{z'}, \tag{4.159}$$

into (4.156). We obtain

$$I_{G_{\mathbf{S}^7}}(x,z,z') = \frac{(\frac{1}{z}+z'+z+\frac{1}{z'})(x-x^7)+(2+\frac{z'}{z}+\frac{1}{zz'}+zz'+\frac{z}{z'})(x^6-x^2)}{(1-x^4)(1-xz')(1-xz)(1-x/z')(1-x/z)}. \tag{4.160}$$

We expand this index with respect to z and z' as

$$I_{G_{\mathbf{S}^7}}(x,z,z') = \sum_{P,P'} I^{\text{grav}}_{P,P'}(x) z^P z'^{P'}. \tag{4.161}$$

The coefficients $I^{\text{grav}}_{P,P'}(x)$ are given by

$$I^{\text{grav}}_{P,P'}(x) = (1-\delta_{P,0}\delta_{P',0})x^{|P|+|P'|} + \delta_{P,0}\frac{x^{|P'|+2}}{1-x^4} + \delta_{P',0}\frac{x^{|P|+2}}{1-x^4}. \tag{4.162}$$

The charges P and P' invariant under (2.52) must satisfy

$$\frac{1}{p}P \in \mathbf{Z}, \quad \frac{1}{q}P' \in \mathbf{Z}, \quad P_M \in \mathbf{Z}. \tag{4.163}$$

The general solution to these conditions is

$$P = pa, \quad P' = q(a+kb), \quad a,b \in \mathbf{Z}. \tag{4.164}$$

The integer b is equal to the M-momentum P_M. As a result, the single-particle index for the bulk gravitons in $AdS_4 \times \mathbf{S}^7_{p,q,k}$ is given by

$$I_{G_{\mathbf{S}^7_{p,q,k}}}(x,z,z') = \sum_{a,b=-\infty}^{\infty} I^{\text{grav}}_{pa,q(a+kb)}(x) z^{pa} z'^{q(a+kb)}, \tag{4.165}$$

which is identified with the bulk contribution $I^{\mathcal{B}}(x,z,z')$.

4.3.2 Action of a vector-multiplet on $AdS_4 \times \mathbf{S}^3$

In this subsection, we construct the action of the vector-multiplet in the seven-dimensional curved background $\widetilde{\mathcal{S}_U} \simeq AdS_4 \times \mathbf{S}^3$. The action is necessary to carry out Kaluza-Klein analysis and obtain the spectrum of the localized modes in $\widetilde{\mathcal{S}_U}$. The spectrum is needed to calculate the contribution to indices from the singularity.

A $D = 7$ $\mathcal{N} = 2$ vector-multiplet consists of a gauge field A_M, a symplectic Majorana spinor field λ, and three real scalar fields ϕ_i $(i = 1, 2, 3)$. If we identify the background spacetime with the singular locus $\widetilde{\mathcal{S}_U}$, the R-symmetry of this theory is $SU(2)_R$ defined in §3.3, and the isometry of $\mathbf{S}^3$ is $SU(2)'_R \times SU(2)'_F$. λ and ϕ_i belong to the $SU(2)_R$ doublet and the triplet, respectively. In advance, we show the supersymmetric action

$$\begin{aligned} S &= \int d^7x\sqrt{-g}\Big[-\frac{1}{4}F_{MN}F^{MN} - \frac{1}{2}\overline{\lambda}\Gamma^M D_M\lambda - \frac{1}{2}\partial_M\phi_i\partial^M\phi_i \\ &\quad + \frac{3}{8L}(\overline{\lambda}\Gamma\lambda) + \frac{1}{L^2}\phi_i\phi_i - \frac{3}{4L}\epsilon^{kmn}A_kF_{mn}\Big], \end{aligned} \tag{4.166}$$

where L is AdS radius. This is invariant under the supersymmetry transformation

$$\delta\phi_i = i(\overline{\epsilon}\sigma_i\lambda), \quad \delta\lambda = -i\sigma_i\Gamma^M\epsilon\partial_M\phi_i + \frac{1}{2}F_{MN}\Gamma^{MN}\epsilon + \frac{i}{L}\phi_i\sigma_i\Gamma\epsilon, \quad \delta A_M = -(\overline{\epsilon}\Gamma_M\lambda). \tag{4.167}$$

There are mass terms for the fermion λ and the scalars ϕ_i. We also have the Chern-Simons coupling for the gauge field. Note that the tachyonic scalar mass $m^2 = -2/L^2$ satisfies the Breitenlohner-Freedman bound $m^2 \geq -9/(4L^2)$. These terms are inevitable to obtain the Kaluza-Klein spectrum consistent with the gauge invariant operators in the boundary theory.

To construct the action (4.166), we use the Noether procedure; we can determine the coupling of the component fields order by order with respect to the background curvature by supersymmetry. We start from the action

$$S_0 = \int d^7x\sqrt{-g}\left[-\frac{1}{4}F_{MN}F^{MN} - \frac{1}{2}\overline{\lambda}\Gamma^M\nabla_M\lambda - \frac{1}{2}\partial_M\phi_i\partial^M\phi_i\right], \tag{4.168}$$

and the transformation laws

$$\delta\phi_i = i(\overline{\epsilon}\sigma_i\lambda), \quad \delta\lambda = -i\sigma_i\Gamma^M\epsilon\partial_M\phi_i + \frac{1}{2}F_{MN}\Gamma^{MN}\epsilon, \quad \delta A_M = -(\overline{\epsilon}\Gamma_M\lambda), \tag{4.169}$$

where $M, N, \ldots$ are seven-dimensional vector indices and $i, j = 1, 2, 3$ are indices for $SU(2)_R$ triplet. σ_i are Pauli matrices acting on $SU(2)_R$ doublets. These are obtained by dimensional reduction from $\mathcal{N} = 1$ supersymmetric Maxwell theory on the ten-dimensional flat background, and the covariantization with respect to diffeomorphism. If the background spacetime is flat and the transformation parameter ϵ is a constant, the action (4.168) is invariant under the transformation (4.169).

The first step of the Noether procedure is to compute the supersymmetry variation without assuming the flatness. For a general curved background and a coordinate dependent parameter ϵ, we obtain the variation in the form $J^M \nabla_M \epsilon$.

$$\delta S_0 = \int \left[i(\overline{\lambda}\sigma_i \Gamma^M \Gamma^N \nabla_M \epsilon)\partial_N \phi_i - \frac{1}{2}(\overline{\lambda}\Gamma^L \Gamma^{MN} \nabla_L \epsilon) F_{MN} \right]. \tag{4.170}$$

If we were constructing a supergravity action, this term would be canceled by introducing the Noether coupling to the gravitino, $J^M \psi_M$. But now, we want the action invariant under the global supersymmetry, whose parameter ϵ satisfies the Killing spinor equations

$$\nabla_\mu \epsilon = a\Gamma\Gamma_\mu \epsilon, \quad \nabla_m \epsilon = b\Gamma\Gamma_m \epsilon, \quad \Gamma = \Gamma_{0123}, \tag{4.171}$$

where we use $\mu, \nu, \ldots = 0, 1, 2, 3$ for AdS_4 and $m, n, \ldots = 4, 5, 6$ for $\mathbf{S}^3$. a and b are parameters with dimension of mass. These parameters are proportional to the curvature of AdS_4 and $\mathbf{S}^3$. By substituting (4.171) into $[\nabla_M, \nabla_N]\epsilon = (1/4)R_{MNPQ}\Gamma^{PQ}\epsilon$ with the curvature

$$R^{\mathrm{AdS}_4}_{\mu\nu\lambda\rho} = -\frac{1}{R^2_{\mathrm{AdS}_4}}(g_{\mu\lambda}g_{\nu\rho} - g_{\nu\lambda}g_{\mu\rho}), \quad R^{\mathbf{S}^3}_{mnpq} = \frac{1}{R^2_{\mathbf{S}^3}}(g_{mp}g_{nq} - g_{np}g_{mq}) \tag{4.172}$$

we obtain

$$a^2 = \frac{1}{4R^2_{\mathrm{AdS}_4}}, \quad b^2 = \frac{1}{4R^2_{\mathbf{S}^3}}. \tag{4.173}$$

If we require the background spacetime $AdS_4 \times \mathbf{S}^3$ is the locus in the M2-brane near horizon geometry $AdS_4 \times \mathbf{S}^7$. From (3.24), two radii R_{AdS_4} and $R_{\mathbf{S}^3}$ are related by $R_{\mathbf{S}^3} = 2R_{AdS_4}$. This means $a = \pm 2b$. In the following, however, we will not use this relation as an input because this relation is automatically obtained by requiring supersymmetry.

Let us first focus on the first term in (4.170). By using (4.171), we obtain

$$(\text{1st term in } \delta S_0) = \int \left[i(2a - 3b)(\overline{\lambda}\sigma_i \Gamma^\nu \Gamma \epsilon)\partial_\nu \phi_i + i(4a - b)(\overline{\lambda}\sigma_i \Gamma^n \Gamma \epsilon)\partial_n \phi_i \right]. \tag{4.174}$$

There are two ways to obtain similar variations to cancel this. One is to introduce a fermion mass term

$$S'_\lambda = \frac{m_\lambda}{2}(\overline{\lambda}\Gamma\lambda). \tag{4.175}$$

The second is to deform the fermion transformation law by

$$\delta'\lambda = iq\phi_i\sigma_i\Gamma\epsilon. \qquad (4.176)$$

m_λ and q are real parameters with mass dimension 1. For the variation (4.174) to be canceled by $\delta' S_0$ and $\delta S'_\lambda$, the parameters m_λ and q should be given by

$$m_\lambda = a + b, \quad q = 3a - 2b. \qquad (4.177)$$

In addition to terms canceling (4.174), $\delta' S_0$ and $\delta S'_\lambda$ provide more terms. One of them is the term in $\delta' S_0$ including $\nabla\epsilon$. By using the Killing spinor equations in (4.171) it becomes

$$i(3a - 2b)(4a + 3b)(\overline{\lambda}\sigma_i\epsilon)\phi_i. \qquad (4.178)$$

We also obtain a similar term from $\delta' S'_\lambda$:

$$\delta' S'_\lambda = -i(3a - 2b)(a + b)(\overline{\lambda}\sigma_i\epsilon)\phi_i. \qquad (4.179)$$

These two terms are proportional to $\delta\phi_i$, and can be canceled by introducing the following scalar mass term:

$$S'_\phi = -\frac{m_\phi^2}{2}\phi_i\phi_i, \quad m_\phi^2 = -9a^2 + 4b^2. \qquad (4.180)$$

Now all variations independent of the gauge field have been canceled. Let us turn to the terms including the gauge field. The second term in (4.170) is rewritten with (4.171) as

$$(\text{2nd term in } \delta S_0) = \bar{\lambda}\left(-\frac{3b}{2}F_{\mu\nu}\Gamma^{\mu\nu} + (2a-b)F_{\nu m}\Gamma^\nu\Gamma^m + \left(2a + \frac{b}{2}\right)F_{mn}\Gamma^{mn}\right)\Gamma\epsilon. \qquad (4.181)$$

A similar term arises from $\delta S'_\lambda$:

$$\frac{a+b}{2}F_{MN}(\lambda\Gamma\Gamma^{MN}\epsilon). \qquad (4.182)$$

The first two terms in (4.181) must be canceled by the corresponding part of (4.182). This requires the relation

$$a = 2b. \qquad (4.183)$$

This is the relation which is expected from the M2-brane near horizon geometry. The term in (4.181) including F_{mn} is canceled by introducing the Chern-Simons term

$$S_{CS} = -\frac{3a}{2}\epsilon^{kmn}A_kF_{mn}. \qquad (4.184)$$

Now all variations are canceled, and we obtain the action (4.166) and the supersymmetry transformation (4.167) by setting $a = 1/(2L)$.

4.3.3 Kaluza-Klein analysis

In this subsection, we carry out Kaluza-Klein analysis for component fields in a $D = 7$ $\mathcal{N} = 2$ vector-multiplet. We expand fields in $\mathbf{S}^3$ into spherical harmonics, and determine the conformal dimension for every mode by using equations of motion derived from the action we constructed in §4.3.2.

In advance, let us summarize our convention using the calculation. We take the Poincare coordinates in AdS_4 with the metric

$$ds^2 = L^2 \frac{-dx_0^2 + dx_1^2 + dx_2^2 + dz^2}{z^2}. \tag{4.185}$$

The conformal dimension is defined as an eigenvalue of the Lie derivative associated with the Killing vector

$$D = z\partial_z + x^i \partial_i. \tag{4.186}$$

We denote spin j spherical harmonics in $\mathbf{S}^3$ by $Y^m_{j,(s_1,s_2)}$. Let $SU(2)_1 \times SU(2)_2$ be the isometry of $\mathbf{S}^3$. The index j is the spin of the field, and quantum numbers s_1 and s_2 are $SU(2)_1$ and $SU(2)_2$ angular momenta, which take half integers and satisfy

$$|s_1 - s_2| \leq j \leq s_1 + s_2, \quad s_1 + s_2 - j \in \mathbf{Z}. \tag{4.187}$$

m is the magnetic quantum numbers in the range

$$-j \leq m \leq j. \tag{4.188}$$

$Y^m_{j,(s_1,s_2)}$ actually represents a set of $(2s_1 + 1)(2s_2 + 1)$ harmonics forming the (s_1, s_2) representation of $SU(2)_1 \times SU(2)_2$. We suppress indices for them. Only harmonics with $j = |s_1 - s_2|$ are independent. For example, vector harmonics $\vec{Y}_{1,(s,s)} = (Y^{+1}_{1,(s,s)}, Y^0_{1,(s,s)}, Y^{-1}_{1,(s,s)})$, which do not satisfy this condition, are given as the gradient of the scalar harmonics: $\vec{Y}_{1,(s,s)} \propto \vec{\nabla} Y_{0,(s,s)}$.

These harmonics are eigenmodes of the Laplacian:

$$\Delta Y^m_{j,(s_1,s_2)} = -\frac{1}{R^2}[2s_1(s_1 + 1) + 2s_2(s_2 + 1) - j(j + 1)]Y^m_{j,(s_1,s_2)}, \tag{4.189}$$

where R is the radius of $\mathbf{S}^3$ in which the harmonics are defined. The following formula is also convenient:

$$\mathrm{rot}\, Y_{j,(s_1,s_2)} = \frac{1}{R}[s_2(s_2 + 1) - s_1(s_1 + 1)]Y_{j,(s_1,s_2)}. \tag{4.190}$$

In particular on the three-sphere with radius $2L$, eigenvalue of the Laplacian for the scalar harmonic is

$$\Delta_{\mathbf{S}^3} Y_{0,(s,s)} = -\frac{1}{(2L)^2} 4s(s + 1)Y_{0,(s,s)}. \tag{4.191}$$

The differential operator rot is a generalization of the rotation. For a spin j field ϕ_j, it is defined by

$$\mathrm{rot}\,\phi_j = T_m^{(j)} \nabla_m \phi_j, \tag{4.192}$$

where $T_m^{(j)}$ are $SO(3)$ generators of spin j representation normalized by $[T_m^{(j)}, T_n^{(j)}] = \epsilon_{mnp} T_p^{(j)}$. rot becomes the ordinary rotation for a vector field, and the Dirac's operator for a spinor field.

$$\mathrm{rot}\,\vec{\phi}_1 = \vec{\nabla} \times \vec{\phi}_1, \quad \mathrm{rot}\,\phi_{\frac{1}{2}} = -\frac{i}{2}\gamma^m \nabla_m \phi_{\frac{1}{2}}. \tag{4.193}$$

First, let us consider the scalar fields. The equation of motion of them derived from (4.166) is

$$\Delta\phi_i + \frac{2}{L^2}\phi_i = 0, \tag{4.194}$$

where the Laplacian is defined with the background $AdS_4 \times \mathbf{S}^3$ metric

$$ds_7^2 = L^2 \frac{-dx_0^2 + dx_1^2 + dx_2^2 + dz^2}{z^2} + (2L)^2 d\Omega_3^2. \tag{4.195}$$

The scalar function in $\mathbf{S}^3$ can be expanded by scalar spherical harmonics $Y_{0,(s,s)}$:

$$\phi = \sum_s f_{(s,s)} Y_{0,(s,s)}, \tag{4.196}$$

The quantum number $s = 0, 1/2, 1, \ldots$ is the orbital angular momentum in $\mathbf{S}^3$. $f_{(s,s)}$ are scalar fields on AdS_4. A harmonic $Y_{0,(s,s)}$ belongs to the $SO(4)(= SU(2)'_R \times SU(2)'_F)$ representation with the highest weight (s, s), and it actually has $(2s+1)^2$ components forming the representation. For the purpose of computing the conformal dimension of the corresponding operators, it is sufficient to assume that $f_{(s,s)}$ depends only on the radial coordinate r in AdS$_4$. By substituting this into the equation of motion (4.194), we obtain

$$z^4 \frac{d}{dr}\frac{1}{z^2}\frac{d}{dz} f_{(s,s)} - (s^2 + s - 2) f_{(s,s)} = 0. \tag{4.197}$$

This has two independent solutions

$$f_{(s,s)} \propto z^{s+2}, \quad z^{-s+1}. \tag{4.198}$$

When we discuss field-operator correspondence, we need to take one of these two solutions which is normalizable. If s is sufficiently large, only the former is normalizable. Here we choose the former for every s, which corresponds to an operator with conformal dimension $D = s + 2$.

Next, we consider the gauge field. The linearized equations of motion derived from the action (4.166) are

$$\nabla_M F^{M\mu} = 0, \quad \nabla_M F^{Mk} - \frac{3}{2L}\epsilon^{kmn}F_{mn} = 0. \tag{4.199}$$

The AdS components A_μ of the gauge field are scalars in $\mathbf{S}^3$ while the $\mathbf{S}^3$ components A_m form a vector representation in $\mathbf{S}^3$. They are expanded with scalar and vector spherical harmonics by

$$A_\mu = \sum_{s=0}^{\infty}(a_{s,s})_\mu Y_{0,(s,s)}, \tag{4.200}$$

$$\vec{A} = \sum_{s=1}^{\infty} a_{s-1,s}\vec{Y}_{1,(s-1,s)} + \sum_{s=1}^{\infty} a_{s,s-1}\vec{Y}_{1,(s,s-1)} + \sum_{s=1/2}^{\infty} a_{s,s}\vec{\nabla}Y_{0,(s,s)}. \tag{4.201}$$

We also expand the gauge transformation parameter Λ with the scalar harmonics as

$$\Lambda = \sum_{s=0}^{\infty}\lambda_{s,s}Y_{0,(s,s)}. \tag{4.202}$$

We can set $a_{s,s} = 0$ for $s \geq 1/2$ by using gauge symmetry with parameters $\lambda_{s,s}$ with $s \geq 1/2$. To fix the residual gauge symmetry with parameter $\lambda_{0,0}$ we take the Lorentz gauge in AdS_4.

$$\nabla_\mu (a_{0,0})^\mu = 0. \tag{4.203}$$

We still have residual gauge symmetry with parameter $\lambda_{0,0}$ satisfying $\Delta_{AdS_4}\lambda_{0,0} = 0$, which will be fixed later. With the gauge choice we mentioned above, the equations of motion reduce to the following set of differential equations.

$$\nabla_\mu (a_{s,s})^\mu = 0 \quad (s \geq 1/2), \tag{4.204}$$

$$\left(\Delta_{AdS_4}(a_{s,s})_\mu + \frac{3}{L^2}(a_{s,s})_\mu + (a_{s,s})_\mu\Delta_{\mathbf{S}^3}\right)Y_{0,(s,s)} = 0, \tag{4.205}$$

$$\left(\Delta_{AdS}a_{s-1,s} + a_{s-1,s}\Delta_{\mathbf{S}^3} - \frac{1}{2L^2}a_{s-1,s} - \frac{3}{L}a_{s-1,s}\vec{\nabla}\times\right)\vec{Y}_{1,(s-1,s)} = 0, \tag{4.206}$$

$$\left(\Delta_{AdS}a_{s,s-1} + a_{s,s-1}\Delta_{\mathbf{S}^3} - \frac{1}{2L^2}a_{s,s-1} - \frac{3}{L}a_{s,s-1}\vec{\nabla}\times\right)\vec{Y}_{1,(s,s-1)} = 0. \tag{4.207}$$

To determine the conformal dimension of the corresponding operators, we assume that $(a_{s,s})_\mu$, $a_{s-1,s}$, and $a_{s,s-1}$ depend only on the radial coordinate r. By using this assumption and the formuli (4.189) and (4.192), the equations for the radial component $(a_{s,s})_r$ become

$$\left(z\frac{d}{dz} - 2\right)(a_{s,s})_z = 0, \quad \left(z^2\frac{d}{dz^2} - 2 - s(s+1)\right)(a_{s,s})_r = 0. \tag{4.208}$$

For $s \geq 1/2$, these two equations do not have non-vanishing solutions. For $s = 0$, there is a solution $(a_{0,0})_z \propto z^2$, but we can set $(a_{0,0})_z = 0$ by the residual gauge symmetry $\lambda_{0,0} \propto r^3$, which satisfies $\Delta_{AdS_4}\lambda_{0,0} = 0$. The other equations of motion (4.205), (4.206), and (4.207) reduce to

$$\left(z^2\frac{d^2}{dz^2} - s(s+1)\right)(a_{s,s})_\mu(z) = 0, \tag{4.209}$$

$$\left(z^4\frac{d}{dz}\frac{1}{z^2}\frac{d}{dz} - s^2 - 3s\right)a_{s-1,s}(z) = 0, \tag{4.210}$$

$$\left(z^4\frac{d}{dz}\frac{1}{z^2}\frac{d}{dz} - s^2 + 3s\right)a_{s,s-1}(z) = 0. \tag{4.211}$$

Each of these has two independent solutions.

$$(a_{s,s})_\mu \propto z^{s+1}, \quad (D = s+2), \quad z^{-s}, \quad (D = -s+1), \tag{4.212}$$

$$a_{s-1,s} \propto z^{s+3}, \quad (D = s+3), \quad z^{-s}, \quad (D = -s), \tag{4.213}$$

$$a_{s,s-1} \propto z^{s}, \quad (D = s), \quad z^{-s+3}, \quad (D = -s+3). \tag{4.214}$$

D given above are the corresponding conformal dimensions. The former of each equation is the normalizable mode chosen here.

Finally, let us consider the fermion field λ. The equation of motion is

$$-\Gamma^M\nabla_M\lambda + \frac{3}{4L}\Gamma\lambda = 0. \tag{4.215}$$

λ is an eight-component spinor off shell and each mode of this field is expanded by the direct products of a four-component spinor in AdS_4 and a two-component spinor spherical harmonic in $\mathbf{S}^3$. We take the anzats

$$\lambda = \sum_s \left(\psi_{s-\frac{1}{2},s}(z)\otimes Y_{\frac{1}{2},(s-\frac{1}{2},s)} + \psi_{s,s-\frac{1}{2}}(z)\otimes Y_{\frac{1}{2},(s,s-\frac{1}{2})}\right). \tag{4.216}$$

The coefficient spinors $\psi_{s-\frac{1}{2},s}$ and $\psi_{s,s-\frac{1}{2}}$ have an implicit $SU(2)_R$ index as well as λ. Correspondingly to the factorization of the spinor (4.216), we factorize the Dirac matrices also as

$$\Gamma^m = \gamma^5\otimes\gamma^m, \quad \Gamma^\mu = \gamma^\mu\otimes\mathbf{1}_2, \quad \Gamma = i\gamma^5\otimes\mathbf{1}_2. \tag{4.217}$$

By substituting (4.216) and (4.217) into (4.215), we obtain the differential equations

$$\left(z\frac{d}{dz} - \frac{3}{2}\right)\psi_{s-\frac{1}{2},s} = i\gamma^{\widehat{r}}\gamma^5\,(s+1)\,\psi_{s-\frac{1}{2},s}, \tag{4.218}$$

$$\left(z\frac{d}{dz} - \frac{3}{2}\right)\psi_{s,s-\frac{1}{2}} = i\gamma^{\widehat{r}}\gamma^5\left(-s+\frac{1}{2}\right)\psi_{s,s-\frac{1}{2}}, \tag{4.219}$$

where the index $\widehat{r}$ of $\gamma^{\widehat{r}}$ represents the local Lorentz index along the radial direction. The solutions to these equations and the corresponding conformal dimensions are

$$\psi_{s-\frac{1}{2},s} = \eta^{(+)}_{s-\frac{1}{2},s} z^{s+\frac{5}{2}} + \eta^{(-)}_{s-\frac{1}{2},s} z^{-s+\frac{1}{2}}, \quad D = s + \frac{5}{2}, \quad -s + \frac{1}{2}, \tag{4.220}$$

$$\psi_{s,s-\frac{1}{2}} = \eta^{(+)}_{s,s-\frac{1}{2}} z^{-s+2} + \eta^{(-)}_{s,s-\frac{1}{2}} z^{s+1}, \quad D = -s + 2, \quad s + 1, \tag{4.221}$$

where $\eta^{(\pm)}$ are constant spinors satisfying $i\gamma^{\widehat{r}}\gamma^5\eta^{(\pm)} = \pm\eta^{(\pm)}$. $\eta^{(+)}_{s-\frac{1}{2},s}$ and $\eta^{(-)}_{s,s-\frac{1}{2}}$ correspond to the normalizable modes for all s.

In the end, we obtain the Kaluza-Klein spectrum given in Table 4.1. Notice

Table 4.1: The Kaluza-Klein spectrum of a vector-multiplet (A_M, λ, ϕ_i) in $\widetilde{\mathcal{S}} \simeq AdS_4 \times \mathbf{S}^3$. For each mode shown in the table, there exists the other mode with D replaced by $3 - D$, which is not normalizable for large s.

fields	$SU(2)_J$	$SU(2)_R$	$SU(2)'_R$	$SU(2)'_F$	D
ϕ_i	0	1	s	s	$s+2$
A_M	0	0	$s+1$	s	$s+1$
	1	0	s	s	$s+2$
	0	0	$s-1$	s	$s+3$
λ	$\frac{1}{2}$	$\frac{1}{2}$	$s+\frac{1}{2}$	s	$s+\frac{3}{2}$
	$\frac{1}{2}$	$\frac{1}{2}$	$s-\frac{1}{2}$	s	$s+\frac{5}{2}$

that these Kaluza-Klein modes form 1/2 BPS representations given in Table D.4 in Appendix D. Therefore, the spectrum $V_{\widetilde{\mathcal{S}_U}}$ is written as

$$V_{\widetilde{\mathcal{S}_U}} = \bigoplus_{s=0,\frac{1}{2},\dots} (4, B, +)_{2(s+1)} \otimes (\mathbf{s} \text{ of } SU(2)'_F). \tag{4.222}$$

For every s, we have $2s + 1$ superconformal multiplets, which are transformed as the spin s representation of the flavor group $SU(2)'_F$. This fact suggests that the full Kaluza-Klein spectrum can be obtained by KK analysis of only one field, for example, a scalar field ϕ_i and the knowledge of representation theory. Let us explain this below. Three scalar fields including it form an $SU(2)_R$ triplet, so we identify these modes with the third line in the table. Kaluza-Klein analysis tells us the conformal dimension of the scalar field. By comparison of the conformal dimension and the $SU(2)'_R$ representations, we can relate the quantum number n to the orbital spin s by

$$n = 2s + 2, \quad s = 0, \frac{1}{2}, \cdots . \tag{4.223}$$

Using Table D.4 given by representation theory, we can reproduce the full spectrum $V_{\widetilde{\mathcal{S}_U}}$ as (4.222).

In the same way, we can obtain the spectrum $V_{\widetilde{\mathcal{S}_T}}$ as

$$V_{\widetilde{\mathcal{S}_T}} = \bigoplus_{s=0,\frac{1}{2},\dots} (4, B, -)_{2(s+1)} \otimes (\mathbf{s} \text{ of } SU(2)'_F). \tag{4.224}$$

4.3.4 Twisted sector

Let us derive the character for the Kaluza-Klein modes (4.222) by summing up $\chi_{(4,B,+)_n}$ with an appropriate weight. To include the information of the flavor group $SU(2)'_F$, we introduce a variable z' for the Cartan generator P' of $SU(2)'_F$. We normalize P' in a different way from T_3 and T'_3 so that its eigenvalues are integers. We define a character

$$\chi_V = \mathrm{Tr}_V\left(s^{2D} x^{2j} y^{T_3} y'^{T'_3} z^P z'^{P'}\right), \tag{4.225}$$

for a representation V. From (4.222), we derive the character of a vector-multiplet in $\widetilde{\mathcal{S}}$ as follows:

$$\chi_{V_{\widetilde{\mathcal{S}}}} = \sum_{s=0}^{\infty} \chi_{(4,B,+)_{2s+2}} \chi_s(z'^2). \tag{4.226}$$

Remark that the component fields in a vector-multiplet in $\mathcal{S}_U$ do not carry the charge P. Thus, the index should be the function only of x and z', and is independent of z. By using the relation between a character and an index (D.12), (4.226) leads to the index for a vector-multiplet in $\widetilde{\mathcal{S}}$

$$\begin{aligned} I_{V_{\widetilde{\mathcal{S}}}} &= \sum_{s=0}^{\infty} I_{(4,B,+)_{2s+2}} \chi_s(z'^2) \\ &= \sum_{P'\in\mathbf{Z}} \frac{x^{|P'|+2}}{1-x^4} \left[\frac{(Y_2 - x^2) Y_2^{|P'|}}{1 - x^2 Y_2}\right] z'^{P'}. \end{aligned} \tag{4.227}$$

In the same way, we can obtain the index for a vector-multiplet in $\widetilde{\mathcal{S}}'$

$$\begin{aligned} I_{V_{\widetilde{\mathcal{S}}'}} &= \sum_{s=0}^{\infty} I_{(4,B,-)_{2s+2}} \chi_s(z^2) \\ &= \sum_{P\in\mathbf{Z}} \frac{x^{|P|+2}}{1-x^4} \left[\frac{(Y_2^{-1} - x^2) Y_2^{-|P'|}}{1 - x^2 Y_2^{-1}}\right] z^{P}. \end{aligned} \tag{4.228}$$

The character and index for 1/2 BPS representations $(4, B, \pm)_n$ are both derived in Appendix D.

As noticed at the beginning of this section, we compare this index to that obtained in [104]. In [47] it is shown that operators without magnetic charges correspond to Kaluza-Klein modes of the Cartan part of the vector-multiplets with $P = P' = 0$. Therefore, the perturbative part is obtained by picking up the term independent of z', z from (4.227), (4.228) as

$$I_{V_{\tilde{S}}}|_{z'=0} = \frac{x^2}{1-x^4}\left[\frac{(Y_2 - x^2)}{1-x^2Y_2}\right], \quad I_{V_{\tilde{S}'}}|_{z=0} = \frac{x^2}{1-x^4}\left[\frac{(Y_2^{-1} - x^2)}{1-x^2Y_2^{-1}}\right]. \qquad (4.229)$$

Because every Cartan $U(1)$ gives the same contributions, we need to multiply the total rank of the gauge group of the vector-multiplets. In the theory discussed in [104], $SU(M)$ gauge theory are realized on each of the two singular loci, and we need to introduce the additional factor $(M-1)$, respectively. This result, with the rank factor included, precisely agrees with (4.9). (We need the replacement $x^2 \to x$, $Y_2 \to y_3$ to match the conventions.)

Let us get back to our situation. In the gauge theory side, the chemical potential for the charge H_2 is not introduced. Therefore, it is sufficient to set Y_2 in (4.227) to 1 for comparison. Then, the single-particle index for a single $U(1)$ vector-multiplet in $AdS_4 \times \mathbf{S}^3$ reduces to

$$I_{V_{\tilde{S}}}|_{Y_2=1} = \sum_{P'=-\infty}^{\infty} \frac{x^{|P'|+2}}{1-x^4} z'^{P'}, \qquad (4.230)$$

For later convenience, we denote the coefficient of z' expansion as $I^{\rm vec}_{P'}(x)$:

$$I^{\rm vec}_{P'}(x) = \frac{x^{|P'|+2}}{1-x^4}. \qquad (4.231)$$

When we consider the single-particle index of the covering space of the other locus $\mathcal{S}_T$, we should replace the variable z' by z.

Once we derived the single-particle index for $AdS_4 \times \mathbf{S}^3$, the index for orbifold $AdS_4 \times \mathbf{S}^3/\mathbf{Z}_{kq}$ can be obtained by the projection which leaves only $\mathbf{Z}_{kq}$ invariant modes. The procedure of the $\mathbf{Z}_{kq}$ and $\mathbf{Z}_{kp}$ projections is similar to what we have done for the bulk sector. An important difference, however, is that in general there exist non-trivial Wilson lines turned on the orbifolded three-sphere, in which case the charge quatization conditions are shifted compared to otherwise. This is nothing but the Aharanov-Bohm effect. Topologies of the loci $\mathcal{S}_U$ and $\mathcal{S}_T$ are $S^3/\mathbf{Z}_{kq}$ and $S^3/\mathbf{Z}_{kp}$, respectively. Both the orbifold groups are generated by the third generator in (2.52). Associated with the fundamental groups $\pi_1(\mathcal{S}_U) = \mathbf{Z}_{kq}$ and $\pi_1(\mathcal{S}_T) = \mathbf{Z}_{kp}$, we have in general non-trivial monodromy matrix

$$\mathrm{diag}(e^{2\pi i\eta_1}, \ldots, e^{2\pi i\eta_p}) \in U(p), \quad \mathrm{diag}(e^{2\pi i\eta'_1}, \ldots, e^{2\pi i\eta'_q}) \in U(q). \qquad (4.232)$$

Each diagonal component of the monodromy matrix corresponds to the Wilson line turned on each of singular loci $\mathcal{S}_{Ui}$ or $\mathcal{S}_{Ti}$. Note that these are not elements of

$SU(p)$ and $SU(q)$ because we do not impose the condition that their determinants are one. This does not cause any problem because there are no particles coupling to the $U(1)$ part. η_i and $\eta_{i'}$ must be quantized by

$$\eta_i \in \frac{1}{kq}\mathbf{Z}, \quad \eta_{i'} \in \frac{1}{kp}\mathbf{Z}. \tag{4.233}$$

When we compute the contribution of twisted sectors to a multi-particle index, we should take account of the momentum shift due to these discrete Wilson lines.

Before considering the projection for single-particle states, let us consider that for a general multi-particle state. Let ρ_i and $\rho_{i'}$ be the $H_{\mathcal{S}_U}$ and $H_{\mathcal{S}_T}$ charges of the multi-particle state. They are the sum of charges of constituent particles in the state. Because every particle belongs to the adjoint representation of $G_{\mathcal{S}}$, these charges satisfy

$$\sum_{i=1}^{p} \rho_i = \sum_{i'=1}^{q} \rho_{i'} = 0. \tag{4.234}$$

When we act an element of the orbifold group which rotates the cycles in $\mathcal{S}_U$ and $\mathcal{S}_T$ by r and s times, respectively, the state picks up the phase

$$2\pi i \left(r \sum_{i=1}^{p} \rho_i \eta_i + s \sum_{i'=1}^{q} \rho_{i'} \eta_{i'} \right), \tag{4.235}$$

and this must be canceled by the phase factor associated with the momentum.

Because $(r, s) = (0, -k)$, $(k, 0)$, and $(1, 1)$ for the three generators in (2.52), the cancellation of the phases requires

$$\exp\left(\frac{2\pi i}{p} P\right) = \exp\left(2\pi i k \sum_{i'=1}^{q} \rho_{i'} \eta_{i'}\right), \tag{4.236}$$

$$\exp\left(\frac{2\pi i}{q} P'\right) = \exp\left(-2\pi i k \sum_{i=1}^{p} \rho_i \eta_i\right), \tag{4.237}$$

$$\exp(2\pi i P_M) = \exp\left(-2\pi i \sum_{i=1}^{p} \rho_i \eta_i - 2\pi i \sum_{i'=1}^{q} \rho_{i'} \eta_{i'}\right). \tag{4.238}$$

P and P' satisfying these conditions are given by

$$P = p\left(a + k \sum_{i'=1}^{q} \rho'_{\alpha i'} \eta'_{i'}\right), \quad P' = q\left(a + kb - k \sum_{i=1}^{p} \rho_{\alpha i} \eta_i\right), \quad a, b \in \mathbf{Z}. \tag{4.239}$$

These are the selection rules in the gravity side. We generalize the result obtained for $U(1)^r$ case in [80] to non-abelian case. The M-momentum P_M is

$$P_M = b - \sum_{i=1}^{p} \rho_{\alpha i} \eta_i - \sum_{i'=1}^{q} \rho'_{\alpha i'} \eta'_{i'}, \quad b \in \mathbf{Z}. \tag{4.240}$$

Unlike the case of bulk sector, the M-momentum P_M is not always an integer. These conditions are imposed on any multi-particle states, including single-particle states. Actually, we obtain the momenta (4.164) for bulk single-particle states by simply setting $\rho_i = \rho_{i'} = 0$ in (4.239).

For a single-particle state in the twisted sector on the locus $\mathcal{S}_U$, $\rho_{i'} = P = 0$. This implies that a in the first equation in (4.239) vanishes, and the second equation gives the momentum

$$P' = kq\left(b - \sum_{i=1}^{p}\rho_{\alpha i}\eta_i\right), \quad b \in \mathbf{Z}. \tag{4.241}$$

Let $\{\rho_{\alpha i}\} = \vec{\rho}_\alpha$ be the charge vector for an $SU(p)$ vector-multiplet living in the locus $\mathcal{S}_U$. $\alpha = 1, \ldots, p^2 - 1$ is the adjoint index of $SU(p)$. These vectors are nothing but the weight vectors for the adjoint representation of $SU(p)$. The single-particle index for vector-multiplets in the locus $\mathcal{S}_U$ is

$$\begin{aligned} I_{V_\mathcal{S}} &= \sum_{\alpha=1}^{p^2-1}\sum_{b=-\infty}^{\infty} I^{\rm vec}_{kq(b-\vec{\rho}_\alpha\cdot\vec{\eta})}(x) z'^{kq(b-\vec{\rho}_\alpha\cdot\vec{\eta})} t_1^{\rho_{\alpha 1}}\cdots t_p^{\rho_{\alpha p}} \\ &= \sum_{\vec{\rho}} \deg(\vec{\rho}) \sum_{b=-\infty}^{\infty} I^{\rm vec}_{kq(b-\vec{\rho}\cdot\vec{\eta})}(x) z'^{kq(b-\vec{\rho}\cdot\vec{\eta})} t_1^{\rho_1}\cdots t_p^{\rho_p}, \end{aligned} \tag{4.242}$$

which is nothing but $I^{\mathcal{S}_U}(x, z'; \vec{t})$.

The single-particle index for the $SU(q)$ vector-multiplet localized in the locus $\mathcal{S}_T = AdS_4 \times \mathbf{S}^3/\mathbf{Z}_{kp}$ is obtained in the same way. Because $\rho_i = P' = 0$, the projection restricts the value of the momentum P as

$$P = kp\left(-b + \sum_{i'=1}^{q}\rho'_{\alpha i'}\eta'_{i'}\right), \quad b \in \mathbf{Z}. \tag{4.243}$$

The single-particle index for the $SU(q)$ vector-multiplet in $\mathcal{S}_T$ is given by

$$\begin{aligned} I_{V_{\mathcal{S}'}} &= \sum_{\alpha=1}^{q^2-1}\sum_{a=-\infty}^{\infty} I^{\rm vec}_{kp(a+\vec{\rho}'_\alpha\cdot\vec{\eta})}(x) z^{kq(p+\vec{\rho}'_\alpha\cdot\eta'_\alpha)} t_1'^{\rho'_{\alpha 1}}\cdots t_q'^{\rho'_{\alpha q}} \\ &= \sum_{\vec{\rho}'} \deg(\vec{\rho}') \sum_{a=-\infty}^{\infty} I^{\rm vec}_{kp(a+\vec{\rho}'\cdot\vec{\eta})}(x) z^{kq(p+\vec{\rho}'\cdot\eta'_\alpha)} t_1'^{\rho'_1}\cdots t_q'^{\rho'_q}, \end{aligned} \tag{4.244}$$

which is nothing other than $I^{\mathcal{S}_T}(x, z; \vec{t}')$. Note that due to the constraint (4.234) these indices are invariant under the overall rescaling of $\vec{t}$ and $\vec{t}'$;

$$I^{\mathcal{S}_U}(x, z', c\vec{t}) = I^{\mathcal{S}_U}(x, z', \vec{t}), \quad I^{\mathcal{S}_T}(x, z, c\vec{t}') = I^{\mathcal{S}_T}(x, z, \vec{t}'). \tag{4.245}$$

By summing up the contribution of the bulk and the twisted sectors, we obtain the total single-particle index as (4.147) given at the beginning of this subsection.

4.4 Identification and comparison

In this subsection, we compare the indices of both sides obtained in the previous sections. To do this, we first establish the correspondence between both data in gauge theory side and gravity side. The correspondence is relatively easy to guess by comparing (4.133), (4.134) and (4.239) as follows:

$$d = a, \quad b = m_\bullet, \tag{4.246}$$

$$l_i = kq\eta_i, \quad l_{i'} = kp\eta_{i'}, \tag{4.247}$$

$$\mu_i = \rho_i, \quad \mu_{i'} = \rho_{i'}. \tag{4.248}$$

By using the correspondence (4.246) and (4.240), we can write the M-momentum as

$$P_M + \sum_{i=1}^{p} \rho_{\alpha i}\eta_i + \sum_{i'=1}^{q} \rho'_{\alpha i'}\eta'_{i'} = m_\bullet. \tag{4.249}$$

Since $m_\bullet$ is identical to the diagonal magnetic charge, the relation (4.249) generalizes the correspondence between the M-momentum and the diagonal magnetic charge to the form including the contribution coming from the singularities. The other relations (4.247), (4.248) are also important meanings in AdS/CFT correspondence (§4.4.1), (§4.4.2). By using these relations, we confirm the complete matching of the gauge theory index and the multi-particle index.

In the previous subsection we showed that the gauge theory index is factorized into three parts: neutral, positive, and negative parts. For the two indices to coincides, the multi-particle index on the gravity side should be also factorized in the same way into three parts and it is (§4.4.3). Using the result, we confirm the agreement for each factor. We show the agreement for the neutral part analytically (§4.4.4). Concerning the charged part, we use computers to compute the gauge theory index for many sectors with different charges, and we show that the gauge theory index $I^{(+)}_{\{m_a\}}$ for monopole charges $\{m_a\}$ agrees with the multi-particle index $I^{\mathrm{mp}(+)}_{(P_M,\vec{\rho},\vec{\rho}')}$ (§4.4.5). According to the correspondence (4.248), we set the variables τ_a by using the corresponding variables $\vec{t}, \vec{t}'$ as follows.

$$\prod_{a=1}^{r} \tau_a^{m_a} = \prod_{i=1}^{p} t_i^{\rho_i} \prod_{i'=1}^{q} t_{i'}^{\rho_{i'}} \tag{4.250}$$

4.4.1 Wrapping numbers and magnetic charges

We discuss the relation (4.248). In §2.4, we argue that μ_I are the relative magnetic charges and describe the contribution coming from twisted monopole operators.

On the other hand, ρ_I describe the gauge charges of a vector-multiplet localized in a singular locus. So, (4.248) implies that the gauge charges correspond to the relative magnetic charges.

In the perspective of M-thoery, such gauge charges come from the period integrals on 2-cycles in the internal space and thus correspond to the wrapping numbers of M2-branes on them. To see this, let us consider the world-volume action of a wrapped M2-brane. It couples 3-form field potential minimally, which can be expanded by the harmonic 2-forms as the cohomology basis of $H^2_{\text{free}}(\mathbf{S}^7_{p,q,k},\mathbf{Z}) = \mathbf{Z}^{p+q-2}$, denoted by ω_I, as

$$C_3 = \sum_{I=1}^{p+q-2} \omega_I \wedge \widetilde{A}_I. \tag{4.251}$$

Integrating out 2-cycle in the world-volume, the minimal coupling term reduces to

$$\sum_{I=1}^{p+q-2} \rho_I \int \widetilde{A}_I. \tag{4.252}$$

Here ρ_I is given by the following period, which is interpreted as the wrapping number of the M2-brane on the I-th 2-cycle:

$$\rho_I = \oint \omega_I. \tag{4.253}$$

This period plays a role of the gauge charge with respect to $\widetilde{A}_I$, This argument suggests that the magnetic charges of twisted monopole operators correspond to the wrapping numbers of M2-branes on 2-cycles.

4.4.2 Wilson lines and linking numbers

Let us consider the relation (4.247). Interestingly, this relation suggests that the torsion wilson lines turned on the singular loci correspond to the linking numbers in the field theory side.

This correspondence reproduces the result in [57] as a special case with $k = 1$. Let us follow his discussion in [57]. He considered the type IIB brane system described by Table 2.3 in replacement of $(1,k)$5-branes by D5-branes. By taking T-dual to 6-direction, p NS5-branes are converted into p centered Taub-NUT space, denoted by TN_p, D3-branes become D2-branes, which are dissolved into instantons in TN_p, and D5-branes are turned into D6-branes filling in TN_p with a line-bundle V, which has information of the location of 5-branes. In fact, the 1st Chern class of this line bundle V agrees the linking numbers of NS5-branes.

Furthermore, he showed that by taking the ALE limit of TN_p, monodromy at infinity of V (4.232) agrees the exponent of the linking numbers. By this argument, the relation between the discrete Wilson phase induced on the ALE locus and the linking numbers of D5-branes is established. By taking S-dual for this setup and exchanging NS5-branes and D5-branes, this argument can be also applied to the linking numbers of NS5-branes

Our result (4.247) is a generalization of the result to type IIB setup including $(1,k)$5-branes. We define the linking number by inserting k so that these linking numbers are invariant in the brane creation processes [61]. We establish the relation between the discrete Wilson lines and the linking numbers for general k by requiring the agreement of indices in both sides.

4.4.3 Factorization

We show that the multi-particle index can be factorized into three parts. Namely, I^{mp} should be factorized as

$$I^{\mathrm{mp}} = I^{\mathrm{mp}(0)} I^{\mathrm{mp}(+)} I^{\mathrm{mp}(-)}. \tag{4.254}$$

Let us first confirm this factorization.

The factorization of the multi-particle index is equivalent to the following decomposition of the single-particle index

$$I^{\mathrm{sp}} = I^{\mathrm{sp}(0)} + I^{\mathrm{sp}(+)} + I^{\mathrm{sp}(-)}. \tag{4.255}$$

Let us consider a single-particle state with quantum numbers $(P_M, \vec{\rho}, \vec{\rho}')$. By the relations in (4.246), (4.248) we can determine the corresponding magnetic charges m_a. The decomposability (4.255) claims that the magnetic charges m_a determined in this way for every single-particle state do not include positive and negative components at the same time. This is confirmed easily as follows.

For a bulk graviton state, which has vanishing vectors $\vec{\rho} = \vec{\rho}' = 0$, all the components of the corresponding magnetic charge are the same and are given by

$$m_1 = \cdots = m_r = P_M, \tag{4.256}$$

and thus they never include both positive and negative charges. This is also the case for the Cartan part of the twisted sectors.

For an H_S-charged particle in a twisted sector, one of $\vec{\rho}$ and $\vec{\rho}'$ is non-vanishing. If the particle corresponds to an $SU(p)$ root vector, ρ_i has two non-vanishing components, and one of them is $+1$ and the other is -1. In this case the second relation in (4.248) means that the minimum and the maximum components of the magnetic charges m_a differ by only one. Therefore, the r magnetic charges cannot include both positive and negative charges.

We can always classify single-particle states into neutral, positive, and negative parts according to the magnetic charges, and correspondingly, we can decompose the single-particle index into the three parts as (4.255).

4.4.4 Neutral part

We prove the agreement of indices for the neutral part analytically. The neutral part of the multi-particle index, $I^{\mathrm{mp}(0)}$, is given by

$$I^{\mathrm{mp}(0)}(x,z,z') = \exp\sum_{n=1}^{\infty}\frac{1}{n}I^{\mathrm{sp}}_{(0,\vec{0},\vec{0})}(x^n,z^n,z'^n) \tag{4.257}$$

where $I^{\mathrm{sp}}_{(P_M,\vec{\rho},\vec{\rho}')}$ is given by (4.149) and its explicit form for $(P_M,\vec{\rho},\vec{\rho}') = (0,\vec{0},\vec{0})$ is

$$\begin{aligned} I^{\mathrm{sp}}_{(0,\vec{0},\vec{0})}(x,z,z') &= \sum_{a=-\infty}^{\infty} I^{\mathrm{grav}}_{pa,qa}(x)z^{pa}z'^{qa} + (p-1)I^{\mathrm{vec}}_0(x) + (q-1)I^{\mathrm{vec}}_0(x) \\ &= \frac{x^{p+q}z^p z'^q}{1-x^{p+q}z^p z'^q} + \frac{x^{p+q}z^{-p}z'^{-q}}{1-x^{p+q}z^{-p}z'^{-q}} + (p+q)\frac{x^2}{1-x^4}. \end{aligned} \tag{4.258}$$

The corresponding multi-particle index defined by (4.257) is

$$I^{\mathrm{mp}(0)} = \prod_{i=1}^{\infty}\frac{(1+x^2)^{i(p+q)}}{(1-(x^{p+q}z^p z'^q)^i)(1-(x^{p+q}z^{-p}z'^{-q})^i)}, \tag{4.259}$$

where we used Euler's partition identity to obtain this expression.

On the gauge theory side, the corresponding index (4.111) is

$$I^{(0)}(x,z,z') = \prod_{n=1}^{\infty}\frac{1}{\det M(x^n,z^n,z'^n)}, \tag{4.260}$$

where M is the matrix defined in (4.109). We can easily compute the determinant by rewriting the matrix M as

$$M = \frac{1}{1+x^2}(1-xA)(1-xA^{-1}) \tag{4.261}$$

with the matrix

$$A(z,z') = \begin{pmatrix} \ddots & & & & & \\ & 0 & z_{I-1} & & & \\ & & 0 & z_I & & \\ & & & 0 & z_{I+1} & \\ & & & & 0 & \\ z_r & & & & & \ddots \end{pmatrix}. \tag{4.262}$$

The determinant

$$\frac{1}{\det M} = \frac{(1+x^2)^{p+q}}{(1-x^{p+q}z^p z'^q)(1-x^{p+q}z^{-p}z'^{-q})} \tag{4.263}$$

does not depend on the order of the untwisted and twisted hyper-multiplets in the quiver diagram. On substituting this into (4.260), we see that the neutral part of the gauge theory index actually coincides with the corresponding part of the graviton index;

$$I^{(0)}(x,z,z') = I^{\text{mp}(0)}(x,z,z'). \tag{4.264}$$

This result is consistent with the result in [104]. If we set $z = z' = 1$ and $p = q = M$, we reproduce the index (4.6) with $y_3 = 1$ substituted.

4.4.5 Charged part

Next, let us confirm the agreement of the charged part:

$$I^{(\pm)}(x,z,z') = I^{\text{mp}(\pm)}(x,z,z'). \tag{4.265}$$

We can easily show the following relations between positive and the negative parts:

$$I^{(+)}(x,z,z') = I^{(-)}(x,z^{-1},z'^{-1}), \quad I^{\text{mp}(+)}(x,z,z') = I^{\text{mp}(-)}(x,z^{-1},z'^{-1}). \tag{4.266}$$

Therefore, it is enough to show the relation for the positive part of the indices:

$$I^{(+)}_{\{m_a\}_+}(x,z,z') = I^{\text{mp}(+)}_{(P_M,\vec{\rho},\vec{\rho}')}(x,z,z'). \tag{4.267}$$

Unfortunately, we have not succeeded in proving (4.267) analytically. In the following, we consider three examples of $\mathcal{N}=4$ Chern-Simons theories specified by $\{s_I\} = \{0,0,1\}$, $\{0,0,1,1\}$, and $\{0,1,0,1\}$ [4]. (The simplest case with $\{s_I\} = \{0,1\}$ (ABJM model) has already been investigated in [105].) For each theory we compute $I^{(+)}_{\{m_a\}}$ numerically for many sectors specified by the charges, and confirm the agreement with $I^{\text{mp}(+)}_{(P_M,\vec{\rho},\vec{\rho}')}$.

First, we consider the theory defined by

$$\{s_I\} = \{0,0,1\}. \tag{4.268}$$

[4]When we describe a set of numbers x_a assigned to vertices in the quiver diagram, we choose a reference vertex $a = \bullet$, which is also used for the definition of the linking numbers, and represent $\{x_a\}$ as the vector $\{x_\bullet, x_{R^2(\bullet)}, \ldots, x_{L^2(\bullet)}\}$, where $R^2(\bullet) \equiv R(R(\bullet))$, and $L^n(\bullet)$ and $R^n(\bullet)$ are similarly defined. For a set of numbers y_I assigned to edges, we represent them as $\{y_{R(\bullet)}, y_{R^3(\bullet)}, \ldots, y_{L(\bullet)}\}$.

The background geometry of this theory is $(\mathbf{C}^2/\mathbf{Z}_2\times\mathbf{C}^2)/\mathbf{Z}_k$. The internal space $\mathbf{S}^7_{p,q,k}$ includes a $\mathbf{Z}_2$-fixed singular locus, and there exists one two-cycle at the locus $\mathcal{S}_U$. The vectors $\vec{\rho}=\{\rho_i\}$ and $\vec{\rho}'=\{\rho_{i'}\}$ are parameterized by a single winding number $\rho\in\mathbf{Z}$ as

$$\vec{\rho}=\{\rho_1,\rho_2\}=\{-\rho,\rho\},\quad \vec{\rho}'=\{\rho_3\}=\{0\}. \tag{4.269}$$

We introduce chemical potential t for the charge ρ. This is related to the potentials t_I introduced in §4.3 by $t=t_2/t_1$. By using the identidication, the magnetic charges are determined as

$$\{m_1,m_2,m_3\}=\{P_M,P_M+\rho,P_M\}. \tag{4.270}$$

The Wilson lines η_I vanish up to integers, and this is consistent with the fact that there is no three-cycles in the dual geometry. The quantization rules (4.133) and (4.134) for the charges P and P' are

$$P=2a,\quad P'=a+kP_M,\quad a,P_M\in\mathbf{Z}. \tag{4.271}$$

The positive part of the single-particle index is defined by $m_a\geq 0$ and $\{m_1,m_2,m_3\}\neq\{0,0,0\}$. These conditions mean

$$P_M\geq 0,\quad P_M+\rho\geq 0,\quad (P_M,\rho)\neq(0,0). \tag{4.272}$$

For every pair of charges (P_M,ρ) satisfying (4.272) we would like to confirm

$$I^{(+)}_{\{P_M,P_M+\rho,P_M\}}(x,z,z')=I^{\mathrm{mp}(+)}_{(P_M,\rho)}(x,z,z'). \tag{4.273}$$

Single-particle states exist only for $|\rho|\leq 1$. Eq. (4.149) gives

$$I^{\mathrm{sp}}_{(P_M,0)} = \sum_{a=-\infty}^{\infty} I^{\mathrm{grav}}_{2a,a+kP_M}(x)z^{2a}z'^{kP_M+a}+I^{\mathrm{vec}}_{kP_M}(x)z'^{kP_M}, \tag{4.274}$$

$$I^{\mathrm{sp}}_{(P_M,\pm1)} = I^{\mathrm{vec}}_{kP_M}(x)z'^{kP_M}. \tag{4.275}$$

It is relatively easy to compute indices when one of two bounds in (4.272) is saturated. Let us first consider $P_M=0$ case. In this case, we should confirm

$$I^{(+)}_{\{0,\rho,0\}}(x,z,z')=I^{\mathrm{mp}(+)}_{(0,\rho)}(x,z,z'). \tag{4.276}$$

Because the single-particle index depends on the level k only through the combination P_Mk, the multi-particle index on the right hand side in (4.276) is independent of k. We can easily see that this is also the case for the gauge theory index on the left hand side in (4.276) from the expression (4.112).

The only non-vanishing single-particle index for $P_M = 0$ contributing to $I^{\mathrm{mp}(+)}$ is

$$I^{\mathrm{sp}}_{(0,1)} = \frac{x^2}{1-x^4}, \tag{4.277}$$

and the multi-particle index with $P_M = 0$ is defined by

$$\sum_{\rho=0}^{\infty} I^{\mathrm{mp}(+)}_{(0,\rho)}(x,z,z')t^\rho = \exp\sum_{n=1}^{\infty}\frac{1}{n}I^{\mathrm{sp}}_{(0,1)}(x^n,z^n,z'^n)t^n. \tag{4.278}$$

By using the identity

$$\prod_{i=0}^{\infty}\frac{1}{1-tx^i} = \sum_{i=0}^{\infty}t^i\prod_{j=1}^{i}\frac{1}{1-x^j}, \tag{4.279}$$

we obtain

$$I^{\mathrm{mp}(+)}_{(0,\rho)} = \prod_{i=1}^{\rho}\frac{x^2}{1-x^{4i}}. \tag{4.280}$$

Let us confirm that the gauge theory index agrees with this for small ρ. For $\rho = 1$, we can easily compute the corresponding gauge theory index by hand, and confirm the agreement.

$$I^{(+)}_{\{0,1,0\}} = I^{(+)}_{(\cdot,\ydiagram{1},\cdot)} = \frac{x^2}{1-x^4}. \tag{4.281}$$

For $\rho = 2$, there are two contribution with different monopole backgrounds.

$$I^{(+)}_{\{0,2,0\}} = I^{(+)}_{\{\cdot,\ydiagram{2},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{1,1},\cdot\}}. \tag{4.282}$$

It is again easy to compute these two contributions by hand. They are

$$I^{(+)}_{\{\cdot,\ydiagram{2},\cdot\}} = \frac{x^4}{1-x^8}, \quad I^{(+)}_{\{\cdot,\ydiagram{1,1},\cdot\}} = \frac{x^8}{(1-x^4)(1-x^8)}, \tag{4.283}$$

and the summation agrees with the multi-particle index

$$I^{(+)}_{\{0,2,0\}} = \frac{x^2}{1-x^4}\frac{x^2}{1-x^8} = I^{\mathrm{mp}(+)}_{(0,2)}. \tag{4.284}$$

As the charge becomes large, the computation of the gauge theory index becomes complicated rapidly. For $\rho \geq 3$, we use computers to generate gauge theory index as series expansion with respect to the variable x, and check the agreement for small ρ up to certain order of x. The result is as follows.

$$\begin{aligned}
I^{(+)}_{\{0,3,0\}} &= I^{(+)}_{\{\cdot,\ydiagram{3},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{2,1},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{1,1,1},\cdot\}} \\
&= I^{\mathrm{mp}(+)}_{(0,3)} + \mathcal{O}(x^{101}), && (4.285) \\
I^{(+)}_{\{0,4,0\}} &= I^{(+)}_{\{\cdot,\ydiagram{4},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{3,1},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{2,2},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{2,1,1},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{1,1,1,1},\cdot\}} \\
&= I^{\mathrm{mp}(+)}_{(0,4)} + \mathcal{O}(x^{101}), && (4.286) \\
I^{(+)}_{\{0,5,0\}} &= I^{(+)}_{\{\cdot,\ydiagram{5},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{4,1},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{3,2},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{3,1,1},\cdot\}} \\
&\quad + I^{(+)}_{\{\cdot,\ydiagram{2,2,1},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{2,1,1,1},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{1,1,1,1,1},\cdot\}} \\
&= I^{\mathrm{mp}(+)}_{(0,5)} + \mathcal{O}(x^{31}). && (4.287)
\end{aligned}$$

All these results are consistent with (4.280) up to the order we have computed.

Next let us consider the case with $P_M \geq 1$ and $P_M + \rho = 0$. The relation we would like to confirm is

$$I^{(+)}_{\{P_M,0,P_M\}}(x,z,z') = I^{\mathrm{mp}(+)}_{(P_M,-P_M)}(x,z,z'). \tag{4.288}$$

The single-particle index contributing to this part is

$$I^{\mathrm{sp}}_{(1,-1)} = \frac{x^2(xz')^k}{1-x^4}. \tag{4.289}$$

With the help of the identity (4.279) we obtain

$$I^{\mathrm{mp}(+)}_{(P_M,-P_M)} = \prod_{i=1}^{P_M} \frac{x^{k+2} z'^k}{1-x^{4i}}. \tag{4.290}$$

We have confirmed the following relations up to the indicated order of x for $k = 1,2,3,4,5$.

$$\begin{aligned}
I^{(+)}_{\{1,0,1\}} &= I^{(+)}_{\{\ydiagram{1},\cdot,\ydiagram{1}\}} \\
&= I^{\mathrm{mp}(+)}_{(1,-1)} + \mathcal{O}(x^{101}),
\end{aligned} \tag{4.291}$$

$$\begin{aligned}
I^{(+)}_{\{2,0,2\}} &= I^{(+)}_{\{\ydiagram{2},\cdot,\ydiagram{2}\}} + I^{(+)}_{\{\ydiagram{1,1},\cdot,\ydiagram{2}\}} + I^{(+)}_{\{\ydiagram{2},\cdot,\ydiagram{1,1}\}} + I^{(+)}_{\{\ydiagram{1,1},\cdot,\ydiagram{1,1}\}} \\
&= I^{\mathrm{mp}(+)}_{(2,-2)} + \mathcal{O}(x^{31}),
\end{aligned} \tag{4.292}$$

$$\begin{aligned}
I^{(+)}_{\{3,0,3\}} &= I^{(+)}_{\{\ydiagram{3},\cdot,\ydiagram{3}\}} + I^{(+)}_{\{\ydiagram{2,1},\cdot,\ydiagram{3}\}} + I^{(+)}_{\{\ydiagram{1,1,1},\cdot,\ydiagram{3}\}} + I^{(+)}_{\{\ydiagram{3},\cdot,\ydiagram{2,1}\}} + I^{(+)}_{\{\ydiagram{2,1},\cdot,\ydiagram{2,1}\}} \\
&\quad + I^{(+)}_{\{\ydiagram{1,1,1},\cdot,\ydiagram{2,1}\}} + I^{(+)}_{\{\ydiagram{3},\cdot,\ydiagram{1,1,1}\}} + I^{(+)}_{\{\ydiagram{2,1},\cdot,\ydiagram{1,1,1}\}} + I^{(+)}_{\{\ydiagram{1,1,1},\cdot,\ydiagram{1,1,1}\}} \\
&= I^{\mathrm{mp}(+)}_{(3,-3)} + \mathcal{O}(x^{11}).
\end{aligned} \tag{4.293}$$

All these results are consistent with (4.290).

Finally, let us consider a few examples in which all magnetic charges are positive. For $k = 1,2,3,4,5$ we have checked

$$\begin{aligned}
I^{(+)}_{\{1,1,1\}} &= I^{(+)}_{\{\ydiagram{1},\ydiagram{1},\ydiagram{1}\}} \\
&= I^{\mathrm{mp}(+)}_{(1,0)} + \mathcal{O}(x^{31}),
\end{aligned} \tag{4.294}$$

$$\begin{aligned}
I^{(+)}_{\{2,1,2\}} &= I^{(+)}_{\{\ydiagram{2},\ydiagram{1},\ydiagram{2}\}} + I^{(+)}_{\{\ydiagram{1,1},\ydiagram{1},\ydiagram{2}\}} + I^{(+)}_{\{\ydiagram{2},\ydiagram{1},\ydiagram{1,1}\}} + I^{(+)}_{\{\ydiagram{1,1},\ydiagram{1},\ydiagram{1,1}\}} \\
&= I^{\mathrm{mp}(+)}_{(2,-1)} + \mathcal{O}(x^{21}),
\end{aligned} \tag{4.295}$$

$$\begin{aligned}
I^{(+)}_{\{2,2,2\}} &= I^{(+)}_{\{\ydiagram{2},\ydiagram{2},\ydiagram{2}\}} + I^{(+)}_{\{\ydiagram{2},\ydiagram{2},\ydiagram{1,1}\}} + I^{(+)}_{\{\ydiagram{2},\ydiagram{1,1},\ydiagram{2}\}} + I^{(+)}_{\{\ydiagram{2},\ydiagram{1,1},\ydiagram{1,1}\}} \\
&\quad + I^{(+)}_{\{\ydiagram{1,1},\ydiagram{2},\ydiagram{2}\}} + I^{(+)}_{\{\ydiagram{1,1},\ydiagram{2},\ydiagram{1,1}\}} + I^{(+)}_{\{\ydiagram{1,1},\ydiagram{1,1},\ydiagram{2}\}} + I^{(+)}_{\{\ydiagram{1,1},\ydiagram{1,1},\ydiagram{1,1}\}} \\
&= I^{\mathrm{mp}(+)}_{(2,0)} + \mathcal{O}(x^{11}).
\end{aligned} \tag{4.296}$$

where

$$\begin{aligned}
I^{\mathrm{mp}(+)}_{(1,0)} &= I^{\mathrm{sp}}_{(1,0)} + I^{\mathrm{sp}}_{(0,1)} I^{\mathrm{sp}}_{(1,-1)}, &(4.297)\\
I^{\mathrm{mp}(+)}_{(2,-1)} &= I^{\mathrm{sp}}_{(2,-1)} + I^{\mathrm{sp}}_{(1,0)} I^{\mathrm{sp}}_{(1,-1)} + I^{\mathrm{sp}}_{(0,1)} \left(\frac{1}{2}(I^{\mathrm{sp}}_{(1,-1)})^2 + \frac{1}{2} I^{\mathrm{sp}}_{(1,-1)}(\cdot^2)\right), &(4.298)\\
I^{\mathrm{mp}(+)}_{(2,0)} &= I^{\mathrm{sp}}_{(2,0)} + I^{\mathrm{sp}}_{(2,-1)} I^{\mathrm{sp}}_{(0,1)} + I^{\mathrm{sp}}_{(1,1)} I^{\mathrm{sp}}_{(1,-1)}\\
&\quad + \left(\frac{1}{2}(I^{\mathrm{sp}}_{(1,-1)})^2 + \frac{1}{2} I^{\mathrm{sp}}_{(1,-1)}(\cdot^2)\right)\left(\frac{1}{2}(I^{\mathrm{sp}}_{(0,1)})^2 + \frac{1}{2} I^{\mathrm{sp}}_{(0,1)}(\cdot^2)\right)\\
&\quad + \frac{1}{2}(I^{\mathrm{sp}}_{(1,0)})^2 + \frac{1}{2} I^{\mathrm{sp}}_{(1,0)}(\cdot^2) + I^{\mathrm{sp}}_{(1,0)} I^{\mathrm{sp}}_{(1,-1)} I^{\mathrm{sp}}_{(0,1)}. &(4.299)
\end{aligned}$$

Secondly, let us consider the cases with $p = 2$ and $q = 2$. There are two cases with $\{s_I\} = \{0, 0, 1, 1\}$ and $\{s_I\} = \{0, 1, 0, 1\}$, which we call UUTT and UTUT theories, respectively. These are simplest examples that are distinguished by the order of two kinds of hyper-multiplets in the quiver diagrams.

We first consider UUTT theory with $\{s_I\} = \{0, 0, 1, 1\}$. The inking numbers are

$$\vec{l} = \{l_1, l_2\} = \{2k, 2k\}, \quad \vec{l}' = \{l_3, l_4\} = \{-2k, -2k\}, \tag{4.300}$$

and the Wilson line parameters η_I vanishes up to integers. On the gravity side, we have two A_1 type singular loci. We parameterize the vectors $\vec{\rho}$ and $\vec{\rho}'$ by two integers ρ and ρ' as

$$\vec{\rho} = \{\rho_1, \rho_2\} = \{-\rho, \rho\}, \quad \vec{\rho}' = \{\rho_3, \rho_4\} = \{-\rho', \rho'\}, \tag{4.301}$$

We introduce chemical potentials t and t' for the charges ρ and ρ', respectively. These are related to the potentials t_I introduced in §4.3 by $t = t_2/t_1$ and $t' = t_4/t_3$. From the identification, we obtain

$$\{m_1, m_2, m_3, m_4\} = \{P_M, P_M + \rho, P_M, P_M + \rho'\}. \tag{4.302}$$

The positive part is defined by

$$m_a \geq 0, \quad \{m_1, m_2, m_3, m_4\} \neq \{0, 0, 0, 0\}, \tag{4.303}$$

and these are equivalent to

$$P_M \geq 0, \quad P_M + \rho \geq 0, \quad P_M + \rho' \geq 0, \quad (P_M, \rho, \rho') \neq (0, 0, 0). \tag{4.304}$$

We would like to show

$$I^{(+)}_{\{P_M, P_M+\rho, P_M, P_M+\rho'\}}(x, z, z') = I^{\mathrm{mp}(+)}_{(P_M,\rho,\rho')}(x, z, z'), \tag{4.305}$$

for every set of charges (P_M,ρ,ρ') satisfying (4.304). Eq. (4.149) gives the single-particle index

$$\begin{aligned}
I^{\rm sp}_{(P_M,0,0)}(x,z,z') &= \sum_{a=-\infty}^{\infty} I^{\rm grav}_{2a,2(kP_M+a)}(x)z^{2a}z'^{2(kP_M+a)} \\
&\quad + I^{\rm vec}_{2kP_M}(x)(z'^{2kP_M}+z^{-2kP_M}), \qquad (4.306)\\
I^{\rm sp}_{(P_M,-1,0)}(x,z,z') &= I^{\rm vec}_{2kP_M}(x)z'^{2kP_M}, \qquad (4.307)\\
I^{\rm sp}_{(P_M,0,-1)}(x,z,z') &= I^{\rm vec}_{-2kP_M}(x)z^{-2kP_M}, \qquad (4.308)\\
I^{\rm sp}_{(P_M,1,0)}(x,z,z') &= I^{\rm vec}_{2kP_M}(x)z'^{2kP_M}, \qquad (4.309)\\
I^{\rm sp}_{(P_M,0,1)}(x,z,z') &= I^{\rm vec}_{-2kP_M}(x)z^{-2kP_M}. \qquad (4.310)
\end{aligned}$$

When some of the inequalities in (4.304) are saturated, the computation of $I^{\rm mp(+)}$ and $I^{(+)}$ are relatively easy, and we first consider such cases. The last condition in (4.304) means that the first three inequalities are not saturated at the same time. If $P_M=0$, only single-particle states saturating the same inequality can contribute to the multi-particle index. There are only two such single-particle charges, $(P_M,\rho,\rho')=(0,1,0)$ and $(0,0,1)$, and thus the multi-particle index is given by

$$\sum_{\rho=0}^{\infty}\sum_{\rho'=0}^{\infty} I^{\rm mp(+)}_{(0,\rho,\rho')}t^{\rho}t'^{\rho'} = \exp\left(\sum_{n\geq 1}\frac{1}{n}\left[I^{\rm sp}_{(0,1,0)}(x^n,z^n,z'^n)t^n + I^{\rm sp}_{(0,0,1)}(x^n,z^n,z'^n)t'^n\right]\right) \qquad (4.311)$$

By using the identity (4.279), we obtain

$$I^{\rm mp(+)}_{(0,\rho,\rho')} = \left(\prod_{i=1}^{\rho}\frac{x^2}{1-x^{4i}}\right)\left(\prod_{i'=1}^{\rho'}\frac{x^2}{1-x^{4i'}}\right). \qquad (4.312)$$

We can easily show that for $I^{(+)}_{\{0,\rho,0,\rho'\}}$ the integrals in (4.112) are factorized into two parts, and the relation

$$I^{(+)}_{\{0,\rho,0,\rho'\}} = I^{(+)}_{\{0,\rho,0,0\}}I^{(+)}_{\{0,0,0,\rho'\}} \qquad (4.313)$$

holds. In general, if the cyclic sequence of the magnetic charges splits into several parts by vanishing components, the integrals in (4.112) are factorized, and we obtain a relation like (4.313). Furthermore, each of two factors in (4.313) is the same as the index $I^{(+)}_{\{0,\rho,0\}}$ for the UUT theory. By using the results in the last subsection, we can confirm $I^{\rm mp(+)}_{(0,\rho,\rho')} = I^{(+)}_{\{0,\rho,0,\rho'\}}$.

Next, let us consider the case in which $P_M\geq 1$ and the second or the third bounds in (4.304) are saturated. Namely, $P_M+\rho=0$ or $P_M+\rho'=0$. Because there is no one-particle state saturating both the bounds, the multi-particle index for such charges vanishes; $I^{\mathrm{mp}(+)}_{(P_M,-P_M,-P_M)} = 0$. On the gauge theory side, we can show $I^{(+)}_{\{P_M,0,P_M,0\}} = 0$ by using the factorization $I^{(+)}_{\{P_M,0,P_M,0\}} = I^{(+)}_{\{P_M,0,0,0\}} I^{(+)}_{\{0,0,P_M,0\}}$, and applying the selection rules to the two factors.

When only one of $P_M + \rho = 0$ or $P_M + \rho' = 0$ in (4.304) is saturated, only single-particle states with charges $(1,0,-1)$ or $(1,-1,0)$ contribute to the multi-particle index, and we obtain

$$I^{\mathrm{mp}(+)}_{(\rho',0,-\rho')} = \left(\prod_{i'=1}^{\rho'} \frac{x^2(xz^{-1})^{pk}}{1-x^{4i'}}\right), \quad I^{\mathrm{mp}(+)}_{(\rho,-\rho,0)} = \left(\prod_{i=1}^{\rho} \frac{x^2(xz')^{qk}}{1-x^{4i}}\right). \tag{4.314}$$

These are easily generalized to

$$I^{\mathrm{mp}(+)}_{(\rho+\rho',-\rho,-\rho')} = \left(\prod_{i'=1}^{\rho'} \frac{x^2(xz^{-1})^{pk}}{1-x^{4i'}}\right)\left(\prod_{i=1}^{\rho} \frac{x^2(xz')^{qk}}{1-x^{4i}}\right). \tag{4.315}$$

We confirm for $k = 1, \ldots, 5$ that this index is correctly reproduced as the gauge theory index for small ρ and ρ' as follows.

$$\begin{aligned}
I^{(+)}_{\{1,1,1,0\}} &= I^{(+)}_{\{\ydiagram{1},\ydiagram{1},\ydiagram{1},\cdot\}} \\
&= I^{\mathrm{mp}(+)}_{(1,0,-1)} + \mathcal{O}(x^{101}), && (4.316)\\
I^{(+)}_{\{2,2,2,0\}} &= I^{(+)}_{\{\ydiagram{2},\ydiagram{2},\ydiagram{2},\cdot\}} + I^{(+)}_{\{\ydiagram{1,1},\ydiagram{2},\ydiagram{2},\cdot\}} + I^{(+)}_{\{\ydiagram{2},\ydiagram{1,1},\ydiagram{2},\cdot\}} + I^{(+)}_{\{\ydiagram{2},\ydiagram{2},\ydiagram{1,1},\cdot\}} \\
&\quad + I^{(+)}_{\{\ydiagram{1,1},\ydiagram{1,1},\ydiagram{2},\cdot\}} + I^{(+)}_{\{\ydiagram{1,1},\ydiagram{2},\ydiagram{1,1},\cdot\}} + I^{(+)}_{\{\ydiagram{2},\ydiagram{1,1},\ydiagram{1,1},\cdot\}} + I^{(+)}_{\{\ydiagram{1,1},\ydiagram{1,1},\ydiagram{1,1},\cdot\}} \\
&= I^{\mathrm{mp}(+)}_{(2,0,-2)} + \mathcal{O}(x^{21}), && (4.317)\\
I^{(+)}_{\{1,0,1,1\}} &= I^{(+)}_{\{\ydiagram{1},\cdot,\ydiagram{1},\ydiagram{1}\}} \\
&= I^{\mathrm{mp}(+)}_{(1,-1,0)} + \mathcal{O}(x^{101}), && (4.318)\\
I^{(+)}_{\{2,0,2,2\}} &= I^{(+)}_{\{\ydiagram{2},\cdot,\ydiagram{2},\ydiagram{2}\}} + I^{(+)}_{\{\ydiagram{1,1},\cdot,\ydiagram{2},\ydiagram{2}\}} + I^{(+)}_{\{\ydiagram{2},\cdot,\ydiagram{1,1},\ydiagram{2}\}} + I^{(+)}_{\{\ydiagram{2},\cdot,\ydiagram{2},\ydiagram{1,1}\}} \\
&\quad + I^{(+)}_{\{\ydiagram{1,1},\cdot,\ydiagram{1,1},\ydiagram{2}\}} + I^{(+)}_{\{\ydiagram{1,1},\cdot,\ydiagram{2},\ydiagram{1,1}\}} + I^{(+)}_{\{\ydiagram{2},\cdot,\ydiagram{1,1},\ydiagram{1,1}\}} + I^{(+)}_{\{\ydiagram{1,1},\cdot,\ydiagram{1,1},\ydiagram{1,1}\}} \\
&= I^{\mathrm{mp}(+)}_{(2,-2,0)} + \mathcal{O}(x^{21}), && (4.319)\\
I^{(+)}_{\{2,1,2,1\}} &= I^{(+)}_{\{\ydiagram{2},\ydiagram{1},\ydiagram{2},\ydiagram{1}\}} + I^{(+)}_{\{\ydiagram{1,1},\ydiagram{1},\ydiagram{2},\ydiagram{1}\}} + I^{(+)}_{\{\ydiagram{2},\ydiagram{1},\ydiagram{1,1},\ydiagram{1}\}} + I^{(+)}_{\{\ydiagram{1,1},\ydiagram{1},\ydiagram{1,1},\ydiagram{1}\}} \\
&= I^{\mathrm{mp}(+)}_{(2,-1,-1)} + \mathcal{O}(x^{21}). && (4.320)
\end{aligned}$$

Finally, we give more examples without vanishing magnetic charges.

$$\begin{aligned}
I^{(+)}_{\{1,1,1,1\}} &= I^{(+)}_{\{\square,\square,\square,\square\}} \\
&= I^{\mathrm{mp}(+)}_{(1,0,0)} + \mathcal{O}(x^{101}), && (4.321)\\
I^{(+)}_{\{1,2,1,1\}} &= I^{(+)}_{\{\square,\square\!\square,\square,\square\}} + I^{(+)}_{\{\square,\substack{\square\\\square},\square\!\square,\square\}} \\
&= I^{\mathrm{mp}(+)}_{(1,1,0)} + \mathcal{O}(x^{31}), && (4.322)\\
I^{(+)}_{\{1,1,1,2\}} &= I^{(+)}_{\{\square,\square,\square,\square\!\square\}} + I^{(+)}_{\{\square,\square,\square,\substack{\square\\\square}\}} \\
&= I^{\mathrm{mp}(+)}_{(1,0,1)} + \mathcal{O}(x^{31}), && (4.323)\\
I^{(+)}_{\{1,2,1,2\}} &= I^{(+)}_{\{\square,\square\!\square,\square,\square\!\square\}} + I^{(+)}_{\{\square,\substack{\square\\\square},\square,\square\!\square\}} + I^{(+)}_{\{\square,\square\!\square,\square,\substack{\square\\\square}\}} + I^{(+)}_{\{\square,\substack{\square\\\square},\square,\substack{\square\\\square}\}} \\
&= I^{\mathrm{mp}(+)}_{(1,1,1)} + \mathcal{O}(x^{31}), && (4.324)
\end{aligned}$$

where

$$\begin{aligned}
I^{\mathrm{mp}(+)}_{(1,0,0)} &= I^{\mathrm{sp}}_{(1,0,0)} + I^{\mathrm{sp}}_{(1,-1,0)} I^{\mathrm{sp}}_{(0,1,0)} + I^{\mathrm{sp}}_{(1,0,-1)} I^{\mathrm{sp}}_{(0,0,1)}, && (4.325)\\
I^{\mathrm{mp}(+)}_{(1,1,0)} &= I^{\mathrm{sp}}_{(1,1,0)} + I^{\mathrm{sp}}_{(1,0,0)} I^{\mathrm{sp}}_{(0,1,0)} + I^{\mathrm{sp}}_{(1,0,-1)} I^{\mathrm{sp}}_{(0,0,1)} I^{\mathrm{sp}}_{(0,0,1)} \\
&\quad + \left(\frac{1}{2}(I^{\mathrm{sp}}_{(0,1,0)})^2 + \frac{1}{2} I^{\mathrm{sp}}_{(0,1,0)}(\cdot^2)\right) I^{\mathrm{sp}}_{(1,0,-1)}, && (4.326)\\
I^{\mathrm{mp}(+)}_{(1,0,1)} &= I^{\mathrm{sp}}_{(1,0,1)} + I^{\mathrm{sp}}_{(1,0,0)} I^{\mathrm{sp}}_{(0,0,1)} + I^{\mathrm{sp}}_{(1,-1,0)} I^{\mathrm{sp}}_{(0,1,0)} I^{\mathrm{sp}}_{(0,1,0)} \\
&\quad + \left(\frac{1}{2}(I^{\mathrm{sp}}_{(0,0,1)})^2 + \frac{1}{2} I^{\mathrm{sp}}_{(0,0,1)}(\cdot^2)\right) I^{\mathrm{sp}}_{(1,-1,1)}, && (4.327)\\
I^{\mathrm{mp}(+)}_{(1,1,1)} &= I^{\mathrm{sp}}_{(1,0,0)} I^{\mathrm{sp}}_{(0,1,0)} I^{\mathrm{sp}}_{(0,0,1)} + I^{\mathrm{sp}}_{(1,0,1)} I^{\mathrm{sp}}_{(0,1,0)} + I^{\mathrm{sp}}_{(1,1,0)} I^{\mathrm{sp}}_{(0,0,1)} \\
&\quad + \left(\frac{1}{2}(I^{\mathrm{sp}}_{(0,1,0)})^2 + \frac{1}{2} I^{\mathrm{sp}}_{(0,1,0)}(\cdot^2)\right) I^{\mathrm{sp}}_{(0,0,1)} I^{\mathrm{sp}}_{(1,-1,0)} \\
&\quad + \left(\frac{1}{2}(I^{\mathrm{sp}}_{(0,0,1)})^2 + \frac{1}{2} I^{\mathrm{sp}}_{(0,0,1)}(\cdot^2)\right) I^{\mathrm{sp}}_{(0,1,0)} I^{\mathrm{sp}}_{(1,0,-1)}. && (4.328)
\end{aligned}$$

Finally, we move on to the UTUT theory with $\{s_I\} = \{0,1,0,1\}$. The linking numbers for this theory are

$$\vec{l} = \{l_1, l_3\} = \{2k, k\}, \quad \vec{l'} = \{l_2, l_4\} = \{-k, -2k\}, \tag{4.329}$$

and the Wilson line parameters are given by

$$\vec{\eta} = \{\eta_1, \eta_3\} = \{0, \frac{1}{2}\}, \quad \vec{\eta'} = \{\eta_2, \eta_4\} = \{\frac{1}{2}, 0\}. \tag{4.330}$$

This theory is the simplest example with the non-trivial Wilson lines on the singular loci. We parameterize $\vec{\rho}$ and $\vec{\rho'}$ by two integers ρ and ρ' as

$$\vec{\rho} = \{\rho_1, \rho_3\} = \{-\rho, \rho\}, \quad \vec{\rho'} = \{\rho_2, \rho_4\} = \{-\rho', \rho'\}. \tag{4.331}$$

We introduce chemical potentials t and t' for the charges ρ and ρ', respectively. These are related to the potentials t_I introduced in §4.3 by $t = t_3/t_1$ and $t' = t_4/t_2$. Then the magnetic charges are given by

$$\begin{aligned}\{m_a\} &= \{P_M - \frac{\rho+\rho'}{2}, P_M + \frac{\rho-\rho'}{2}, P_M + \frac{\rho+\rho'}{2}, P_M - \frac{\rho-\rho'}{2}\} \\ &= \{m_\bullet, m_\bullet+\rho, m_\bullet+\rho+\rho', m_\bullet+\rho'\},\end{aligned} \tag{4.332}$$

where $m_\bullet$ is the magnetic charge for the reference vertex, and is related to P_M by

$$P_M = m_\bullet + \frac{1}{2}(\rho+\rho'). \tag{4.333}$$

The relation we would like to confirm is

$$I^{(+)}_{\{P_M-\frac{\rho+\rho'}{2}, P_M+\frac{\rho-\rho'}{2}, P_M+\frac{\rho+\rho'}{2}, P_M-\frac{\rho-\rho'}{2}\}}(x,z,z') = I^{\mathrm{mp}(+)}_{(P_M,\rho,\rho')}(x,z,z'). \tag{4.334}$$

The positive part of the single-particle index is defined by

$$m_a \geq 0, \quad \{m_1, m_2, m_3, m_4\} \neq \{0,0,0,0\}, \tag{4.335}$$

and these are equivalent to

$$m_\bullet \geq 0, \quad m_\bullet + \rho \geq 0, \quad m_\bullet + \rho' \geq 0, \quad (m_\bullet, \rho, \rho') \neq (0,0,0). \tag{4.336}$$

The single-particle index is given by

$$\begin{aligned}I^{\mathrm{sp}}_{(m_\bullet,0,0)}(x,z,z') &= \sum_{a=-\infty}^{\infty} I^{\mathrm{grav}}_{2a,2(km_\bullet+a)}(x) z^{2a} z'^{2(km_\bullet+a)} \\ &\quad + I^{\mathrm{vec}}_{2km_\bullet}(x)(z'^{2km_\bullet} + z^{-2km_\bullet}), &(4.337)\\ I^{\mathrm{sp}}_{(m_\bullet-\frac{1}{2},-1,0)}(x,z,z') &= I^{\mathrm{vec}}_{k(2m_\bullet-1)}(x) z'^{k(2m_\bullet-1)}, &(4.338)\\ I^{\mathrm{sp}}_{(m_\bullet-\frac{1}{2},0,-1)}(x,z,z') &= I^{\mathrm{vec}}_{-k(2m_\bullet-1)}(x) z^{-k(2m_\bullet-1)}, &(4.339)\\ I^{\mathrm{sp}}_{(m_\bullet+\frac{1}{2},1,0)}(x,z,z') &= I^{\mathrm{vec}}_{k(2m_\bullet+1)}(x) z'^{k(2m_\bullet+1)}, &(4.340)\\ I^{\mathrm{sp}}_{(m_\bullet+\frac{1}{2},0,1)}(x,z,z') &= I^{\mathrm{vec}}_{-k(2m_\bullet+1)}(x) z^{-k(2m_\bullet+1)}. &(4.341)\end{aligned}$$

As we did in the UUTT theory, let us first consider the cases in which some of the magnetic charges m_a vanish. Because all four vertices in the quiver diagram are on an equal footing, we can assume $m_\bullet = m_1 = 0$ without loosing generality. This means that the first bound in (4.336) is saturated, and on the gravity side only single-particle states with $(m_\bullet, \rho, \rho') = (0,1,0)$ or $(0,0,1)$ can contribute to the index. The multi-particle index in this case is determined with the relation

$$\begin{aligned}&\sum_{\rho,\rho'} I^{\mathrm{mp}(+)}_{(\frac{\rho+\rho'}{2},\rho,\rho')}(x,z,z') t^\rho t'^{\rho'} \\ &= \exp\left(\sum_{n\geq 1}\frac{1}{n}\left[I^{\mathrm{sp}}_{(1/2,1,0)}(x^n,z^n,z'^n)t^n + I^{\mathrm{sp}}_{(1/2,0,1)}(x^n,z^n,z'^n)t'^n\right]\right). \end{aligned} \tag{4.342}$$

By using the identity (4.279), we obtain

$$I^{\mathrm{mp}(+)}_{(\frac{\rho+\rho'}{2},\rho,\rho')} = \left(\prod_{i=1}^{\rho} \frac{x^2(xz')^k}{1-x^{4i}}\right)\left(\prod_{i'=1}^{\rho'} \frac{x^2(xz^{-1})^k}{1-x^{4i'}}\right). \tag{4.343}$$

For $k = 1, \ldots, 5$ and small ρ and ρ', we confirmed that this multi-particle index is reproduced as the gauge theory index

$$\begin{aligned}
I^{(+)}_{\{0,0,1,1\}} &= I^{(+)}_{\{\cdot,\cdot,\ydiagram{1},\ydiagram{1}\}} \\
&= I^{\mathrm{mp}(+)}_{(1/2,0,1)} + \mathcal{O}(x^{101}), && (4.344)\\
I^{(+)}_{\{0,0,2,2\}} &= I^{(+)}_{\{\cdot,\cdot,\ydiagram{2},\ydiagram{2}\}} + I^{(+)}_{\{\cdot,\cdot,\ydiagram{2},\ydiagram{1,1}\}} + I^{(+)}_{\{\cdot,\cdot,\ydiagram{1,1},\ydiagram{2}\}} + I^{(+)}_{\{\cdot,\cdot,\ydiagram{1,1},\ydiagram{1,1}\}} \\
&= I^{\mathrm{mp}(+)}_{(1,0,2)} + \mathcal{O}(x^{31}), && (4.345)\\
I^{(+)}_{\{0,0,3,3\}} &= +I^{(+)}_{\{\cdot,\cdot,\ydiagram{3},\ydiagram{3}\}} + I^{(+)}_{\{\cdot,\cdot,\ydiagram{3},\ydiagram{2,1}\}} + I^{(+)}_{\{\cdot,\cdot,\ydiagram{3},\ydiagram{1,1,1}\}} + I^{(+)}_{\{\cdot,\cdot,\ydiagram{2,1},\ydiagram{3}\}} + I^{(+)}_{\{\cdot,\cdot,\ydiagram{2,1},\ydiagram{2,1}\}} \\
&\quad +I^{(+)}_{\{\cdot,\cdot,\ydiagram{2,1},\ydiagram{1,1,1}\}} + I^{(+)}_{\{\cdot,\cdot,\ydiagram{1,1,1},\ydiagram{3}\}} + I^{(+)}_{\{\cdot,\cdot,\ydiagram{1,1,1},\ydiagram{2,1}\}} + I^{(+)}_{\{\cdot,\cdot,\ydiagram{1,1,1},\ydiagram{1,1,1}\}} \\
&= I^{\mathrm{mp}(+)}_{(3/2,0,3)} + \mathcal{O}(x^{11}), && (4.346)\\
I^{(+)}_{\{0,1,1,0\}} &= I^{(+)}_{\{\cdot,\ydiagram{1},\ydiagram{1},\cdot\}} \\
&= I^{\mathrm{mp}(+)}_{(1/2,1,0)} + \mathcal{O}(x^{101}), && (4.347)\\
I^{(+)}_{\{0,2,2,0\}} &= I^{(+)}_{\{\cdot,\ydiagram{2},\ydiagram{2},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{2},\ydiagram{1,1},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{1,1},\ydiagram{2},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{1,1},\ydiagram{1,1},\cdot\}} \\
&= I^{\mathrm{mp}(+)}_{(1,2,0)} + \mathcal{O}(x^{31}), && (4.348)\\
I^{(+)}_{\{0,3,3,0\}} &= I^{(+)}_{\{\cdot,\ydiagram{3},\ydiagram{3},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{3},\ydiagram{2,1},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{3},\ydiagram{1,1,1},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{2,1},\ydiagram{3},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{2,1},\ydiagram{2,1},\cdot\}} \\
&\quad +I^{(+)}_{\{\cdot,\ydiagram{2,1},\ydiagram{1,1,1},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{1,1,1},\ydiagram{3},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{1,1,1},\ydiagram{2,1},\cdot\}} + I^{(+)}_{\{\cdot,\ydiagram{1,1,1},\ydiagram{1,1,1},\cdot\}} \\
&= I^{\mathrm{mp}(+)}_{(3/2,3,0)} + \mathcal{O}(x^{11}), && (4.349)\\
I^{(+)}_{\{0,1,2,1\}} &= I^{(+)}_{\{\cdot,\ydiagram{1},\ydiagram{2},\ydiagram{1}\}} + I^{(+)}_{\{\cdot,\ydiagram{1},\ydiagram{1,1},\ydiagram{1}\}} \\
&= I^{\mathrm{mp}(+)}_{(1,1,1)} + \mathcal{O}(x^{41}), && (4.350)\\
I^{(+)}_{\{0,2,3,1\}} &= I^{(+)}_{\{\cdot,\ydiagram{2},\ydiagram{3},\ydiagram{1}\}} + I^{(+)}_{\{\cdot,\ydiagram{2},\ydiagram{2,1},\ydiagram{1}\}} + I^{(+)}_{\{\cdot,\ydiagram{2},\ydiagram{1,1,1},\ydiagram{1}\}} + I^{(+)}_{\{\cdot,\ydiagram{1,1},\ydiagram{3},\ydiagram{1}\}} + I^{(+)}_{\{\cdot,\ydiagram{1,1},\ydiagram{2,1},\ydiagram{1}\}} + I^{(+)}_{\{\cdot,\ydiagram{1,1},\ydiagram{1,1,1},\ydiagram{1}\}} \\
&= I^{\mathrm{mp}(+)}_{(3/2,2,1)} + \mathcal{O}(x^{21}), && (4.351)\\
I^{(+)}_{\{0,1,3,2\}} &= I^{(+)}_{\{\cdot,\ydiagram{1},\ydiagram{3},\ydiagram{2}\}} + I^{(+)}_{\{\cdot,\ydiagram{1},\ydiagram{3},\ydiagram{1,1}\}} + I^{(+)}_{\{\cdot,\ydiagram{1},\ydiagram{2,1},\ydiagram{2}\}} + I^{(+)}_{\{\cdot,\ydiagram{1},\ydiagram{2,1},\ydiagram{1,1}\}} + I^{(+)}_{\{\cdot,\ydiagram{1},\ydiagram{1,1,1},\ydiagram{2}\}} + I^{(+)}_{\{\cdot,\ydiagram{1},\ydiagram{1,1,1},\ydiagram{1,1}\}} \\
&= I^{\mathrm{mp}(+)}_{(3/2,1,2)} + \mathcal{O}(x^{21}). && (4.352)
\end{aligned}$$

We also check in some sectors with magnetic charges without vanishing components the gauge theory index correctly reproduces the corresponding multi-particle index for $k = 1, \ldots, 5$.

$$
\begin{aligned}
I^{(+)}_{\{1,1,1,1\}} &= I^{(+)}_{\{\square,\square,\square,\square\}} \\
&= I^{\mathrm{mp}(+)}_{(1,0,0)} + \mathcal{O}(x^{101}), && (4.353) \\
I^{(+)}_{\{1,2,2,1\}} &= I^{(+)}_{\{\square,\square\!\square,\square\!\square,\square\}} + I^{(+)}_{\{\square,\substack{\square\\\square},\square\!\square,\square\}} + I^{(+)}_{\{\square,\square\!\square,\substack{\square\\\square},\square\}} + I^{(+)}_{\{\square,\substack{\square\\\square},\substack{\square\\\square},\square\}} \\
&= I^{\mathrm{mp}(+)}_{(3/2,1,0)} + \mathcal{O}(x^{31}), && (4.354) \\
I^{(+)}_{\{1,1,2,2\}} &= I^{(+)}_{\{\square,\square,\square\!\square,\square\!\square\}} + I^{(+)}_{\{\square,\square,\substack{\square\\\square},\square\!\square\}} + I^{(+)}_{\{\square,\square,\square\!\square,\substack{\square\\\square},\square\}} + I^{(+)}_{\{\square,\square,\substack{\square\\\square},\substack{\square\\\square}\}} \\
&= I^{\mathrm{mp}(+)}_{(3/2,0,1)} + \mathcal{O}(x^{31}), && (4.355)
\end{aligned}
$$

where

$$
\begin{aligned}
I^{\mathrm{mp}(+)}_{(1,0,0)} &= I^{\mathrm{sp}}_{(1,0,0)} + I^{\mathrm{sp}}_{(1/2,-1,0)} I^{\mathrm{sp}}_{(1/2,1,0)} + I^{\mathrm{sp}}_{(1/2,0,-1)} I^{\mathrm{sp}}_{(1/2,0,1)}, && (4.356) \\
I^{\mathrm{mp}(+)}_{(3/2,1,0)} &= I^{\mathrm{sp}}_{(3/2,1,0)} + I^{\mathrm{sp}}_{(1,0,0)} I^{\mathrm{sp}}_{(1/2,1,0)} + I^{\mathrm{sp}}_{(1/2,1,0)} I^{\mathrm{sp}}_{(1/2,0,1)} I^{\mathrm{sp}}_{(1/2,0,-1)} \\
&\quad + \left(\frac{1}{2} (I^{\mathrm{sp}}_{(1/2,1,0)})^2 + \frac{1}{2} I^{\mathrm{sp}}_{(1/2,1,0)}(\cdot^2) \right) I^{\mathrm{sp}}_{(1/2,-1,0)}, && (4.357) \\
I^{\mathrm{mp}(+)}_{(3/2,0,1)} &= I^{\mathrm{sp}}_{(3/2,0,1)} + I^{\mathrm{sp}}_{(1,0,0)} I^{\mathrm{sp}}_{(1/2,0,1)} + I^{\mathrm{sp}}_{(1/2,0,1)} I^{\mathrm{sp}}_{(1/2,1,0)} I^{\mathrm{sp}}_{(1/2,-1,0)} \\
&\quad + \left(\frac{1}{2} (I^{\mathrm{sp}}_{(1/2,0,1)})^2 + \frac{1}{2} I^{\mathrm{sp}}_{(1/2,0,1)}(\cdot^2) \right) I^{\mathrm{sp}}_{(1/2,0,-1)}. && (4.358)
\end{aligned}
$$

Chapter 5

Fractional M2-branes and ranks of gauge groups

The richness of the class is also reflected in the dual geometry.

5.1 Type IIB brane analysis

In this chapter, the gauge group is both unitary or special unitary group. Let us perform a certain classification of the $\mathcal{N} = 4$ Chern-Simons theories only by their ranks, which means that the other parameters of a equivalent class are fixed . To do such a classification, it is helpful to use the type IIB brane system given in §2.3. In this setup, let us move a five-brane, say i-th NS5-brane, in the positive direction along $\mathbf{S}^1$ continuously until it comes back to the original position. Let us call this process one i-th NS5 movement. It is evident that the parameter p, q, k, s_I are fixed under this movement. However, ranks of gauge groups are changed because the D3-brane creation occurs when an NS5-brane gets across an $(1, k)$5-brane. This is called the Hanany-Witten effect [61]. When the NS5-brane passes through an $(1, k)$5-brane, k D3-branes are created after the cross. See also Figure 5.1.

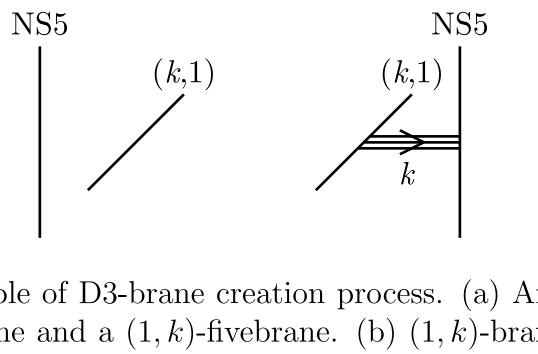

Figure 5.1: An example of D3-brane creation process. (a) An initial setup consisting of an NS5-brane and a $(1, k)$-fivebrane. (b) $(1, k)$-brane is moved on the other side of the NS5-brane. k D3-branes are created.

For simplicity, let us study the $k = 1$ case, which is constituted by D3, NS5-, and $(1,1)$ 5-branes. The set of these three kinds of branes is equivalent to the set D3, NS5, and D5-branes up to a certain $SL(2,\mathbf{Z})$ duality transformation. Thus we use the latter set of branes. The directions of these branes are shown in Table 2.3.

Let us define n_I as the number of D3-branes emanating from the I-th five-brane. By definition N_I and n_I are related by

$$n_I = N_{L(I)} - N_{R(I)}. \tag{5.1}$$

Because n_I are invariant under an overall shift $N_a \to N_a + c$, $(c \in \mathbf{Z})$, we cannot uniquely determine N_a from n_I. This degree of freedom represents integral D3-branes wrapping around the whole $\mathbf{S}^1$. We focus only on the fractional brane charges and use n_I to represent D3-brane distributions.

With a fixed ordering, a D3-brane distribution is specified by a vector

$$(n_1, n_2, \ldots, n_p | n_{1'}, n_{2'}, \ldots, n_{q'}). \tag{5.2}$$

We call this vector "charge vector." By definition, the components of a charge vector must satisfy the constraint

$$\sum_{i=1}^{p} n_i + \sum_{i'=1'}^{q'} n_{i'} = 0. \tag{5.3}$$

The set of charge vectors, whose components are constrained by (5.3), forms the group

$$\Gamma = \mathbf{Z}^{p+q-1}. \tag{5.4}$$

We should not regard the group Γ as the group characterizing the conserved charge of fractional D3-branes because D3-brane distributions corresponding to different elements of Γ may be continuously deformed to one another. We should regard charges of such brane configurations as the same.

Under one i-th NS5-movement, the charge vector changes by

$$\mathbf{v}_i = (0, \ldots, -q, \ldots, 0 | 1, \ldots, 1) = -q\mathbf{e}_i + \sum_{j'=1'}^{q'} \mathbf{e}_{j'} \in \Gamma, \tag{5.5}$$

where $\mathbf{e}_i$ ($\mathbf{e}_{j'}$) is the unit vectors whose i-th (j'-th) component is 1. Note that $\mathbf{e}_i$ and $\mathbf{e}_{j'}$ themselves are not elements of Γ because they do not satisfy the constraint (5.3). Similarly, under one i'-th D5-movement, the charge vector changes by

$$\mathbf{w}_{i'} = (1, \ldots, 1 | 0, \ldots, -p, \ldots, 0) = \sum_{j=1}^{p} \mathbf{e}_j - p\mathbf{e}_{i'} \in \Gamma. \tag{5.6}$$

When we identify configurations deformed by continuous deformation to one another, these vectors should be identified with 0. Therefore, the group describing the charge of fractional branes is the quotient group Γ/H where H is the subgroup of Γ generated by the vectors $\mathbf{v}_i$ and $\mathbf{w}_{j'}$. This is given by

$$\Gamma/H = (\mathbf{Z}_p^{q-1} \oplus \mathbf{Z}_q^{p-1} \oplus \mathbf{Z}_{pq})/(\mathbf{Z}_p \oplus \mathbf{Z}_q). \tag{5.7}$$

In the rest of this section, we explain how this expression of the quotient group is obtained.

As we mentioned above, the $p+q$ vectors $\mathbf{e}_i$ and $\mathbf{e}_{i'}$ are not elements of Γ. Let us choose $p+q-1$ linearly independent basis in Γ. We take the following vectors.

$$\begin{aligned}
\mathbf{f}_i &= \mathbf{e}_i - \mathbf{e}_p \quad (i = 1, \ldots, p-1), && (5.8)\\
\mathbf{g}_{i'} &= \mathbf{e}_{i'} - \mathbf{e}_{q'} \quad (i' = 1', \ldots, (q-1)'), && (5.9)\\
\mathbf{h} &= \mathbf{e}_p - \mathbf{e}_{q'}. && (5.10)
\end{aligned}$$

We can easily check that these vectors span Γ. In order to obtain (5.7), we define the subgroup $H' \subset H$ generated by the following elements in H.

$$\begin{aligned}
\mathbf{v}_p - \mathbf{v}_i &= q\mathbf{f}_i \quad (i = 1, \ldots, p-1), && (5.11)\\
\mathbf{w}_{q'} - \mathbf{w}_{i'} &= p\mathbf{g}_{i'} \quad (i' = 1', \ldots, (q-1)'), && (5.12)\\
-p\mathbf{v}_p + q\mathbf{w}_{q'} - \sum_{j'=1'}^{q'} \mathbf{w}_{j'} &= pq\mathbf{h}. && (5.13)
\end{aligned}$$

We can easily show that

$$\Gamma/H' = \mathbf{Z}_p^{q-1} \oplus \mathbf{Z}_q^{p-1} \oplus \mathbf{Z}_{pq}, \quad H/H' = \mathbf{Z}_p \oplus \mathbf{Z}_q, \tag{5.14}$$

and the relation $\Gamma/H = (\Gamma/H')/(H/H')$ gives (5.7).

We can easily generalize the analysis in the case of $k=1$ to that of $k \geq 2$. Let us again consider fractional D3-brane charges in the type IIB brane setup. We can realize the Chern-Simons theory at level k by replacing D5-branes by $(1,k)$ fivebranes. We can again represent distributions of D3-branes by charge vectors (5.2) with their components constrained by (5.3). In this case, the vectors (5.5) and (5.6) are multiplied by the extra factor k. Namely, we should replace the subgroup H by kH which is generated by $k\mathbf{v}_i$ and $k\mathbf{w}_{i'}$, and the quotient group becomes

$$\Gamma/(kH) = (\Gamma/kH')/(H/H') = (\mathbf{Z}_{kq}^{p-1} \oplus \mathbf{Z}_{kp}^{q-1} \oplus \mathbf{Z}_{kpq})/(\mathbf{Z}_p \oplus \mathbf{Z}_q). \tag{5.15}$$

5.2 Third homology

In this section, we derive the $H_3(\mathbf{S}^7_{p,q,k})$ and show that this fractional brane charge (5.15) precisely agree with the homology $H_3(\mathbf{S}^7_{p,q,k})$ obtained by the geometric side.

For simplicity, we first consider the case of $k = 1$. Let us remind that $\mathbf{S}^7_{p,q,1}$ can be represented as a $\mathbf{T}^2$ fibration over $B = \mathbf{S}^5$. This fibration is defined in the following way. By introducing the real coordinate $0 \le t \le 1$ by (3.45), the manifold $\mathbf{S}^7_{p,q,1}$ is represented as $L_p \times L_q$ fibration over the segment $0 \le t \le 1$. Each of Lens spaces L_p and L_q can be represented as $\mathbf{S}^1$ fibration over 2-sphere. For L_p, which is rotated by the $SU(2)_U$ R-symmetry, we refer to the base manifold and the fiber as $\mathbf{S}^2_A$ and α-cycle, respectively. We also define $\mathbf{S}^2_B$ and β-cycle for the other Lens space L_q, which is rotated by $SU(2)_T$. (Figure 5.2) Due to the

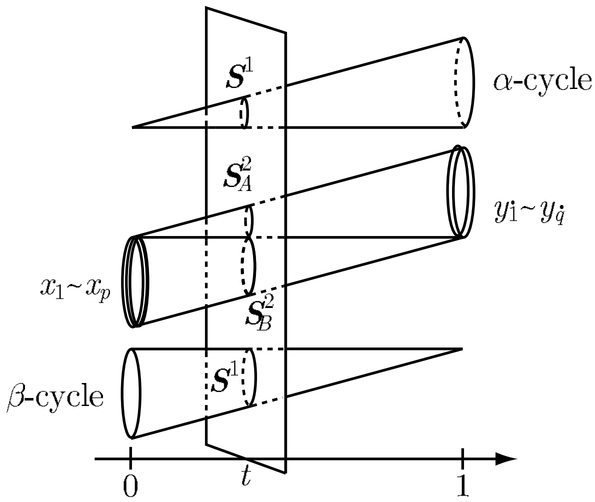

Figure 5.2: The orbifold is represented as a fibration over the segment $0 \le t \le 1$.

$\mathbf{Z}_p \times \mathbf{Z}_q$ orbifolding, the periods of α and β-cycles ara $2\pi/p$ and $2\pi/q$, respectively. If we combine $\mathbf{S}^2_A$, $\mathbf{S}^2_B$, and the segment parameterized by t, they form a 5-sphere $B = \mathbf{S}^5$. We can regard the orbifold $\mathbf{S}^7_{p,q,1}$ as a $\mathbf{T}^2$ fibration over B.

At $t = 0$, which defines $\mathbf{S}^2 \subset B$, the Lens space L_p shrinks and so does the α-cycle. Similarly, on $\mathbf{S}^2 \subset B$ with $t = 1$ the β-cycle shrinks. These $\mathbf{S}^2$ link to each other in B. By blowing up the singularities, these $\mathbf{S}^2$s split into p and q $\mathbf{S}^2$s, respectively.[1] We call them x_i $(i = 1, \dots, p)$ and $y_{i'}$ $(i' = 1', \dots, q')$. (Figure 5.3) We can follow the IIB/M duality to see that each of them corresponds to the fivebrane with the same index.

[1]We blow-up the singularities only to make cycles well-defined. When we compute the volume of five-cycles later, we consider the orbifold limit.

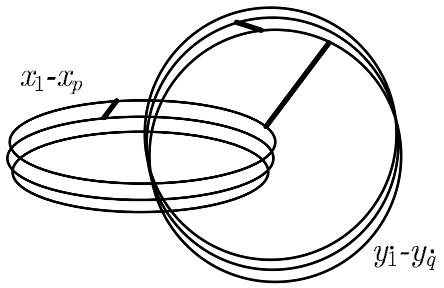

Figure 5.3: The three segments connecting cycles are examples of three types of three-cycles in the orbifold.

3-cycles in $\mathbf{S}^7_{p,q,1}$ can be represented as $\mathbf{T}^2$ fibrations over segments in the base manifold $B = \mathbf{S}^5$. There are three types of segments connecting two loci of degenerate fiber. (Figure 5.3) We denote a segment connecting a point in x_i and a point in x_j by $[x_i, x_j]$. We similarly define $[y_{i'}, y_{j'}]$ and $[x_i, y_{j'}]$. We also adopt the notation

$$S^\alpha, S^\beta, S^{\alpha\beta} \subset \mathbf{S}^7_{p,q,1} \tag{5.16}$$

for the manifold obtained by combining a subset $S \subset B$ and fibers indicated as superscripts. $S^{\alpha\beta}$ is the $\mathbf{T^2}$ fibration over S. $S^{\alpha\beta}$ can be regarded as β-fibration over a certain base manifold, which is isomorphic to S^α. S^α is a global section in this fiber bundle. Therefore, S^α can be defined only when β-cycle fibration over S has a global section. Similarly, we can define S^β when α-cycle fiber has trivial topology over S. With these notations, we can represent 3-cycles generating H_3 as

$$[x_i, x_j]^{\alpha\beta}, \quad [y_{i'}, y_{j'}]^{\alpha\beta}, \quad [x_i, y_{j'}]^{\alpha\beta}. \tag{5.17}$$

Because one of α and β cycles shrinks at the endpoints of the segments, these are closed 3-cycles. The topology of $[x_i, x_j]^{\alpha\beta}$ and $[y_{i'}, y_{j'}]^{\alpha\beta}$ is $\mathbf{S}^2 \times \mathbf{S}^1$, and that of $[x_i, y_{j'}]^{\alpha\beta}$ is $\mathbf{S}^3$.

These 3-cycles are not linearly independent. There are combinations of cycles which can be unwrapped. Let us consider

$$\sum_{i=1}^{p} [x_i, y_{j'}]^{\alpha\beta}. \tag{5.18}$$

This union of 3-cycles can be unwrapped in $\mathbf{S}^7_{p,q,1}$. This can be shown by giving a 4-chain whose boundary is (5.18). Such an “unwrapping chain” is constructed in the following way. Because $\pi_2(\mathbf{S}^5) = 0$, there is a three-dimensional disk $\mathbf{D}^3 \subset B$ whose boundary is $y_{j'}$. (The gray disk in Figure 5.4) We call this $Y_{j'}$. (We also define X_i in the same way for x_i.) This disk intersects once with every x_i $(i = 1, \ldots, p)$. Let $\bar{Y}_{j'}$ be the subset of $Y_{j'}$ obtained by removing segments

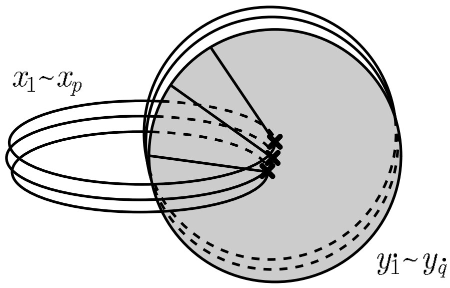

Figure 5.4: The β-cycle fibration over the gray disk with the segments removed is an example of unwrapping four-chains.

connecting these intersecting points and $y_{j'}$ (the segments in Fig 5.4) from the disk.

$$\bar{Y}_{j'} = Y_{j'} \backslash \sum_{i=1}^{p} [x_i, y_{j'}]. \tag{5.19}$$

Because $\bar{Y}_{j'}$ is contractible, we can define $\bar{Y}_{j'}^{\beta}$. We can see that the boundary of the manifold $\bar{Y}_{j'}^{\beta}$ is

$$\partial \bar{Y}_{j'}^{\beta} = \sum_{i=1}^{p} [x_i, y_{j'}]^{\alpha\beta}. \tag{5.20}$$

Before we explain this relation, let us first consider Hopf fibration of $\mathbf{S}^3$ as a simple example. By the Hopf fibration $\mathbf{S}^3$ is described as an $\mathbf{S}^1$ fibration over $\mathbf{S}^2$. Let (θ, ϕ) be the polar coordinates of the base $\mathbf{S}^2$. The first Chern class of this fiber bundle is 1, so that we cannot globally define the coordinate of the fiber. We cover the base $\mathbf{S}^2$ by two patches, north patch $\mathbf{S}^2 \setminus S$ and south patch $\mathbf{S}^2 \setminus N$, where S and N are the south pole ($\theta = \pi$) and the north pole ($\theta = 0$). Let $0 \leq \psi_N < 2\pi$ and $0 \leq \psi_S < 0$ be the fiber coordinates defined in the north and the south patch, respectively. These are paseted by the relation $\psi_N = \psi_S + \phi$. Due to the non-vanishing first Chern class, we cannot take a global section in this fiber bundle. In order to define sections, we need to remove at least one point (0-cycle) from the base $\mathbf{S}^2$. Let us remove the north pole. Then, we can cover the remaining part of the base by the south patch $\mathbf{S}^2 \setminus N$. We can define, for example, the section

$$0 < \theta \leq \pi, \quad 0 \leq \phi < 2\pi, \quad \psi_S = 0. \tag{5.21}$$

We denote this section as Z_N. At the boundary $\theta = 0$ of the south patch $\mathbf{S}^2 \setminus N$, this section wrap the fiber $\mathbf{S}^1$:

$$\partial Z_N = N^{\eta}, \tag{5.22}$$

where η means the fiber $\mathbf{S}^1$. This becomes obvious if we use the coordinate ψ_N, which is suitable for description around the north pole. The boundary is given by

$$\theta = 0, \quad 0 \leq \phi < 2\pi, \quad \psi_N = \phi. \tag{5.23}$$

This winds once along the fiber η. This result makes sense from the fact that the homology $H_1(\mathbf{S}^3)$ vanishes. Any 1-cycle on $\mathbf{S}^3$ can be unwrapped and represented as the boundary of a 2-chain.

Let us return to the case of the $\mathbf{T}^2$ fibration over $Y_{j'}$. In the derivation of (5.20) the β-cycle is simply a spectator, and thus we first neglect it. For simplicity, we consider the case with $p = 1$. Then, we have only one x_i. Let us define the standard polar coordinates (r, θ, ϕ) in $Y_{j'}$ so that $r = 1$ and $r = 0$ correspond to $y_{j'}$ and the intersection with x_i, respectively. Each shell defined by fixed $0 < r < 1$ is $\mathbf{S}^2 \subset Y_{j'}$. This sphere is denoted by $\mathbf{S}^2_A$ in Figure 5.2, and the α-cycle fibration over $\mathbf{S}^2_A$ gives Hopf fibration of $\mathbf{S}^3$. The argument in the last paragraph can be applied for each $\mathbf{S}^3$. We choose the intersection of $\mathbf{S}^2_A$ and the segment $[x_i, y_{j'}]$ as the noth pole N on $\mathbf{S}^2_A$. For each $0 \leq r \leq 1$ we have N^α, the α-cycle fibration over the north pole N, as the boundary of $\mathbf{S}^2_A \backslash N$. By collecting these for every $0 \leq r \leq 1$, we obtain two-dimensional $[x_i, y_{j'}]^\alpha$ as the boundary of $\bar{Y}_{j'} = Y_{j'} \backslash [x_i, y_{j'}]$. Precisely speaking, we also have the boundary $y^\alpha_{j'}$, the α-cycle fibration over $y_{j'}$, at $r = 1$. This part, however, does not survive if we take the β-cycle into account because the β-cycle shrinks on $y_{j'}$. Then, we obtain $[x_i, y_{j'}]^{\alpha\beta}$ as the boundary of $\bar{Y}^\beta_{j'}$. For general p, each segment $[x_i, y_{j'}]$ generates a boundary 3-cycle and we obtain the relation (5.20). By exchanging the role of X_i and $Y_{j'}$, we can also show

$$\partial \bar{X}^\alpha_i = \sum_{j'=1}^{q'} [x_i, y_{j'}]^{\alpha\beta}. \tag{5.24}$$

From (5.20) and (5.24), we obtain the homology relation

$$\sum_{i=1}^{p} [x_i, y_{j'}]^{\alpha\beta} = \sum_{j'=1'}^{q'} [x_i, y_{j'}]^{\alpha\beta} = 0. \tag{5.25}$$

To clarify the relation between the IIB picture and the M-theory picture, we define formal basis $\mathbf{x}_i$ and $\mathbf{y}_{j'}$ and rewrite cycles as $[x_i, x_j]^{\alpha\beta} = \mathbf{x}_i - \mathbf{x}_j$ and so on. A general superposition of cycles, which is depicted as a junction in B, can be written as a linear combination

$$\mathbf{j} = \sum_{i=1}^{p} n_i \mathbf{x}_i + \sum_{i'=1'}^{q'} n_{i'} \mathbf{y}_{i'}, \tag{5.26}$$

where the coefficients must satisfy the constraint (5.3). We have one to one correspondence between 3-cycles in $\mathbf{S}^7_{p,q,1}$ and D3-brane distributions in IIB picture by simply identifying the coefficients in (5.26) to the components of the charge vector (5.2). Via this isomorphism the boundaries (5.20) and (5.24) correspond to $\mathbf{w}_{j'}$ and $\mathbf{v}_i$, the generators of H, and the relation (5.25) defines the homology H_3 as the same coset group Γ/H in (5.7).

Let us construct the homology $H_3(\mathbf{S}^7_{p,q,k}, \mathbf{Z})$ more explicitly. When the level k is greater than 1, we have additional $\mathbf{Z}_k$ factor in the orbifold group. As is shown in (2.56), the generator of $\mathbf{Z}_k$ shifts both α and β cycle by $1/k$ of their periods. Because two cycles nowhere shrink at the same time, this action does not generate fixed points. The $\mathbf{Z}_k$ identification in the $\mathbf{T}^2$ fiber generates new cycles, which are not integral linear combinations of α and β. They are multiples of

$$\gamma = \frac{1}{k}(\alpha - \beta). \tag{5.27}$$

As a result, the 2-cycle defined as the product of α and β is not the fundamental $\mathbf{T}^2$ but its multiple $k\mathbf{T}^2$. Thus the cycles (5.17) are decomposed into k copies of the following elementary cycles.

$$[x_i, x_j]^{\alpha\gamma}, \quad [y_{i'}, y_{j'}]^{\alpha\gamma}, \quad [x_i, y_{j'}]^{\alpha\gamma}. \tag{5.28}$$

Due to this fact, the boundary of unwrapping 4-chains (5.24) and (5.20) are replaced by

$$\partial\bar{X}^\alpha_i = k(-q\mathbf{x}_i + \sum_{j'=1'}^{q'} \mathbf{y}_{j'}), \quad \partial\bar{Y}^\beta_{i'} = k(\sum_{j=1}^{p} \mathbf{x}_j - p\mathbf{y}_{i'}), \tag{5.29}$$

where we defined the formal basis $\mathbf{x}_i$ and $\mathbf{y}_{i'}$ by $[x_i, x_j]^{\alpha\gamma} = \mathbf{x}_i - \mathbf{x}_j$ and so on. These precisely correspond to the vectors $k\mathbf{v}_i$ and $k\mathbf{w}_{i'}$, and thus the homology H_3 becomes isomorphic to the quotient Γ/kH in (5.15).

5.3 Three-form torsion

In this subsection, we relate the fractional brane charge and integrals of the 3-form field on 3-cycles. Let us consider a process in which the number of the fractional branes changes. The fractional brane charge $Q \in H_3$ affects the 3-form field C_3 and measured by the integrals over 3 cycles ζ

$$\oint_\zeta C_3, \quad \zeta \in H_3. \tag{5.30}$$

We define the period integral at $r = r_0$ between the horizon $r = 0$ and AdS boundary $r = \infty$. To change the fractional brane charge by ΔQ, we add an M5-brane wrapped on a 3-cycle $\Delta Q \in H_3$ at AdS boundary, and move it to the horizon. When the M5-brane pass through $r = r_0$, the period integrals changes by

$$\Delta \oint_\zeta C_3 = 2\pi\langle\zeta, \Delta Q\rangle, \tag{5.31}$$

where $\langle *, *\rangle$ is a map $H_3 \times H_3 \to U(1)$, so called the torsion linking form, or, simply, the linking number.

The linking number is defined as follows. Let s be the order of ζ. Namely, s is the smallest positive integer such that $s\zeta$ is homologically trivial. Such an integer always exists because H_3 is pure torsion. There exists a 4-chain D such that

$$s\zeta = \partial D. \tag{5.32}$$

We define the linking number $\langle\zeta, \eta\rangle$ of two 3-cycles ζ and η by

$$\langle\zeta, \eta\rangle = \frac{1}{s}\langle\langle D, \eta\rangle\rangle. \tag{5.33}$$

where $\langle\langle D, \eta\rangle\rangle$ is the intersection number of the 4-chain C and 3-cycle η. Because this number jumps by integers by continuous deformations, only the fractional part of the linking number is a topological invariant.

If we move an M5-brane wrapped on the 3-cycle ΔQ from AdS boundary to the horizon, when it passes through $r = r_0$, the M5-brane intersect with the 4-chain D at $\langle\langle D, \eta\rangle\rangle$ points. In this process, the four-form flux G_4 passing through D, including the contribution of Dirac's string-like objects, changes by $2\pi\langle\langle D, \zeta\rangle\rangle$. By using Stokes' theorem we obtain the relation (5.31).

For the manifold $\mathbf{S}^7_{p,q,k}$, following the definition of the linking number, we can easily obtain

$$k\langle\mathbf{v}_i, \mathbf{j}\rangle = n_i, \quad k\langle\mathbf{w}_{i'}, \mathbf{j}\rangle = n_{i'}, \tag{5.34}$$

for a general 3-cycle $\mathbf{j}$ in (5.26). The linking numbers among the basis are

$$\langle\mathbf{x}_i, \mathbf{x}_j\rangle = -\frac{1}{kq}\delta_{ij}, \quad \langle\mathbf{y}_{i'}, \mathbf{y}_{j'}\rangle = -\frac{1}{kp}\delta_{i'j'}, \quad \langle\mathbf{x}_i, \mathbf{y}_{j'}\rangle = -\frac{1}{2kpq}. \tag{5.35}$$

Due to the constraint (5.3), the linking number among the basis is not unique. For example, a constant shift of all the linking numbers in (5.35) does not affect the linking numbers for 3-cycles which are linear combination of the basis with the coefficient constrained by (5.3).

By "integrating" the relation (5.31) and using (5.34), we obtain

$$n_i - n_i^0 = \frac{k}{2\pi}\oint_{\mathbf{v}_i} C_3, \quad n_{i'} - n_{i'}^0 = \frac{k}{2\pi}\oint_{\mathbf{w}_{i'}} C_3, \tag{5.36}$$

where n_i^0 and $n_{i'}^0$ are integration constants which cannot be determined from (5.31).

Although gauge transformations can change the period integrals of C_3, the relation (5.36) determines a element of Γ/kH in a gauge invariant way if we know n_i^0 and $n_{i'}^0$ because large gauge transformation change the charge vector by an element of kH.

An important fact is that the constants n_i and $n_{i'}$ depend on the ordering, the order of fivebranes. The right hand side of the relations (5.36) are defined on M-theory side, and is independent of the ordering, while n_i and $n_{i'}$ on the left hand side change by multiples of k when we change the order of fivebranes. This means that n_i^0 and $n_{i'}^0$ depends on the ordering, and we cannot simply set them to be zero.

To obtain some information about the constants, we use branes corresponding to baryonic operators. Remember that in the IIB setup baryonic operators correspond to D3-brane disks ending on fivebranes, and when $n_I \neq 0$, they are accompanied by n_I open strings.

A similar phenomenon occurs on the M-theory side. If there is non-trivial background C-field M5-branes wrapped on five-cycles are accompanied by M2-branes attached on their worldvolume, and by identifying these M2-branes to strings in the IIB setup, we obtain relations between n_I and background C-field.

Let us consider the flux conservation on M5-branes and how it relates the background C-field and M2-branes attached on it. The two-form field b_2 on M5-branes couples to the field strength G_4 in the bulk by the coupling

$$S = \frac{1}{2\pi}\int_{M5} b_2 \wedge G_4. \tag{5.37}$$

This implies that the flux behaves as charge on M5-branes. On the worldvolume of an M5-brane wrapped on a five-cycle the total charge coupled by b_2 must cancel due to the flux conservation. This implies that, the cohomology class of the total charge

$$\left[\frac{1}{2\pi}G_4 - \delta(\partial M2)\right] \in H^4(\Omega_I, \mathbf{Z}) \tag{5.38}$$

must be trivial. $\delta(\partial M2)$ is the four-form delta function with support on the boundaries of M2-branes. By the Poincare duality, this is equivalent to

$$[g] = [\partial M2] \in H_1(\Omega_I, \mathbf{Z}), \tag{5.39}$$

where g is the one-cycle Poincare dual to the flux $(2\pi)^{-1}G_4$. The homologies $H_i(\Omega_i, \mathbf{Z})$ in the five-cycle are given by

$$H_0 = \mathbf{Z}, \quad H_1 = \mathbf{Z}_k, \quad H_2 = \mathbf{Z}^{q-1}, \quad H_3 = \mathbf{Z}^{q-1} \oplus \mathbf{Z}_k, \quad H_4 = 0, \quad H_5 = \mathbf{Z}. \tag{5.40}$$

The homologies in $\Omega_{i'}$ are obtained by replacing q in (5.40) by p. Because $H_1(\Omega_I, \mathbf{Z}) = \mathbf{Z}_k$ is pure torsion we can rewrite (5.39) in terms of the linking form $H_3 \times H_1 \to U(1)$ as

$$\frac{1}{2\pi} \oint_\zeta C_3 = \langle \zeta, \partial M2 \rangle, \tag{5.41}$$

where ζ is the generator of the torsion subgroup of $H_3(\Omega_I, \mathbf{Z})$. It is $\zeta = \mathbf{v}_i$ for Ω_i and $\zeta = \mathbf{w}_{i'}$ for $\Omega_{i'}$. If we identify n_I strings ending on a D3-brane disk with a M2-brane wrapped on $n_I\gamma$ where γ is the generator of $H_1(\mathbf{S}^7_{p,q,k}, \mathbf{Z}) = H_1(\Omega_I, \mathbf{Z})$, (5.41) can be rewritten as

$$n_i = \frac{k}{2\pi} \oint_{\mathbf{v}_i} C_3, \quad n_{i'} = \frac{k}{2\pi} \oint_{\mathbf{w}_{i'}} C_3 \qquad \text{mod } k. \tag{5.42}$$

This means that

$$n_i^0 = n_{i'}^0 = 0 \quad \text{mod } k. \tag{5.43}$$

This fixes only the ordering independent part of n_i^0 and $n_{i'}^0$. Although in the $p = q = 1$ case this reproduces the result in [58] for ABJM model, this is not sufficient to establish the relation between the fractional brane charge and the 3-form torsion for $p + q \geq 3$. We leave this problem for future works.

Chapter 6

Wrapped M5-branes and baryonic operators

Baryons are also holographically realized in the M-theory dual.

6.1 Baryonic operators

In this chapter, the gauge group is G_{SU} with the same rank $N_a = N$. Then, as in the case four-dimensional quiver gauge theories, we can construct the following G_{SU} invariant operators

$$B_i^{A_1 A_2 \cdots A_N} = \epsilon_{i_1 \cdots i_N} \epsilon^{j_1 \cdots j_N} q_i^{A_1}{}^{i_1}{}_{j_1} \cdots q_i^{A_N}{}^{i_N}{}_{j_N}, \tag{6.1}$$

$$B_{i'}^{A'_1 A'_2 \cdots A'_N} = \epsilon_{i_1 \cdots i_N} \epsilon^{j_1 \cdots j_N} q_{i'}^{A'_1}{}^{i_1}{}_{j_1} \cdots q_{i'}^{A'_N}{}^{i_N}{}_{j_N}. \tag{6.2}$$

These operators are charged under the baryonic symmetry G_B. Therefore, they cannot be decomposed into mesonic operators, which are G_B neutral. We call them barionic operators. These operators have several remarkable properties.

1. Decomposability into mesons.

   Products of baryonic operators are neutral by combining diagonal monopole operators. For example,

   $$\prod_{i=1}^{p} B_i, \quad \prod_{i'=1}^{q'} B_{i'} \tag{6.3}$$

   carry the same baryonic charge as $e^{iN\tilde{a}}$ and $e^{-iN\tilde{a}}$, respectively. Multiplying the inverses of them, we can construct neutral operators with respect to the baryonic symmetries. This strongly suggests that such operators can be decomposed to the mesonic operators. For $k = 1$ case

   $$e^{-iN\tilde{a}} \prod_{i=1}^{p} B_i \sim b^N, \quad e^{iN\tilde{a}} \prod_{i'=1}^{q'} B_{i'} \sim \tilde{b}^N. \tag{6.4}$$

For general k case

$$e^{-iN\tilde{a}}\prod_{i=1}^{p}B_i^k \sim b^N, \quad e^{iN\tilde{a}}\prod_{i'=1}^{q'}B_{i'}^k \sim \tilde{b}^N, \tag{6.5}$$

$$\prod_{i=1}^{p}B_i^k \prod_{i'=1}^{q'}B_{i'}^k \sim \left(\mathrm{Tr}(\prod_{i=1}^{p} q_i \prod_{i'=1'}^{q'} q_{i'})\right)^N. \tag{6.6}$$

These decomposability suggests that the independent number of baryonic operators is $p+q-2$.

2. Degeneracy.

   Baryonic operators (6.1) and (6.2) have N $SU(2)_R$ indices and N $SU(2)'_R$ indices, respectively. In other words, each baryonic operator (6.1) or (6.2) forms the symmetric representation of $SU(2)_R$ or $SU(2)'_R$, respectively. This suggests that the degeneracy of each baryonic operators is $N+1$.

3. Conformal dimension.

   It can be easily seen that the conformal dimension of the elementary fields q is $\frac{1}{2}$ from the $N=4$ Charn-Simons lagrangean (2.14). Therefore, that of the baryonic operators is given by

$$D = \frac{N}{2}. \tag{6.7}$$

We can check that they are BPS operators. This is because they satisfy $\Delta = D - (j_3 + R) = 0$. Therefore, if there exist their counterparts in the M-theory dual, they are expected to reproduce these properties exactly. We confirm that the counterparts are M5-branes wrapped on 5-cycles.

6.2 Five-cycles

First, let us construct five-cycles. If we represent $\mathbf{S}^7_{p,q,k}$ as the $\mathbf{T}^2$ fibration over $B = \mathbf{S}^5$, the five-cycles can be written as the $\mathbf{T}^2$ fibrations over three-disks. For $k=1$ case,

$$\Omega_i = X_i^{\alpha\beta}, \quad \Omega_{i'} = Y_{i'}^{\alpha\beta}. \tag{6.8}$$

For $k \geq 2$ case,

$$\Omega_i = X_i^{\alpha\gamma}, \quad \Omega_{i'} = Y_{i'}^{\alpha\gamma}. \tag{6.9}$$

The notation is defined in §5.2. These generate the homology $H_5(\mathbf{S}^7_{p,q,1}, \mathbf{Z})$. A natural guess is that these five-cycles are dual to the baryonic operators in the Chern-Simons theory:

$$\Omega_i \leftrightarrow B_i^{\alpha_1\alpha_2\cdots\alpha_N}, \quad \Omega_{i'} \leftrightarrow B_{i'}^{\dot{\alpha}_1\dot{\alpha}_2\cdots\dot{\alpha}_N}. \tag{6.10}$$

We provide several evidences for this correspondence below.

1. Homology relation.

   Let us start $k = 1$ case. It is easy to check that their independent numbers match. The independent number of the cycles in (6.8) is given by the fifth Betti number $b_5(\mathbf{S}^7_{p,q,1}) = H_5(\mathbf{S}^7_{p,q,1}, \mathbf{Z}) = p+q-2$. This fact implies that the decomposability of products of baryons and homology relation are related. Indeed, the number $p + q - 2$ is smaller than that of naive counting of 5-cycles by two, and there should be two relations among the cycles in (6.8). Such homology relations are given by

$$\sum_{i=1}^{p} \Omega_i = \sum_{i'=1'}^{q'} \Omega_{i'} = 0. \tag{6.11}$$

   To see this, as we have done in section §5 for three-cycles, we can give these linear combinations as the boundaries of unwrapping 6-chains. We define a submanifold $\bar{B} \subset B$ by

$$\bar{B} = B \backslash \left(\sum_{i=1}^{p} X_i + \sum_{i'=1'}^{q'} Y_{i'} \right). \tag{6.12}$$

   If we could draw $\mathbf{S}^2$ enclosing x_i in $\bar{B}$, the α-cycle fiber would have non-trivial twist on the $\mathbf{S}^2$. However, such $\mathbf{S}^2$ do not exist in $\bar{B}$ because we removed the disks X_i. Thus the α-cycle fiber over $\bar{B}$ has trivial topology and we can define global sections. Similarly, thanks to the removal of $Y_{i'}$, the β-cycle fiber also have the trivial topology. Because there is a global section associated with α-cycle over $\bar{B}$, the manifold $\bar{B}^\beta$ is well-defined, and its boundary is

$$\partial \bar{B}^\beta = \sum_{i=1}^{p} X_i^{\alpha\beta}. \tag{6.13}$$

   We also obtain

$$\partial \bar{B}^\alpha = \sum_{i'=1}^{q'} Y_{i'}^{\alpha\beta}. \tag{6.14}$$

   As a result, we obtain the relations (6.11). These homology relations are nothing but the decomposition relations (6.4) in the gauge theory side.

   Next, let us consider the relation between baryonic operators and 5-cycle homology H_5 for $k \geq 2$. The $p + q$ generators (6.9) are not linearly independent, and we can take $\bar{B}^\alpha$, $\bar{B}^\beta$, and $\bar{B}^\gamma$ as unwrapping 6-chains which give the relation among these generators. Their boundaries are

$$\partial \bar{B}^\alpha = \sum_{i'=1'}^{q'} Y_{i'}^{\alpha\beta} = k \sum_{i'=1'}^{q'} \Omega_{i'}, \tag{6.15}$$

$$\partial \bar{B}^\beta = \sum_{i=1}^{p} X_i^{\alpha\beta} = k \sum_{i=1}^{p} \Omega_i, \tag{6.16}$$

$$\partial \bar{B}^\gamma = \sum_{i=1}^{p} X_i^{\gamma\beta} + \sum_{i'=1'}^{q'} Y_{i'}^{\gamma\alpha} = \sum_{i=1}^{p} \Omega_i + \sum_{i'=1'}^{q'} \Omega_{i'}. \tag{6.17}$$

Namely, these linear combinations of five-cycles are in trivial element of the homology H_5. By dividing the group $\mathbf{Z}^{p+q}$ generated by the $p+q$ basis Ω_i and $\Omega_{i'}$ by its subgroup $\mathbf{Z}^2$ generated by the above boundaries, we obtain the H_5 homology in (3.43). The first two, (6.15) and (6.16), correspond to the product of B_i and $B_{i'}$ in (6.5), respectively They are decomposed into N-th power of operators defined in (2.59). The third boundary (6.17) corresponds to the product of all the $p+q$ baryonic operators, and it can be decomposed into trace operators as in (6.6).

2. Degeneracy.

The degeneracy of each baryonic operator $N+1$ is also reproduced in the geometry side by considering the collective motion of a wrapped M5-brane. The collective coordinates of five-cycle Ω_i are the coordinates in the transverse direction $\mathbf{S}^2_A$, on which $SU(2)_U$ acts as rotation. The seven-form flux in the background plays a role of magnetic field on $\mathbf{S}^2_A$ and the amount of the flux is N. Therefore, the effective theory of the collective coordinates is the theory of a charged particle in $\mathbf{S}^2_A$ with N unit magnetic flux. The ground states of the particle are the $N+1$ states at the lowest Landau level [112] belonging to the spin $N/2$ representation of $SU(2)_U$. This degeneracy agrees with that of the baryonic operators B_i. In the same way, we can explain the degeneracy of $B_{i'}$ as that of the lowest Landau level of a charged particle in the transverse direction $\mathbf{S}^2_B$.

The degeneracy of baryonic operators for $p+q \geq 3$ are again reproduced in the same way as the $k=1$ case. In the case of $p=q=1$ (ABJM model), we need a special treatment because the global symmetry is enhanced to $SU(4)\times U(1)$ and the motion of collective coordinates are treated as a point particle in $SU(4)/(SU(3)\times U(1))$. This is considered in [69] and the correct multiplicity is obtained.

3. Mass of wrapped M5-branes.

As another non-trivial check of the duality, we compare the mass of the wrapped M5-branes and the conformal dimension of the operators. According to the standard AdS/CFT dictionary, the conformal dimension D of an operator and the mass M of the corresponding object are related by $D = R_{\mathrm{AdS}_4} M$. In the case of an M5-brane wrapped on Ω_i, this relation becomes

$$D = R_{AdS_4} T_{M5} R_{S^7}^5 \,\mathrm{Vol}(\Omega_i) = \frac{Npq}{2\pi^3} \mathrm{Vol}(\Omega_i). \tag{6.18}$$

where $\mathrm{Vol}(\Omega_i)$ is the volume of the 5-cycle Ω_i in $\mathbf{S}^7_{p,q,1}$ with radius 1, and to obtain the last expression we used (3.35) with $k = 1$ and the M5-brane tension $T_{M5} = 2\pi/(2\pi l_p)^6$. Let us calculate the volume of the 5-cycle. The 5-cycle Ω_i, which is represented as a fiber bundle over the segment $0 \le t \le 1$, is illustrated as the shaded region in Figure 6.1. The radii of two

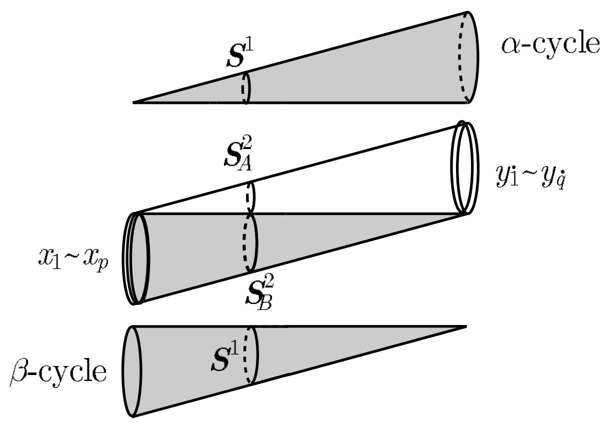

Figure 6.1: The shaded region is a nontrivial 5-cycle $X_a^{\alpha\beta}$.

3-spheres defined by (3.45) are $r_1 = t^{1/2}$ and $r_2 = (1-t)^{1/2}$, respectively. The cross-section at t is $\mathbf{S}^1 \times \mathbf{S}^2 \times \mathbf{S}^1$ with their radii r_1/p, $r_2/2$, r_2/q, respectively.[1] Hence the volume of the 5-cycle is

$$\mathrm{Vol}(\Omega_i) = \int_{t=0}^{t=1} ds \left(\frac{2\pi r_1}{p}\right) \times \left(4\pi \left(\frac{r_2}{2}\right)^2\right) \times \left(\frac{2\pi r_2}{q}\right) = \frac{\pi^3}{pq}, \tag{6.19}$$

where ds is the line element with respect to the parameter t computed as

$$ds^2 = dr_1^2 + dr_2^2 = \frac{1}{4t(1-t)} dt^2. \tag{6.20}$$

(The volume (6.19) is simply $\mathrm{Vol}(\mathbf{S}^5)/pq$ because the five-cycles considered here are orbifolds of large $\mathbf{S}^5$ in $\mathbf{S}^7$.) We obtain the same result for 5-cycles

[1] It is known that when a unit $\mathbf{S}^3$ is represented by the $\mathbf{S}^1$ fibration over $\mathbf{S}^2$, the radii of $\mathbf{S}^1$ and $\mathbf{S}^2$ are 1 and 1/2 respectively.

$Y_{i'}^{\alpha\beta}$. By substituting this into (6.18) we obtain

$$D = \frac{1}{2}N, \tag{6.21}$$

and this agrees with the conformal dimension of the baryonic operators (6.1) and (6.2). (6.21) is consistent with the result of more general analysis in [113] for generic toric tri-Sasakian manifolds.

We can also easily check that the volume of the five-cycles correctly reproduce the conformal dimension $D = N/2$ for $k \geq 2$ case.

6.3 Quark-baryon transition

We can relate the relation between wrapped M5-branes and baryonic operators more directly by using IIB/M duality. By following the duality, we can easily see that an M5-brane wrapped on Ω_I is dual to a D3-brane disk ending on fivebrane I, and as we explain below, the D3-brane disk can be continuously deformed to N open strings corresponding to the constituent bi-fundamental quarks. Similar transition in different brane systems are also considered in [114, 115].

Before we explain the deformation, we comment on a relevant fact about flux conservation on the worldvolume of a D3-brane ending on an NS5-brane. The $U(1)$ gauge field A on an NS5-brane electrically couples to endpoints of D-strings on the NS5-brane. This is the case, too, for magnetic flux $f = da$ on D3-branes, which can be regarded as D-strings dissolved in the D3-brane worldvolume. This coupling is described as the action

$$S = \frac{1}{2\pi} \oint_{\partial D3} A \wedge f. \tag{6.22}$$

By integrating by part, this is rewritten as

$$S = \frac{1}{2\pi} \oint_{\partial D3} a \wedge F, \tag{6.23}$$

and this implies that the flux $F = dA$ on the NS5-brane behaves as an electric charge on the boundary of the D3-brane coupled by the gauge field a. If the D3-brane worldvolume is compact, the electric flux conservation requires the total electric charge vanish. If the integral of flux F over the D3-brane boundary is $2\pi N$, we need N strings ending on the D3-brane worldvolume to compensate the boundary charge. This is also the case for a D3-brane ending on a $(1, k)$ fivebrane.

Bearing this fact in mind, we can show that N open strings and a D3-brane disk can be continuously deformed to each other. In the following we treat three sets of D3-branes, and for distinction we name them as follows:

- X – the coincident N D3-branes between fivebranes I and $I-1$.
- Y – the coincident N D3-branes between fivebranes I and $I+1$.
- D – a D3-brane disk whose boundary is $\mathbf{S}^2$ on fivebrane I.

We here assume that $N_{L(I)} = N_{R(I)} = N$. Let us start from a D3-brane disk D whose boundary is $\mathbf{S}^2$ on the fivebrane I enclosing the both boundaries of X and Y. ((a) in Figure 6.2) Although these boundaries carry magnetic charges

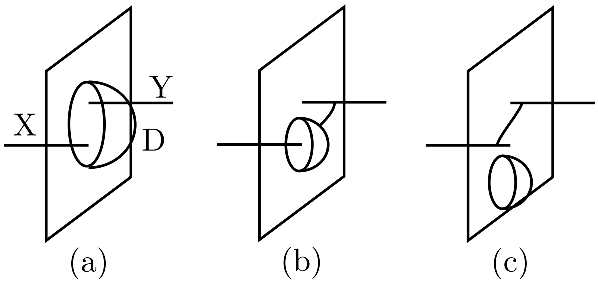

Figure 6.2: Quark-baryon transition

coupled by A, their charges cancel each other, and the net flux passing through the boundary ∂D is zero. There are no open strings ending on D.

We move the disk so that ∂Y, the boundary of Y, gets out of ∂D. When ∂Y passes through ∂D, the flux through ∂D jumps by N, and N open strings stretched between Y and D are generated so that the total electric charge on the disk cancels. ((b) in Figure 6.2)

If we keep moving the disk and ∂X also gets out of the boundary ∂D, the flux through the boundary jumps again by $-N$, and this time N open strings stretched between D and X are generated. Two sets of N strings can be connected to get off from D, and we obtain N open strings connecting X and Y. ((c) in Figure 6.2) The disk can annihilate without any obstructions.

If $n_I = N_{L(I)} - N_{R(I)} \neq 0$, the D3-brane disk D in Figure 6.2 (a) is accompanied by n_I strings attached on it. This corresponds to the fact that we cannot define such $SU(N_{L(I)}) \times SU(N_{R(a)})$ invariant operators as (6.1) and (6.2) due to the mismatch of the number of indices. The n_I open strings attached on the D3-brane disk corresponds to n_I fundamental or $-n_I$ anti-fundamental indices which are not contracted.

Chapter 7

Monopole and baryonic operators

Monopoles and baryons are dual to each other in the M-theory dual.

In this chapter, we discuss the relation between monopole operators and baryonic operators. In §4, we consider that the gauge group is the product of unitary groups. As a result, we can construct gauge invariant monopole operators. Instead, baryonic operators are all gauge variant. On the other hand, in §6 we take the gauge group as the product of special unitary groups to construct gauge invariant baryonic operators. In the result, monopole operators are gauge variant. In other words, we cannot construct both a monopole operator and a baryonic one for each gauge group in the gauge invariant way. This fact suggests that all the operators corresponding to wrapped M-branes cannot be gauge invariant.

Let us discuss it from the dual geometry side. For this purpose, It is helpful to see the reason why symmetry groups which act on wrapped branes are usually regarded as global symmetries. Consider AdS_{d+1} with the metric

$$ds^2 = \frac{R^2}{z^2}((dx^\mu)^2 + dz^2), \quad \mu = 1, \dots, d, \tag{7.1}$$

and let $A_\nu(x^\mu, z)$ be a $U(1)$ gauge field coupling to wrapped branes. We follow [116] and consider the Euclidian AdS space. z is the radial coordinate such that AdS boundary is at $z = 0$. Let us assume the asymptotic behavior of the vector field as

$$A_\nu \propto z^D. \tag{7.2}$$

For the convergence of the Euclidian action, D must satisfy the inequality

$$\frac{d}{2} - 2 < D. \tag{7.3}$$

With the equation of motion $d * dA = 0$ we obtain the asymptotic behavior of the gauge field

$$A_\nu(x^\mu, z) = a_\nu(x^\mu) + z^{d-2} b_\nu(x^\mu). \tag{7.4}$$

On AdS boundary we need to impose boundary condition which fixes one of $a_\nu(x^\mu)$ and $b_\nu(x^\mu)$. When $d \geq 4$, only the second term in (7.4) is allowed by (7.3) and the boundary condition $a_\nu(x^\mu) = 0$ must be imposed. Then the gauge field asymptotically vanishes near the boundary, and this is the reason why the symmetry is global in the boundary CFT.

On the other hand, when $d = 3$, both terms in (7.4) satisfy the inequality (7.3), and we can choose any one of $a_\nu(x^\mu) = 0$ (Dirichlet) and $b_\nu(x^\mu) = 0$ (Neumann) as the boundary condition. Indeed, these two boundary conditions first appeared in [117]. They are used in [118] to construct a pair of Chern-Simons theories which are "S-dual" to each other. Let us take the Neumann boundary condition. In this case, the boundary value of the gauge field $a_\nu(x^\mu) = A_\nu(x^\mu, z = 0)$ does not vanish. The gauge field is dynamical in the sense that it is path integrated. Thus we can regard this as a gauge field in the boundary CFT. The charged objects coupled by A_ν should be also charged in the boundary CFT. Because the Dirichlet and Neumann boundary conditions are exchanged by the duality transformation of the gauge field, both kinds of operators corresponding to electric and magnetic particles in AdS_4 cannot be gauge invariant.

In the M-theory dual, this S-duality transformation is nothing but electro-magnetic duality, which exchanges M2-branes and M5-branes. In a case of taking the gauge group as the unitary groups, the charged objects coupled by the dynamical gauge field is realized by wrapped M5-branes. The gauge fields $A^I = A^I_\nu dx^\nu$ originate in 6-form potential. As the gauge fields $\widetilde{A}_I$ in (4.251), they appear as the coefficient fields of the harmonic expansion of 6-form field:

$$C_6 = \sum_{I=1}^{p+q-2} \widetilde{\omega}_I \wedge A^I, \tag{7.5}$$

where $\widetilde{\omega}_I$ are the harmonic 5-forms in $H^5(\mathbf{S}^7_{p,q,k}, \mathbf{Z}) = \mathbf{Z}^{p+q-2}$. Since C_3, ω_I and $C_6, \widetilde{\omega}_I$ are dual to each other, respectively, the pair of the gauge fields A^I and $\widetilde{A}_I$ are exchanged under the duality transformation. This means that the Dirichlet and Neumann boundary conditions are exchanged under the duality transformation. Therefore, it is impossible to impose the Dirichlet boundary condition on all of them. Consequently, some of wrapped branes correspond to gauge variant operators. Although it may be possible to take some S-dual picture in which wrapped M5-branes correspond to gauge invariant operators, by taking the gauge group as special unitary groups, then we have to relate wrapped M2-branes to gauge variant operators.

Chapter 8

Summary and discussion

In this thesis, we investigated aspects of $\mathcal{N} = 4$ Chern-Simons theories and their gravity duals. We summarize our results and discuss future problems.

In Chapter 2, we made a review on aspects of $\mathcal{N} = 4$ Chern-Simons theories. Compared to the ABJM model, this model has more gauge groups or nodes as a quiver gauge theory. As a result, there are more monopole operators and baryonic operators. We have seen that they produce rich phenomena in $\mathcal{N} = 4$ Chern-Simons theories.

It is known that there are several ways to generalize quiver gauge theories. Another is so-called brane tilings [53, 54, 31]. In such models, a simple prescription to establish the relation between toric data of Calabi-Yau 4-folds and Chern-Simons theories are investigated [55, 56, 41]. It may be interesting to extend our analysis to such a large class of theories.

In general, dual CFTs of toric Calabi-Yau 4-folds cannot be described by brane tilings. In such a case, brane crystals [115, 119, 120] are expected to play an important role. The relation between brane crystals and dual CFT are not fully understood. We expect that such models may be helpful to obtain some information about dual CFT.

In Chapter 3, we investigated gravity duals of $\mathcal{N} = 4$ Chern-Simons theories. As in the four-dimensional case, AdS/CFT duality is also a powerful tool to study strongly-coupled gauge theories in three dimension. It has been more than ten years that AdS/CFT duality has been discovered, but it does not show any sign of slowing down.

In Chapter 4, we confirmed the agreement between a gauge theory index and an AdS_4 multi-particle index. This result strongly suggests that the monopole operators in the twisted sector correspond to wrapped M2-branes on 2-cycles in the internal space.

In the dual of $\mathcal{N} = 4$ models, such 2-cycles come from orbifold singularities, so they are vanishing cycles. This fact suggests that wrapped M2-branes on them preserve supersymmetries. Indeed, we described the collective motion of such wrapped M2-branes by supersymmetric vector-multiplets. The KK modes of them, which form 1/2 BPS $OSp(4|4)$ multiplets, correctly reproduce the spectrum of twisted BPS monopole operators in the gauge theory side.

Let us remark that the exact agreement by using $\mathcal{N} = 4$ models cannot be expected in more generic Chern-Simons theories. This is because a 2-cycle in a generic $\mathcal{N} = 2$ model has its volume and wrapped M2-branes do not preserve a supersymmetry any more [121]. In a general Sasaki-Einstein manifold, such wrapped M2-branes are investigated [122]. They construct supergravity solutions with taking account of the back-reaction of wrapped M2-branes, which we ignored in this analysis. The solutions do not preserve any supersymmetry.

This situation induces an interesting phenomenon. According to the volume of 2-cycles, wrapped M2-branes get their mass. It behaves the order of $N^{1/2}$. From AdS/CFT correspondence, a conformal dimension of a twisted monopole operator can be expected to behave in the same way in such a generic model.

To address such an issue, the localization technique will be also an important skill. As in the computation of indices, the technique enables us to reduce calculation in the strongly coupled region to saddle point method. Indeed, partition functions in several models have been also calculated by using localization technique [123, 124, 125]. As a result, it has been shown that the free energy behaves as the order of $N^{3/2}$, which agrees with that of entropy of membranes in $D = 11$ supergravity. Due to success of the computation of exact partition functions, the exact R-symmetry can be also addressed by extremizing them [126, 127].

It is important to understand the spectrum of more general Chern-Simons theories such as $\mathcal{N} = 2$ theories to understand dynamics of Chern-Simons theories and establish the dual M2-brane description. Even though it is in general difficult to compute the spectrum on the gauge theory side due to large quantum corrections, it seems possible to extend the analysis of $\mathcal{N} = 4$ Chern-Simons theories to $\mathcal{N} = 2$ theories describing M2-branes in orbifold backgrounds. In such a case, the internal space of the dual geometry in general includes many two-cycles and has complicated torsion 4-form cohomology. The comparison of monopole operator spectrum and Kaluza-Klein spectrum in such a model would be useful for the identification of discrete torsion for a given Chern-Simons theory.

In Chapter 5, we studied wrapped M5-branes wrapped on three-cycles in the internal space. They become fractional D3-branes and realize the difference of ranks of the adjacent gauge groups. They are useful to classify the class of $\mathcal{N} = 4$ Chern-Simons theories by whether or not they share the same infra-red fixed point or Seiberg dual to each other. In three-dimension, there exists another type of duality, so-called mirror duality [128, 61, 129]. It is interesting to study the mirror duality including Chern-Simons interaction with arbitrary Chern-Simons levels [130, 131, 132].

In Chapter 6, we studied wrapped M5-branes wrapped on three-cycles in the internal space. Their counterparts are baryonic operators. We checked that several properties agree between both objects. They are also useful to study strongly-coupled aspects of superconformal Chern-Simons theories as in four dimensional case.

In Chapter 7, we discussed the relation between monopole and baryonic operators. We cannot make both operators gauge invariant. This means that all the operators corresponding to wrapped M-branes cannot be constructed in the gauge invariant way. To put it the other way around, the gauge variant operators have their counterpart in the dual geometry.

In §4.3.3, we derived the spectra of localized modes on the singular loci in Table 4.1. Interestingly, for a small s there are two normalized modes. This fact suggests that there are two appropriate boundary conditions on AdS_4. Indeed, we discussed that both Dirichlet and Neumann boundary conditions can be acceptable. The boundary condition of the gauge field determines whether $U(1)$ symmetry is global or local [118, 133]. The mode satisfying Neumann one belongs to a so-called Betti multiplet [134, 135]. It associates baryonic $U(1)$ symmetries in the Chern-Simons theory. In this paper, we chose the boundary condition which makes the $U(1)$ symmetries local. As a result, monopole operators arose in the theory as dynamical objects and baryonic operators are gauge variant [46]. Taking the opposite boundary condition corresponds to the $SU(N)$ gauge symmetries rather than $U(N)$s. It may be interesting to study the relation between the boundary conditions for the Betti multiplets and corresponding boundary theories in more detail. It is useful to draw a lesson by studying it in more general cases [69, 136, 121].

We hope that to solve these issues leads to further development in Chern-Simons theories and M-theory.

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Appendix A

Convention

The signature of three-dimensional Minkowuski metric is

$$\eta_{\mu\nu} = \mathrm{diag}(-,+,+). \tag{A.1}$$

$SO(1,2)$ oscillator matrices should satisfy

$$\gamma^\mu \gamma^\nu = \eta^{\mu\nu} + \gamma_\rho \varepsilon^{\mu\nu\rho}, \quad \varepsilon^{012} = 1. \tag{A.2}$$

We realize $SO(1,2)$ oscillator matrices as real symmetric traceless 2×2 matrices

$$(\gamma^0_{\alpha\beta}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\gamma^1_{\alpha\beta}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\gamma^2_{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.3}$$

Charge conjugation matrix is realized by real antisymmetric 2×2 matrix

$$(C_{\alpha\beta}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (-\varepsilon_{\alpha\beta}), \quad (C^{-1\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\varepsilon^{\alpha\beta}) = (-C^{\alpha\beta}). \tag{A.4}$$

Our rule to raise or lowering spinor indices is given by

$$\psi_\alpha = C_{\alpha\beta}\psi^\beta, \quad (C^{-1})^{\alpha\beta}\psi_\beta = \psi^\alpha = \psi_\beta C^{\beta\alpha}. \tag{A.5}$$

Our contraction rule is the southwest-northeast type.

$$\psi\chi = \psi_\alpha \chi^\alpha. \tag{A.6}$$

For a bilinear of the same spinor, we define

$$\psi^2 = \frac{1}{2}\psi_\alpha \psi^\alpha. \tag{A.7}$$

Appendix B

$\mathcal{N}_{3D} = 2$ formulation

B.1 $\mathcal{N}_{3D} = 2$ superspace

For text books of $\mathcal{N}_{3D} = 2$ formulation, see [137, 138]. Let us consider $\mathcal{N} = 2$ supersymmetry variation

$$\delta_\epsilon x^\mu := \epsilon_a \gamma^\mu \theta^a, \quad \delta_\epsilon \theta^a := \epsilon^a. \tag{B.1}$$

We take the convention of complex conjugation so that $\delta_\epsilon x^\mu$ are real. We denote supercharges as Q^a_α:

$$\delta_\epsilon := \epsilon_a Q^a = \epsilon_{a\alpha} Q^{a\alpha}, \tag{B.2}$$

and grassmannian derivative as ∂^a_α:

$$\epsilon_a \partial^a(\theta^b) := \epsilon^b \Leftrightarrow \partial^{a\alpha}(\theta^{b\beta}) = \delta^{ab} C^{-1\beta\alpha}. \tag{B.3}$$

Then (B.1) leads to

$$Q^\alpha_a = \partial^\alpha_a + \partial^\alpha{}_\beta \theta^\beta_a, \tag{B.4}$$

where $\partial^\alpha{}_\beta$ means $\partial^\alpha{}_\beta = \gamma^{\mu\alpha}{}_\beta \partial_\mu$. Since there are two supersymmetry generators, we can combine them into complex variables:

$$\theta = \frac{1}{\sqrt{2}}(\theta^1 + i\theta^2), \quad \bar\theta = \frac{1}{\sqrt{2}}(\theta^1 - i\theta^2) \tag{B.5}$$

$$\partial = \frac{1}{\sqrt{2}}(\partial^1 - i\partial^2), \quad \bar\partial = \frac{1}{\sqrt{2}}(\partial^1 + i\partial^2) \tag{B.6}$$

$$Q = \frac{1}{\sqrt{2}}(Q^1 - iQ^2), \quad \bar Q = \frac{1}{\sqrt{2}}(Q^1 + iQ^2) \tag{B.7}$$

By using these definitions, we find

$$Q^\alpha = \partial^\alpha - (\bar\theta\partial)^\alpha, \quad \bar Q^\alpha = \bar\partial^\alpha - (\theta\partial)^\alpha. \tag{B.8}$$

Algebraic relations are given

$$\{Q_\alpha, \bar Q_\beta\} = -2\partial_{\alpha\beta}, \quad \{Q_\alpha, Q_\beta\} = \{\bar Q_\alpha, \bar Q_\beta\} = 0. \tag{B.9}$$

By using $U = \theta\partial\bar{\theta}$, we can write $Q, \bar{Q}$ as

$$Q_\alpha = e^{-U}\partial_\alpha e^{U}, \quad \bar{Q}_\alpha = e^{U}\bar{\partial}_\alpha e^{-U}. \tag{B.10}$$

B.2 Chiral superfield

A chiral superfield, Q, is defined by

$$\bar{D}_\alpha Q = 0. \tag{B.11}$$

Here we define

$$D_\alpha := \partial_\alpha + (\bar{\theta}\partial)_\alpha, \quad \bar{D}_\alpha := \bar{\partial}_\alpha + (\theta\partial)_\alpha. \tag{B.12}$$

As a result,

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_\beta\} = \{\bar{D}_\alpha, Q_\beta\} = \{\bar{D}_\alpha, \bar{Q}_\beta\} = 0. \tag{B.13}$$

By using $U = \theta\partial\bar{\theta}$, we can write $D, \bar{D}$ as

$$D_\alpha = e^{U}\partial_\alpha e^{-U}, \quad \bar{D}_\alpha = e^{-U}\bar{\partial}_\alpha e^{U}. \tag{B.14}$$

Therefore a chiral superfield is written as

$$Q = e^{-U}Q_0(x, \theta). \tag{B.15}$$

$Q_0(x, \theta)$ can be expanded by θ as

$$Q_0(x, \theta) = q(x) + \theta\psi(x) - 2\theta^2 F(x). \tag{B.16}$$

Due to $\bar{D}$ commuting with δ_ϵ, $\delta_\epsilon Q$ is also a chiral superfield.

$$\delta_\epsilon Q = e^{-U}(\Delta_\epsilon q(x) + \theta\Delta_\epsilon\psi(x) - 2\theta^2\Delta_\epsilon F(x)). \tag{B.17}$$

This reduces to

$$\Delta_\epsilon q = \epsilon\psi, \tag{B.18}$$

$$\Delta_\epsilon\psi_\alpha = 2(\bar{\epsilon}_\beta\partial^\beta{}_\alpha q - \epsilon_\alpha F), \tag{B.19}$$

$$\Delta_\epsilon F = \bar{\epsilon}\partial\psi. \tag{B.20}$$

This is nothing but the supersymmetry transformation of a chiral superfield.

B.3 Vector superfield

Let us couple a chiral superfield to a gauge field in a supersymmetric way. Gauge transformation of a fundamental chiral superfield, Q, is given by

$$Q^{\Lambda} = e^{\Lambda} Q \Leftrightarrow \Delta_{\Lambda} Q = \Lambda Q. \tag{B.21}$$

To keep Q^{Λ} as a chiral superfield, the gauge parameter Λ has to be also a chiral superfield. In order to make the canonical kinetic energy of Q, $Q^{\dagger}Q$, gauge-invariant, we need a new superfield, V. This is because a chiral superfield takes values in complex number. The finite gauge transformation of V is given by

$$(e^{V})^{\Lambda} = e^{-\Lambda^{\dagger}} e^{V} e^{-\Lambda}. \tag{B.22}$$

The gauge-invariant kinetic energy is written as $Q^{\dagger} e^{V} Q$. Reality condition requires the superfield V as real superfield: $V^{\dagger} = V$. The infinitesimal gauge transformation is

$$\Delta_{\Lambda} V = -(\Lambda + \Lambda^{\dagger} + \frac{1}{2}[V, \Lambda - \Lambda^{\dagger}] + \cdots). \tag{B.23}$$

By using this gauge degrees of freedom, we can set the following gauge-fixing condition, so-called WZ gauge,

$$V| = DV| = \bar{D}V| = D^2 V| = \bar{D}^2 V| = 0. \tag{B.24}$$

Here $|$ means that we set to $\theta = \bar{\theta} = 0$. Higher grassmannian derivatives are component fields.

$$\widetilde{A}_{\alpha\beta} = \frac{1}{2}\bar{D}_{(\alpha} D_{\beta)} V|, \quad \sigma = -\frac{1}{4} D_{\alpha}\bar{D}^{\alpha} V|, \tag{B.25}$$

$$\lambda_{\alpha} = \frac{1}{4} D_{\alpha}\bar{D}^2 V|, \quad \bar{\lambda}_{\alpha} = \frac{1}{4}\bar{D}_{\alpha} D^2 V|, \quad D = \frac{1}{8} D_{\alpha}\bar{D}^2 D^{\alpha} V|. \tag{B.26}$$

Supersymmetry variation breaks the gauge-fixing condition. However, combining gauge degrees of freedom, we can define a modified supersymmetry variation, which preserves WZ gauge.

$$\Delta_{\epsilon}^{WZ} := \Delta_{\epsilon} + \Delta_{\Lambda}, \tag{B.27}$$

where the gauge parameter is fixed as

$$\Lambda| = 0, \quad \Lambda^{\dagger}| = 0, \quad D^2\Lambda| = 4\bar{\epsilon}\bar{\lambda}, \quad \bar{D}^2\Lambda^{\dagger}| = 4\epsilon\lambda, \tag{B.28}$$

$$D^{\alpha}\Lambda| = \bar{\epsilon}_{\beta} 2(A^{\beta\alpha} + C^{\beta\alpha}\sigma), \quad \bar{D}^{\alpha}\Lambda^{\dagger}| = \epsilon_{\beta} 2(-A^{\beta\alpha} + C^{\beta\alpha}\sigma). \tag{B.29}$$

Supersymmetry transformation is obtained as

$$
\begin{aligned}
\Delta_\epsilon^{WZ} A_{\alpha\beta} &= 2(\epsilon_{(\alpha}\bar{\lambda}_{\beta)} - \bar{\epsilon}_{(\alpha}\lambda_{\beta)}), && \text{(B.30)}\\
\Delta_\epsilon^{WZ} \sigma &= (\epsilon\bar{\lambda} + \bar{\epsilon}\lambda), && \text{(B.31)}\\
\Delta_\epsilon^{WZ} \lambda_\alpha &= \epsilon_\gamma(\frac{1}{2}[\not{D}^\gamma{}_\beta, \not{D}^\beta{}_\alpha] + [\not{D}^\gamma{}_\alpha, \sigma]) - \epsilon_\alpha D, && \text{(B.32)}\\
\Delta_\epsilon^{WZ} \bar{\lambda}_\alpha &= \bar{\epsilon}_\gamma(-\frac{1}{2}[\not{D}^\gamma{}_\beta, \not{D}^\beta{}_\alpha] + [\not{D}^\gamma{}_\alpha, \sigma]) - \bar{\epsilon}_\alpha D, && \text{(B.33)}\\
\Delta_\epsilon^{WZ} D &= \epsilon_\gamma[\not{D}^\gamma{}_\beta, \bar{\lambda}^\beta] + \bar{\epsilon}_\gamma[\not{D}^\gamma{}_\beta, \lambda^\beta] + \epsilon[\bar{\lambda}, \sigma] + \bar{\epsilon}[\sigma, \lambda], && \text{(B.34)}
\end{aligned}
$$

where the covariant derivative $\not{D}_{\alpha\beta}$ is defined by

$$\not{D}_{\alpha\beta} = \partial_{\alpha\beta} + \widetilde{A}_{\alpha\beta}. \qquad \text{(B.35)}$$

Here $\widetilde{A}_{\alpha\beta}$ is anti-hermitian. From a simple calculation, we obtain

$$[\not{D}_{\alpha\gamma}, \not{D}^\gamma{}_\beta] = \gamma^{\mu\nu}_{\alpha\beta}\widetilde{F}_{\mu\nu} = \varepsilon^{\mu\nu\rho}\gamma_{\rho\alpha\beta}\widetilde{F}_{\mu\nu}. \qquad \text{(B.36)}$$

By using this equation, we get

$$
\begin{aligned}
\Delta_\epsilon^{WZ} \lambda_\alpha &= \epsilon_\gamma\gamma_\rho{}^\gamma{}_\alpha(\frac{1}{2}\varepsilon^{\mu\nu\rho}\widetilde{F}_{\mu\nu} + [D^\rho, \sigma]) - \epsilon_\alpha D, && \text{(B.37)}\\
\Delta_\epsilon^{WZ} \bar{\lambda}_\alpha &= \bar{\epsilon}_\gamma\gamma_\rho{}^\gamma{}_\alpha(-\frac{1}{2}\varepsilon^{\mu\nu\rho}\widetilde{F}_{\mu\nu} + [D^\rho, \sigma]) - \bar{\epsilon}_\alpha D. && \text{(B.38)}
\end{aligned}
$$

For a chiral superfield,

$$
\begin{aligned}
\Delta_\epsilon^{WZ} q &= \epsilon\psi, && \text{(B.39)}\\
\Delta_\epsilon^{WZ} \psi_\alpha &= 2(\bar{\epsilon}_\beta(\not{D}^\beta{}_\alpha + \delta^\beta{}_\alpha\sigma)q - \epsilon_\alpha F), && \text{(B.40)}\\
\Delta_\epsilon^{WZ} F &= \bar{\epsilon}(\not{D} + \sigma)\psi + 2\bar{\epsilon}\bar{\lambda}q. && \text{(B.41)}
\end{aligned}
$$

For an anti-chiral superfield,

$$
\begin{aligned}
\Delta_\epsilon^{WZ} q^\dagger &= \bar{\epsilon}\psi^\dagger, && \text{(B.42)}\\
\Delta_\epsilon^{WZ} \psi_\alpha^\dagger &= 2(\epsilon_\beta(\not{D}^\beta{}_\alpha q^\dagger + \delta^\beta{}_\alpha q^\dagger\sigma) - \bar{\epsilon}_\alpha F^\dagger), && \text{(B.43)}\\
\Delta_\epsilon^{WZ} F^\dagger &= \epsilon(\not{D}\psi^\dagger + C\psi^\dagger\sigma) + 2q^\dagger\epsilon\lambda. && \text{(B.44)}
\end{aligned}
$$

B.4 Supersymmetric Lagrangians

We define the grassmann integral as

$$\int d^2\theta = \frac{1}{2}\partial^2, \quad \int d^4\theta = \frac{1}{4}\bar{\partial}^2\partial^2. \qquad \text{(B.45)}$$

Note that

$$\partial^2\theta^2 = -1. \qquad \text{(B.46)}$$

Supersymmetric Lagrangians are given as follows.

1. Kinetic term of an $\mathcal{N}_{3D}=2$ vector-multiplet $v=(\widetilde{A}_\mu,\sigma,\lambda,D)$ (Yang-Mills term)

$$\begin{aligned} -\mathcal{L}_v^{YM} &= \mathrm{Tr}\int d^2\theta\,(-F^2) &\text{(B.47)}\\ &= \mathrm{Tr}\left[\lambda^\dagger[\not{D}+\sigma,\lambda]+\frac{1}{4}\widetilde{F}_{\mu\nu}\widetilde{F}^{\mu\nu}-\frac{1}{2}[D_\mu,\sigma]^2+\frac{1}{2}D^2\right], &\text{(B.48)}\end{aligned}$$

where F_α is given by

$$F_\alpha=\frac{1}{4}\bar{D}^2e^{-v}[D_\alpha,e^v]. \tag{B.49}$$

2. Mass term of an $\mathcal{N}_{3D}=2$ vector-multiplet v (Chern-Simons term)

$$\begin{aligned} -\mathcal{L}_v^{CS} &= -\frac{1}{4}\mathrm{Tr}\int_0^1 dt\int d^4\theta\,(-v)\bar{D}_\alpha(e^{-tv}D^\alpha e^{tv}) &\text{(B.50)}\\ &= -\mathrm{Tr}\left[-\varepsilon^{\mu\nu\rho}\left(\frac{1}{2}\widetilde{A}_\mu\partial_\nu\widetilde{A}_\rho+\frac{1}{3}\widetilde{A}_\mu\widetilde{A}_\nu\widetilde{A}_\rho\right)+\bar{\lambda}\lambda+D\sigma\right]. &\text{(B.51)}\end{aligned}$$

3. Kinetic term of a fundamental chiral multiplet $Q=(q,\psi,F_q)$

$$\begin{aligned} -\mathcal{L}_Q^F &= \mathrm{Tr}\int d^4\theta\,Q^\dagger e^vQ &\text{(B.52)}\\ &= \mathrm{Tr}\Big[-D_\mu q^\dagger D^\mu q+\frac{1}{2}\bar{\psi}(\not{D}+\sigma)\psi\\ &\qquad +\bar{\psi}\lambda^\dagger q+q^\dagger\lambda\psi+q^\dagger(D-\sigma^2)q+F_q^\dagger F_q\Big]. &\text{(B.53)}\end{aligned}$$

4. Kinetic term of a bifundamental chiral multiplet Q. Q is fundamental for G_h and anti-fundamental for G_t.

$$\begin{aligned} -\mathcal{L}_Q^B &= \mathrm{Tr}\int d^4\theta\,Q^\dagger e^{v_h}Qe^{-v_t} &\text{(B.54)}\\ &= \mathrm{Tr}\Big[-D_\mu q^\dagger D^\mu q+\frac{1}{2}\psi^\dagger\not{D}\psi+\frac{1}{2}(\psi^\dagger\sigma_h\psi+\psi\sigma_t\psi^\dagger)\\ &\qquad +(\psi^\dagger{\lambda_h}^\dagger q+q^\dagger\lambda_h\psi)+q^\dagger(D_h-\sigma_h\sigma_h)q\\ &\qquad -(q{\lambda_t}^\dagger\psi^\dagger+\psi\lambda_tq^\dagger)+q(-D_t-\sigma_t\sigma_t)q^\dagger\\ &\qquad +q^\dagger\sigma_tq\sigma_h\Big]. &\text{(B.55)}\end{aligned}$$

5. Kinetic term of an adjoint chiral multiplet $\Phi=(\phi,\chi,F_\phi)$

$$\begin{aligned} -\mathcal{L}_\Phi^A &= \mathrm{Tr}\int d^4\theta\,\Phi^\dagger e^v\Phi e^{-v} &\text{(B.56)}\\ &= \mathrm{Tr}\Big[-D_\mu\phi^\dagger D^\mu\phi+\frac{1}{2}\bar{\chi}[(\not{D}+\sigma),\chi]+F_\phi^\dagger F_\phi\\ &\qquad +\bar{\chi}[\lambda^\dagger,\phi]+\phi^\dagger[\lambda,\chi]+\phi^\dagger[D,\phi]-\phi^\dagger[\sigma,[\sigma,\phi]]\Big]. &\text{(B.57)}\end{aligned}$$

6. Mass term of an adjoint chiral multiplet Φ

$$-\mathcal{L}_{\Phi}^{mass} = -\mathrm{Tr}\int d^2\theta\,\frac{1}{2}\Phi^2 = -\mathrm{Tr}\left[F_\phi\phi + \frac{1}{2}\chi^2\right]. \tag{B.58}$$

7. Superpotential of a fundamental hyper-multiplet

$$-\mathcal{L}_H^{F,int} = \mathrm{Tr}\int d^2\theta\,\widetilde{Q}\Phi Q \tag{B.59}$$

$$= \mathrm{Tr}\left[F_\phi q\widetilde{q} + \phi F_q\widetilde{q} + \phi q F_{\widetilde{q}} + \frac{1}{2}\Big((\chi\psi)\widetilde{q} + (\psi\chi)q + \phi(\psi\widetilde{\psi})\Big)\right]. \tag{B.60}$$

8. Superpotential of a bifundamental hyper-multiplet

$$-\mathcal{L}_H^{B,int} = \mathrm{Tr}\int d^2\theta\,\widetilde{Q}\Phi_h Q - Q\Phi_t\widetilde{Q} \tag{B.61}$$

$$= \mathrm{Tr}\Bigg[\widetilde{q}F_{\phi_h}q + \phi_h F_q\widetilde{q} + \phi_h q F_{\widetilde{q}} + \frac{1}{2}\Big((\chi_h\psi)\widetilde{q} + (\widetilde{\psi}\chi_h)q + \phi_h(\psi\widetilde{\psi})\Big)$$
$$-\Big(qF_{\phi_t}\widetilde{q} + F_q\phi_t\widetilde{q} + q\phi_t F_{\widetilde{q}} + \frac{1}{2}\Big((\psi\chi_t)\widetilde{q} + q(\chi_t\widetilde{\psi}) + \phi_t(\widetilde{\psi}\psi)\Big)\Big)\Bigg]. \tag{B.62}$$

9. Kinetic term of an $\mathcal{N}_{3D}=4$ vector-multiplet $V=(v,\Phi)$

$$-\mathcal{L}_V^{YM} = -\mathcal{L}_v^{YM} - \mathcal{L}_\Phi^{A} \tag{B.63}$$

$$= \mathrm{Tr}\Bigg[\frac{1}{4}\widetilde{F}_{\mu\nu}\widetilde{F}^{\mu\nu} + \frac{1}{2}\lambda^{A\dot{B}}[\not{D},\lambda_{\dot{B}A}] - \frac{1}{4}D_\mu\sigma_{\dot{A}}{}^{\dot{B}}D^\mu\sigma_{\dot{B}}{}^{\dot{A}}$$
$$+\frac{1}{2}F^A{}_BF^B{}_A - \frac{1}{2}\lambda^\dagger_{\dot{A}B}[\lambda^{B\dot{C}}\sigma_{\dot{C}}{}^{\dot{A}}] + \frac{1}{4}[\sigma_{\dot{A}}{}^{\dot{B}},\sigma_{\dot{C}}{}^{\dot{D}}][\sigma_{\dot{B}}{}^{\dot{A}},\sigma_{\dot{D}}{}^{\dot{C}}]\Bigg]. \tag{B.64}$$

Here we set

$$(q^A) = \begin{pmatrix} q \\ \widetilde{q}^\dagger \end{pmatrix}, \quad (\psi_{\dot{A}}) = \begin{pmatrix} \psi \\ \widetilde{\psi}^\dagger \end{pmatrix}, \tag{B.65}$$

$$(\sigma_{\dot{A}}{}^{\dot{B}}) = \begin{pmatrix} \sigma & \sqrt{2}\phi \\ \sqrt{2}\phi^\dagger & -\sigma \end{pmatrix}, \quad (\lambda^{A\dot{B}}) = \begin{pmatrix} \lambda & \frac{1}{\sqrt{2}}\bar{\chi} \\ \frac{1}{\sqrt{2}}\chi & -\bar{\lambda} \end{pmatrix}, \tag{B.66}$$

$$(F^A{}_B) = \begin{pmatrix} \frac{1}{\sqrt{2}}D' & \bar{F}_\phi \\ F_\phi & -\frac{1}{\sqrt{2}}D' \end{pmatrix}, \quad (D' = D + [\phi,\phi^\dagger]). \tag{B.67}$$

The supersymmetry variation is give by

$$\Delta_\epsilon\widetilde{A}_\mu = \epsilon^\dagger_{\dot{A}B}\gamma^\mu\lambda^{B\dot{A}}, \tag{B.68}$$

$$\Delta_\epsilon\sigma^{T\dot{B}}{}_{\dot{A}} = 2\epsilon^\dagger_{\dot{A}C}\lambda^{C\dot{B}} - \text{(to be traceless)}, \tag{B.69}$$

$$\Delta_\epsilon\lambda^{A\dot{B}} = \epsilon^{A\dot{D}}(\frac{1}{2}\gamma^{\mu\nu}\widetilde{F}_{\mu\nu}\delta_{\dot{D}}{}^{\dot{B}} + [\not{D},\sigma_{\dot{D}}{}^{\dot{B}}] + \frac{1}{2}[\sigma_{\dot{D}}{}^{\dot{C}},\sigma_{\dot{C}}{}^{\dot{B}}]) - D^A{}_C\epsilon^{C\dot{B}}, \tag{B.70}$$

$$\Delta_\epsilon F^A{}_B = \sqrt{2}\epsilon^{A\dot{C}}[\not{D}\delta_{\dot{C}}{}^{\dot{D}} - \sigma_{\dot{C}}{}^{\dot{D}},\lambda^\dagger_{\dot{D}B}] - \text{(to be traceless)}, \tag{B.71}$$

where

$$(\epsilon^{A\dot{B}}) = \begin{pmatrix} \epsilon & 0 \\ 0 & \bar{\epsilon} \end{pmatrix}, \quad (\epsilon^\dagger_{\dot{A}B}) = \begin{pmatrix} \bar{\epsilon} & 0 \\ 0 & \epsilon \end{pmatrix}. \tag{B.72}$$

10. Mass term of an $\mathcal{N}_{3D}=4$ vector-multiplet $V=(v,\Phi)$

$$
\begin{aligned}
-\mathcal{L}_V^{CS} &= -\mathcal{L}_v^{CS}-\mathcal{L}_\Phi^{mass} &\text{(B.73)}\\
&= \mathrm{Tr}\Big[-\varepsilon^{\mu\nu\rho}\left(\frac{1}{2}\widetilde{A}_\mu\partial_\nu\widetilde{A}_\rho+\frac{1}{3}\widetilde{A}_\mu\widetilde{A}_\nu\widetilde{A}_\rho\right)\\
&\qquad +\frac{1}{2}(\lambda^{A\dot{B}}\lambda^\dagger_{\dot{A}B}+\sigma_{\dot{A}}{}^{\dot{B}}D^B{}_A)-\frac{1}{6}\sigma_{\dot{A}}{}^{\dot{B}}\sigma_{\dot{B}}{}^{\dot{C}}\sigma_{\dot{C}}{}^{\dot{A}}\Big]. &\text{(B.74)}
\end{aligned}
$$

We set

$$(D^A{}_B)=(\sqrt{2}F^A{}_B). \qquad \text{(B.75)}$$

11. Kinetic term of a fundamental hyper-multiplet $H=(Q,\widetilde{Q})$

$$
\begin{aligned}
-\mathcal{L}_H^F &= -\mathcal{L}_Q^F-\mathcal{L}_Q^{F,int}-\mathcal{L}_{\widetilde{Q}}^F-\mathcal{L}_{\widetilde{Q}}^{F,int} &\text{(B.76)}\\
&= \mathrm{Tr}\Big[-D_\mu q_A^\dagger D^\mu q^A+\frac{1}{2}\psi^{\dagger\dot{A}}(\delta_{\dot{A}}{}^{\dot{B}}\not{D}+\sigma_{\dot{A}}{}^{\dot{B}})\psi_{\dot{B}}\\
&\qquad +\psi^{\dagger\dot{B}}\lambda^\dagger_{\dot{B}A}q^A+q_A^\dagger\lambda^{A\dot{B}}\psi_{\dot{B}}+q_A^\dagger(D^A{}_B-\delta^A{}_B\frac{1}{2}\sigma_{\dot{D}}{}^{\dot{C}}\sigma_{\dot{C}}{}^{\dot{D}})q^B\Big] &\text{(B.77)}
\end{aligned}
$$

Here we eliminate auxiliary fields $F_q, F_{\widetilde{q}}$ by using equations of motion. The supersymmetry transformation is given by

$$
\begin{aligned}
\Delta_\epsilon q^A &= \epsilon^{A\dot{B}}\psi_{\dot{B}}, &\text{(B.78)}\\
\Delta_\epsilon\psi_{\dot{A}} &= 2(\not{D}\delta_{\dot{A}}{}^{\dot{B}}+\sigma_{\dot{A}}{}^{\dot{B}})\epsilon^\dagger_{\dot{B}C}q^C. &\text{(B.79)}
\end{aligned}
$$

12. Lagrangian of a bifundamental hyper-multiplet $H=(Q,\widetilde{Q})$. Q is fundamental for G_h and anti-fundamental for G_t. $\widetilde{Q}$ is opposite.

$$
\begin{aligned}
-\mathcal{L}_H^B &= -\mathcal{L}_Q^B-\mathcal{L}_Q^{B,int}-\mathcal{L}_{\widetilde{Q}}^B-\mathcal{L}_{\widetilde{Q}}^{B,int} &\text{(B.80)}\\
&= \mathrm{Tr}\Big[-D_\mu q_A^\dagger D^\mu q^A+\frac{1}{2}\psi^{\dagger\dot{A}}\not{D}\psi_{\dot{A}}+\frac{1}{2}(\psi^{\dagger\dot{A}}\sigma_h{}_{\dot{A}}{}^{\dot{B}}\psi_{\dot{B}}+\psi_{\dot{B}}\sigma_t{}_{\dot{A}}{}^{\dot{B}}\psi^{\dagger\dot{A}})\\
&\qquad +(\psi^{\dagger\dot{B}}\lambda_h{}^\dagger_{\dot{B}A}q^A+q_A^\dagger\lambda_h{}^{A\dot{B}}\psi_{\dot{B}})+q_A^\dagger(D_h{}^A{}_B-\delta^A{}_B\frac{1}{2}\sigma_h{}_{\dot{D}}{}^{\dot{C}}\sigma_h{}_{\dot{C}}{}^{\dot{D}})q^B\\
&\qquad -(q^A\lambda_t{}^\dagger_{\dot{B}A}\psi^{\dagger\dot{B}}+\psi_{\dot{B}}\lambda_t{}^{A\dot{B}}q_A^\dagger)+q^B(-D_t{}^A{}_B-\delta^A{}_B\frac{1}{2}\sigma_t{}_{\dot{D}}{}^{\dot{C}}\sigma_t{}_{\dot{C}}{}^{\dot{D}})q_A^\dagger\\
&\qquad +q_A^\dagger\sigma_t{}_{\dot{C}}{}^{\dot{B}}q^A\sigma_h{}_{\dot{B}}{}^{\dot{C}}\Big]. &\text{(B.81)}
\end{aligned}
$$

The supersymmetry transformation is given by

$$
\begin{aligned}
\Delta_\epsilon q^A &= \epsilon^{A\dot{B}}\psi_{\dot{B}}, &\text{(B.82)}\\
\Delta_\epsilon\psi_{\dot{A}} &= 2(\not{D}\delta_{\dot{A}}{}^{\dot{B}}\epsilon^\dagger_{\dot{B}C}q^C+\sigma_h{}_{\dot{A}}{}^{\dot{B}}\epsilon^\dagger_{\dot{B}C}q^C-q^C\epsilon^\dagger_{\dot{B}C}\sigma_t{}_{\dot{A}}{}^{\dot{B}}). &\text{(B.83)}
\end{aligned}
$$

Appendix C

$OSp(\mathcal{N}|4)$ superconformal algebra

In Appendix C, we give a list of $OSp(\mathcal{N}|4)$ superconformal algebra following the convention in [139]. We focus on the case of $\mathcal{N} = 2N$.

First, let us give the bosonic subalgebra, $Sp(4, \mathbf{R}) \times SO(2N)$. $Sp(4, \mathbf{R})$ conformal algebra is given by

$$
\begin{aligned}
[M^\alpha{}_\beta, M^\gamma{}_\delta] &= -\delta^\alpha{}_\delta M^\gamma{}_\beta + \delta^\gamma{}_\beta M^\alpha{}_\delta, && \text{(C.1)}\\
\left[K^{\alpha\beta}, P_{\gamma\delta}\right] &= 4\delta^\alpha{}_{(\gamma}\delta^\beta{}_{\delta)} D + 4\delta^{(\alpha}{}_{(\gamma} M^{\beta)}{}_{\delta)}, && \text{(C.2)}\\
[M^\alpha{}_\beta, P_{\gamma\delta}] &= \delta^\alpha{}_\beta P_{\gamma\delta} - \delta^\alpha{}_\gamma P_{\beta\delta} - \delta^\alpha{}_\delta P_{\beta\gamma}, && \text{(C.3)}\\
\left[M^\alpha{}_\beta, K^{\gamma\delta}\right] &= \delta^\alpha{}_\beta K^{\gamma\delta} - \delta^\alpha{}_\gamma K^{\beta\delta} - \delta^\alpha{}_\delta K^{\beta\gamma}, && \text{(C.4)}\\
[D, P_{\gamma\delta}] &= P_{\gamma\delta}, \quad [D, K^{\gamma\delta}] = -K^{\gamma\delta}, \quad [D, M^\alpha{}_\beta] = 0. && \text{(C.5)}
\end{aligned}
$$

Here $P_{\gamma\delta}$ is a translation operator, $M^\alpha{}_\beta$ is a rotation operator, D is the dilatation operator, and $K^{\gamma\delta}$ is a special conformal generator. The indices $\alpha, \beta, \cdots$ take the values of $\uparrow, \downarrow$ or, in the same meaning, $+, -$, respectively. They are related by the usual notation $P_\mu, M_{\mu\nu}, K^\mu$ by

$$
P_{\alpha\beta} = \gamma^\mu_{\alpha\beta} P_\mu, \quad K^{\alpha\beta} = \gamma_\mu^{\alpha\beta} K^\mu, \quad M^\alpha{}_\beta = \frac{i}{2}\gamma^{\mu\nu\alpha}{}_\beta M_{\mu\nu}. \tag{C.6}
$$

$SO(2N)$ R-symmetry algebra is defined by

$$
[R^{ab}, R^{cd}] = i\delta^{ac} R^{bd} \pm (perm), \tag{C.7}
$$

where $a, b, \cdots = 1, \cdots, 2N$. To discuss highest weight representations later, it is useful to rewrite $SO(2N)$ generators in terms of the Cartan generators $\mathcal{H}_l$ ($l = 1, \cdots, N$) and raising (lowering) operators $E^{+\pm}_{mn}(E^{-\pm}_{mn})$ $(1 \le m < n \le N)$. They are defined by

$$
\begin{aligned}
\mathcal{H}_l &:= R_{2l-1,2l}, && \text{(C.8)}\\
E^{+\pm}_{mn} &:= R_{2m-1,2n-1} + iR_{2m,2n-1} \pm iR_{2m-1,2n} \mp R_{2m,2n}, && \text{(C.9)}\\
E^{-\mp}_{mn} &:= R_{2m-1,2n-1} - iR_{2m,2n-1} \mp iR_{2m-1,2n} \mp R_{2m,2n}. && \text{(C.10)}
\end{aligned}
$$

Easily we obtain

$$E^{++}_{mm} = E^{--}_{mm} = 0, \qquad E^{-+}_{mm} = -E^{+-}_{mm} = 2i\mathcal{H}_m, \tag{C.11}$$

$$E^{-+}_{mn} = -E^{+-}_{nm}, \quad E^{--}_{mn} = -E^{--}_{nm}, \quad E^{++}_{mn} = -E^{++}_{nm}, \quad E^{+-}_{mn} = -E^{-+}_{nm}. \tag{C.12}$$

Their commutation relations are

$$[\mathcal{H}_l, E^{+\pm}_{mn}] = (\delta_{lm} \pm \delta_{ln})E^{+\pm}_{mn}, \qquad [\mathcal{H}_l, E^{-\mp}_{mn}] = -(\delta_{lm} \pm \delta_{ln})E^{-\mp}_{mn}. \tag{C.13}$$

These commutators mean that $E^{\pm\pm}_{mn}$ are eigenvectors of Cartan generators (roots). In the case of $n = m$, we obtain

$$[\mathcal{H}_l, \mathcal{H}_m] = 0, \tag{C.14}$$

which is consistent with that fact that $\mathcal{H}_l$ are Cartan generators. Other commutation relations are

$$\begin{aligned}
\left[E^{+\pm}_{mn}, E^{-\mp}_{mn}\right] &= 4(1-\delta_{mn})(\mathcal{H}_m \pm \mathcal{H}_n), && \text{(C.15)}\\
\left[E^{+\pm}_{lm}, E^{-\pm}_{ln}\right] &= (2 + (-1 \mp 1)\delta_{lm} + (-1 \pm 1)\delta_{ln})iE^{\pm\pm}_{mn}, && \text{(C.16)}\\
\left[E^{+\pm}_{lm}, E^{-\mp}_{ln}\right] &= (2 + (-1 \mp 1)(\delta_{lm} + \delta_{ln}))iE^{\pm\mp}_{mn} + 4\delta_{mn}\mathcal{H}_l. && \text{(C.17)}
\end{aligned}$$

In particular,

$$\begin{aligned}
[E^{++}_{lm}, E^{--}_{ln}] &= 2(1 - (\delta_{lm} + \delta_{ln}))iE^{+-}_{mn} + 4\delta_{mn}\mathcal{H}_l, && \text{(C.18)}\\
\left[E^{+-}_{lm}, E^{-+}_{ln}\right] &= 2iE^{\pm\mp}_{mn} + 4\delta_{mn}\mathcal{H}_l, && \text{(C.19)}\\
\left[E^{+\pm}_{lm}, E^{-\mp}_{lm}\right] &= 4(1-\delta_{lm})(\mathcal{H}_l \pm \mathcal{H}_m). && \text{(C.20)}
\end{aligned}$$

Second, we give the algebra including fermionic generators. The anti-commutation relations of fermionic generators $\mathcal{Q}, \bar{\mathcal{Q}}, \mathcal{S}, \bar{\mathcal{S}}$ are given by

$$\begin{aligned}
\{\mathcal{Q}^l_\alpha, \mathcal{Q}^m_\beta\} &= \{\bar{\mathcal{Q}}^l_\alpha, \bar{\mathcal{Q}}^m_\beta\} = 0, \quad \{\mathcal{S}^l_\alpha, \mathcal{S}^m_\beta\} = \{\bar{\mathcal{S}}^l_\alpha, \bar{\mathcal{S}}^m_\beta\} = 0, && \text{(C.21)}\\
\{\mathcal{Q}^l_\alpha, \bar{\mathcal{Q}}^m_\beta\} &= 2\delta^{lm}P_{\alpha\beta}, \qquad \{\mathcal{S}^l_\alpha, \bar{\mathcal{S}}^m_\beta\} = 2\delta^{lm}K_{\alpha\beta}, && \text{(C.22)}\\
\{\mathcal{Q}^l_\alpha, \mathcal{S}^{m\beta}\} &= \delta_\alpha{}^\beta E^{++}_{lm}, \qquad \{\bar{\mathcal{Q}}^l_\alpha, \bar{\mathcal{S}}^{m\beta}\} = \delta_\alpha{}^\beta E^{--}_{lm}, && \text{(C.23)}\\
\{\mathcal{Q}^l_\alpha, \bar{\mathcal{S}}^{m\beta}\} &= 2i\delta^{lm}(M_\alpha{}^\beta + \delta_\alpha{}^\beta D) + \delta_\alpha{}^\beta E^{+-}_{lm}, && \text{(C.24)}\\
\{\bar{\mathcal{Q}}^l_\alpha, \mathcal{S}^{m\beta}\} &= 2i\delta^{lm}(M_\alpha{}^\beta + \delta_\alpha{}^\beta D) + \delta_\alpha{}^\beta E^{-+}_{lm}. && \text{(C.25)}
\end{aligned}$$

In particular,

$$\begin{aligned}
\{\mathcal{Q}^l_\alpha, \mathcal{S}^{l\beta}\} &= 0, \qquad \{\bar{\mathcal{Q}}^l_\alpha, \bar{\mathcal{S}}^{m\beta}\} = 0, && \text{(C.26)}\\
\{\mathcal{Q}^l_\alpha, \bar{\mathcal{S}}^{l\beta}\} &= 2i(M_\alpha{}^\beta + \delta_\alpha{}^\beta D - \delta_\alpha{}^\beta H^l), && \text{(C.27)}\\
\{\bar{\mathcal{Q}}^l_\alpha, \mathcal{S}^{l\beta}\} &= 2i(M_\alpha{}^\beta + \delta_\alpha{}^\beta D + \delta_\alpha{}^\beta H^l), && \text{(C.28)}\\
\{\mathcal{Q}^l_\pm, \bar{\mathcal{S}}^{l\mp}\} &= 2iJ_\pm, \quad \{\mathcal{Q}^l_\pm, \bar{\mathcal{S}}^{l\pm}\} = 2i(\pm J_2 + D - H^l), && \text{(C.29)}\\
\{\bar{\mathcal{Q}}^l_\pm, \mathcal{S}^{l\mp}\} &= 2iJ_\pm, \quad \{\bar{\mathcal{Q}}^l_\pm, \mathcal{S}^{l\pm}\} = 2i(\pm J_2 + D + H^l), && \text{(C.30)}
\end{aligned}$$

where we set $J_\pm = M_\pm{}^\mp, J_2 = M_+{}^+$.

Commutators between fermionic generators and conformal generators are

$$[P_{\alpha\beta}, \mathcal{Q}^a_\alpha] = 0, \qquad [K^{\alpha\beta}, \mathcal{S}^{a\alpha}] = 0, \tag{C.31}$$

$$[D, \mathcal{Q}^a_\alpha] = \tfrac{1}{2}\mathcal{Q}^a_\alpha, \qquad [D, \mathcal{S}^{a\alpha}] = -\tfrac{1}{2}\mathcal{S}^{a\alpha}, \tag{C.32}$$

$$\left[K^{\alpha\beta}, \mathcal{Q}^a_\gamma\right] = i(\delta^\alpha{}_\gamma \mathcal{S}^{a\beta} + \delta^\beta{}_\gamma \mathcal{S}^{a\alpha}), \qquad [P_{\alpha\beta}, \mathcal{S}^{a\gamma}] = -i(\delta_\alpha{}^\gamma \mathcal{Q}^a_\beta + \delta_\beta{}^\gamma \mathcal{Q}^a_\alpha), \tag{C.33}$$

$$\left[M^\alpha{}_\beta, \mathcal{Q}^a_\gamma\right] = \tfrac{1}{2}(\delta^\alpha{}_\gamma \mathcal{Q}^a_\beta - \varepsilon_{\beta\gamma}\mathcal{Q}^{a\alpha}), \qquad \left[M^\alpha{}_\beta, \mathcal{S}^a_\gamma\right] = -\tfrac{1}{2}(\delta^\alpha{}_\gamma \mathcal{S}^a_\beta - \varepsilon_{\beta\gamma}\mathcal{S}^{a\alpha}). \tag{C.34}$$

Commutators between fermionic generators and R-symmetry generators are

$$[\mathcal{H}_m, \mathcal{Q}_n] = \delta_{mn}\mathcal{Q}_n, \qquad \left[\mathcal{H}_m, \bar{\mathcal{Q}}_n\right] = -\delta_{mn}\bar{\mathcal{Q}}_n, \tag{C.35}$$

$$\left[E^{-+}_{lm}, \mathcal{Q}_n\right] = 2i\delta_{ln}\mathcal{Q}_m, \qquad \left[E^{+-}_{lm}, \bar{\mathcal{Q}}_n\right] = 2i\delta_{ln}\bar{\mathcal{Q}}_m, \tag{C.36}$$

$$\left[E^{+-}_{lm}, \mathcal{Q}_n\right] = -2i\delta_{mn}\mathcal{Q}_l, \qquad \left[E^{-+}_{lm}, \bar{\mathcal{Q}}_n\right] = -2i\delta_{mn}\bar{\mathcal{Q}}_l, \tag{C.37}$$

$$\left[E^{--}_{lm}, \mathcal{Q}_n\right] = 2i(\delta_{ln}\bar{\mathcal{Q}}_m - \delta_{mn}\bar{\mathcal{Q}}_l), \qquad \left[E^{++}_{lm}, \bar{\mathcal{Q}}_n\right] = 2i(\delta_{ln}\mathcal{Q}_m - \delta_{mn}\mathcal{Q}_l), \tag{C.38}$$

$$\left[E^{++}_{lm}, \mathcal{Q}_n\right] = 0, \qquad \left[E^{--}_{lm}, \bar{\mathcal{Q}}_n\right] = 0. \tag{C.39}$$

The same equations are also the case for substituting $\mathcal{Q}, \bar{\mathcal{Q}}$ for $\mathcal{S}, \bar{\mathcal{S}}$, respectively. That is,

$$[\mathcal{H}_m, \mathcal{S}_n] = \delta_{mn}\mathcal{S}_n, \qquad \left[\mathcal{H}_m, \bar{\mathcal{S}}_n\right] = -\delta_{mn}\bar{\mathcal{S}}_n, \tag{C.40}$$

$$\left[E^{-+}_{lm}, \mathcal{S}_n\right] = 2i\delta_{ln}\mathcal{S}_m, \qquad \left[E^{+-}_{lm}, \bar{\mathcal{S}}_n\right] = 2i\delta_{ln}\bar{\mathcal{S}}_m, \tag{C.41}$$

$$\left[E^{+-}_{lm}, \mathcal{S}_n\right] = -2i\delta_{mn}\mathcal{S}_l, \qquad \left[E^{-+}_{lm}, \bar{\mathcal{S}}_n\right] = -2i\delta_{mn}\bar{\mathcal{S}}_l, \tag{C.42}$$

$$\left[E^{--}_{lm}, \mathcal{S}_n\right] = 2i(\delta_{ln}\bar{\mathcal{S}}_m - \delta_{mn}\bar{\mathcal{S}}_l), \qquad \left[E^{++}_{lm}, \bar{\mathcal{S}}_n\right] = 2i(\delta_{ln}\mathcal{S}_m - \delta_{mn}\mathcal{S}_l), \tag{C.43}$$

$$\left[E^{++}_{lm}, \mathcal{S}_n\right] = 0, \qquad \left[E^{--}_{lm}, \bar{\mathcal{S}}_n\right] = 0. \tag{C.44}$$

Appendix D

$1/2$ BPS representations of $OSp(8|4)$ and $OSp(4|4)$

In Appendix D, we review 1/2 BPS representations of the superconformal group $OSp(8|4)$ and $OSp(4|4)$, their characters and indices [140, 111, 139].

To describe highest weights of irreducible representations, we use Cartan generators D (dilatation) and $j \equiv J_2$ (spin) for the conformal group $Sp(4, \mathbf{R})$, and $h_l\,(l = 1,2,3,4)$ for $SO(8)_R$. We define four supercharges q^l_α and $\overline{q}^l_\alpha$ ($\alpha = \uparrow, \downarrow$) carrying the Cartan charges shown in Table D.1.

Table D.1: Sixteen supercharges.

	D	j	h_1	h_2	h_3	h_4
$\bar{q}^4$	$\frac{1}{2}$	$\pm\frac{1}{2}$	0	0	0	-1
$\bar{q}^3$	$\frac{1}{2}$	$\pm\frac{1}{2}$	0	0	-1	0
$\bar{q}^2$	$\frac{1}{2}$	$\pm\frac{1}{2}$	0	-1	0	0
$\bar{q}^1$	$\frac{1}{2}$	$\pm\frac{1}{2}$	-1	0	0	0
q^4	$\frac{1}{2}$	$\pm\frac{1}{2}$	0	0	0	1
q^3	$\frac{1}{2}$	$\pm\frac{1}{2}$	0	0	1	0
q^2	$\frac{1}{2}$	$\pm\frac{1}{2}$	0	1	0	0
q^1	$\frac{1}{2}$	$\pm\frac{1}{2}$	1	0	0	0

As usual, we use highest weights to specify representations. The highest weights of an $SO(8)_R$ representation should satisfy the following inequality:

$$h_1 \geq h_2 \geq h_3 \geq |h_4|. \tag{D.1}$$

The highest weight states of 1/2 BPS representations saturate this bound. There are two series of such representations, which are called $(\mathcal{N}, B, \pm)$ in [139]. $\mathcal{N}$ in the first slot refers to the number of supersymmetry. A representation in each series is specified by one integer n, and the components of its highest weight are

$$D = \frac{n}{2}, \quad j = 0, \quad h_1 = h_2 = h_3 = \pm h_4 = \frac{n}{2}, \quad n = 1, 2, \ldots \tag{D.2}$$

The last component h_4 is positive for $(8, B, +)$, and negative for $(8, B, -)$. We denote the 1/2 BPS representation with the highest weight (D.2) by $(8, B, \pm)_n$. The highest weight states $|0\rangle_{(8,B,\pm)}$ satisfy

$$q^1|0\rangle_{(8,B,+)} = q^2|0\rangle_{(8,B,+)} = q^3|0\rangle_{(8,B,+)} = q^4|0\rangle_{(8,B,+)} = 0, \tag{D.3}$$

$$q^1|0\rangle_{(8,B,-)} = q^2|0\rangle_{(8,B,-)} = q^3|0\rangle_{(8,B,-)} = \bar{q}^4|0\rangle_{(8,B,-)} = 0. \tag{D.4}$$

The spectra of $(8, B, \pm)_n$ representations is decomposed into fifteen irreducible representations of the bosonic subgroup $Sp(4, \mathbf{R}) \times SO(8)$. The decomposition of $(8, B, -)_n$ is shown in Table D.2. In the table, Δ is defined by

$$\Delta = \{Q, S\} = D - (j + h_1), \tag{D.5}$$

where we choose $Q = q^1_\downarrow$. That of $(8, B, +)_n$ is obtained from this by flipping the sign of h_4.

Table D.2: The spectrum of the 1/2 BPS representation $(8, B, -)_n$. The representation is decomposed into fifteen irreducible representations of the bosonic subgroup $Sp(4, \mathbf{R}) \times SO(8)_R$.

D	j	h_1	h_2	h_3	h_4	Δ	range
$\frac{n}{2}$	0	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{-n}{2}$	0	$(n \geq 1)$
$\frac{n+1}{2}$	$\frac{1}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{-n+2}{2}$	0	$(n \geq 1)$
$\frac{n+2}{2}$	1	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n-2}{2}$	$\frac{-n+2}{2}$	0	$(n \geq 2)$
$\frac{n+3}{2}$	$\frac{3}{2}$	$\frac{n}{2}$	$\frac{n-2}{2}$	$\frac{n-2}{2}$	$\frac{-n+2}{2}$	0	$(n \geq 2)$
$\frac{n+4}{2}$	2	$\frac{n-2}{2}$	$\frac{n-2}{2}$	$\frac{n-2}{2}$	$\frac{-n+2}{2}$	1	$(n \geq 2)$
$\frac{n+2}{2}$	0	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{-n+4}{2}$	1	$(n \geq 2)$
$\frac{n+3}{2}$	$\frac{1}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n-2}{2}$	$\frac{-n+4}{2}$	1	$(n \geq 3)$
$\frac{n+4}{2}$	1	$\frac{n}{2}$	$\frac{n-2}{2}$	$\frac{n-2}{2}$	$\frac{-n+4}{2}$	1	$(n \geq 3)$
$\frac{n+5}{2}$	$\frac{3}{2}$	$\frac{n-2}{2}$	$\frac{n-2}{2}$	$\frac{n-2}{2}$	$\frac{-n+4}{2}$	2	$(n \geq 3)$
$\frac{n+4}{2}$	0	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n-4}{2}$	$\frac{-n+4}{2}$	2	$(n \geq 4)$
$\frac{n+5}{2}$	$\frac{1}{2}$	$\frac{n}{2}$	$\frac{n-2}{2}$	$\frac{n-4}{2}$	$\frac{-n+4}{2}$	2	$(n \geq 4)$
$\frac{n+6}{2}$	1	$\frac{n-2}{2}$	$\frac{n-2}{2}$	$\frac{n-4}{2}$	$\frac{-n+4}{2}$	3	$(n \geq 4)$
$\frac{n+6}{2}$	0	$\frac{n}{2}$	$\frac{n-4}{2}$	$\frac{n-4}{2}$	$\frac{-n+4}{2}$	3	$(n \geq 4)$
$\frac{n+7}{2}$	$\frac{1}{2}$	$\frac{n-2}{2}$	$\frac{n-4}{2}$	$\frac{n-4}{2}$	$\frac{-n+4}{2}$	4	$(n \geq 4)$
$\frac{n+8}{2}$	0	$\frac{n-4}{2}$	$\frac{n-4}{2}$	$\frac{n-4}{2}$	$\frac{-n+4}{2}$	6	$(n \geq 4)$

We define the superconformal character for a representation R by

$$\chi_R = \mathrm{Tr}_R\left(s^{2D} x^{2j} y_1^{h_1} y_2^{h_2} y_3^{h_3} y_4^{h_4}\right), \tag{D.6}$$

where Tr_R means the trace over the representation R. As is shown in Table D.2, $(8, B, -)_n$ is decomposed into fifteen irreducible representations of the bosonic subgroup $Sp(4,\mathbf{R})\times SO(8)_R$. The character is obtained by summing up those for the fifteen representations.

$$\begin{aligned}
\chi_{(8,B,-)_n} \;=\;& \chi^{\rm conf}_{(\frac{n}{2},0)}(s^2,x^2)\chi^{(4)}_{(\frac{n}{2},\frac{n}{2},\frac{n}{2},\frac{-n}{2})}(y)+\chi^{\rm conf}_{(\frac{n+1}{2},\frac{1}{2})}(s^2,x^2)\chi^{(4)}_{(\frac{n}{2},\frac{n}{2},\frac{n}{2},\frac{-n+2}{2})}(y)\\
&+\chi^{\rm conf}_{(\frac{n+2}{2},1)}(s^2,x^2)\chi^{(4)}_{(\frac{n}{2},\frac{n}{2},\frac{n-2}{2},\frac{-n+2}{2})}(y)+\chi^{\rm conf}_{(\frac{n+3}{2},\frac{3}{2})}(s^2,x^2)\chi^{(4)}_{(\frac{n}{2},\frac{n-2}{2},\frac{n-2}{2},\frac{-n+2}{2})}(y)\\
&+\chi^{\rm conf}_{(\frac{n+4}{2},2)}(s^2,x^2)\chi^{(4)}_{(\frac{n-2}{2},\frac{n-2}{2},\frac{n-2}{2},\frac{-n+2}{2})}(y)+\chi^{\rm conf}_{(\frac{n+2}{2},0)}(s^2,x^2)\chi^{(4)}_{(\frac{n}{2},\frac{n}{2},\frac{n}{2},\frac{-n+4}{2})}(y)\\
&+\chi^{\rm conf}_{(\frac{n+3}{2},\frac{1}{2})}(s^2,x^2)\chi^{(4)}_{(\frac{n}{2},\frac{n}{2},\frac{n-2}{2},\frac{-n+4}{2})}(y)+\chi^{\rm conf}_{(\frac{n+4}{2},1)}(s^2,x^2)\chi^{(4)}_{(\frac{n}{2},\frac{n-2}{2},\frac{n-2}{2},\frac{-n+4}{2})}(y)\\
&+\chi^{\rm conf}_{(\frac{n+5}{2},\frac{3}{2})}(s^2,x^2)\chi^{(4)}_{(\frac{n-2}{2},\frac{n-2}{2},\frac{n-2}{2},\frac{-n+4}{2})}(y)+\chi^{\rm conf}_{(\frac{n+4}{2},0)}(s^2,x^2)\chi^{(4)}_{(\frac{n}{2},\frac{n-4}{2},\frac{n-4}{2},\frac{-n+4}{2})}(y)\\
&+\chi^{\rm conf}_{(\frac{n+5}{2},\frac{1}{2})}(s^2,x^2)\chi^{(4)}_{(\frac{n}{2},\frac{n-2}{2},\frac{n-4}{2},\frac{-n+4}{2})}(y)+\chi^{\rm conf}_{(\frac{n+6}{2},1)}(s^2,x^2)\chi^{(4)}_{(\frac{n-2}{2},\frac{n-2}{2},\frac{n-4}{2},\frac{-n+4}{2})}(y)\\
&+\chi^{\rm conf}_{(\frac{n+6}{2},0)}(s^2,x^2)\chi^{(4)}_{(\frac{n}{2},\frac{n-4}{2},\frac{n-4}{2},\frac{-n+4}{2})}(y)+\chi^{\rm conf}_{(\frac{n+7}{2},\frac{1}{2})}(s^2,x^2)\chi^{(4)}_{(\frac{n-2}{2},\frac{n-4}{2},\frac{n-4}{2},\frac{-n+4}{2})}(y)\\
&+\chi^{\rm conf}_{(\frac{n+8}{2},0)}(s^2,x^2)\chi^{(4)}_{(\frac{n-4}{2},\frac{n-4}{2},\frac{n-4}{2},\frac{-n+4}{2})}(y),
\end{aligned}\tag{D.7}$$

where $\chi^{\rm conf}_{(D,j)}$ is the character of the irreducible representation of the conformal group with highest weight (D,j),

$$\chi^{\rm conf}_{(D,j)}={\rm Tr}_{(D,j)}(s^{2D}x^{2j})=\frac{s^{2D}\chi_j(x^2)}{(1-s^2x^2)(1-s^2)(1-s^2x^{-2})},\tag{D.8}$$

and $\chi_j(t)$ is the $SU(2)$ character for the spin j representation

$$\chi_j(t)=\frac{t^j-t^{-j-1}}{1-t^{-1}}=t^j+\cdots+t^{-j},\quad (\chi_j(t)=0\text{ for }j<0),\tag{D.9}$$

and $\chi^{(4)}_h(y)$ is the $SO(8)$ character

$$\chi^{(4)}_h(y)=\frac{\det[y_I^{h_J+4-J}+y_I^{-h_J-4+J}]+\det[y_I^{h_J+4-J}-y_I^{-h_J-4+J}]}{2\times\Pi_{1\le I<J\le 4}(y_I+1/y_I-(y_J+1/y_J))}.\tag{D.10}$$

In general, it is difficult to calculate a character directly on the gauge theory side due to quantum corrections. We can avoid this by choosing the arguments of the character so that the contributions of two states connected by a supercharge Q have opposite signs and cancel each other. Such a character is an index. Let us choose $Q=q^1_\downarrow$. In this case, $SO(2)$ generated by h_1 plays a role of R-symmetry, and $OSp(8|4)$ algebra tells us that four Cartan generators $D+j,h_2,h_3$ and h_4 commute with Q. By using these generators, the index is defined by

$$I_R \;=\; {\rm Tr}_R\bigg((-)^F e^{-\beta'\Delta}x^{2(D+j)}y_2^{h_2}y_3^{h_3}y_4^{h_4}\bigg),\tag{D.11}$$

where F is the fermion number operator. It can be shown that this index does not depend on β' [103]. This means that the states which contribute to the index satisfy $\Delta=0$, which is equivalent to the BPS condition in a unitary representation. By setting $e^{-\beta'}$ to b^2, we can obtain the relation between a character and an index by

$$I_R = \mathrm{Tr}_R\left((-)^F b^{2D} b^{-2j} x^{2(D+j)} b^{-2h_1} y_2^{h_2} y_3^{h_3} y_4^{h_4}\right) = \chi_R(s \to bx, x \to \frac{-x}{b}, y_1 \to \frac{1}{b^2}). \tag{D.12}$$

where we used $(-)^F = (-)^{2j}$ to show the second equality. By substituting (D.7) into the relation (D.12), we obtain the superconformal index for a half BPS representation $(8, B, -)_n$ by

$$I_{(8,B,-)_n} = \frac{\text{(numerator)}}{\text{(denominator)}}, \tag{D.13}$$

where the numerator is

$$\begin{aligned}
&\text{(numerator)} \\
= \quad & \left[x^n (y_2 y_3 y_4)^{-n/2} y_2^{4+n} y_3 y_4 (-1 + y_3 y_4)\right] \times \\
& \Bigg[\left((-1 + x^2 y_3)(x^2 - y_4) y_4^n - (x^2 - y_3) y_3^n (-1 + x^2 y_4)\right) \\
& + x^2 y_2^{1+n} (-1 + y_3 y_4)\left((1 - x^2 y_3)(x^2 - y_4) y_4^{2+n} + (x^2 - y_3) y_3^{2+n} (-1 + x^2 y_4)\right) \\
& + y_3 (y_3 - y_4) y_4 \left(y_3^n (-x^2 + y_3)(x^2 - y_4) y_4^n + (-1 + x^2 y_3)(-1 + x^2 y_4)\right) \\
& + x^2 y_2^3 \left((x^2 - y_3) y_3^{2+n} (x^2 - y_4)(y_3 - y_4) y_4^{2+n} + (-1 + x^2 y_3)(-y_3 + y_4)(-1 + x^2 y_4)\right) \\
& + y_2 (y_3 - y_4)\Big(y_3^{1+n} (x^2 - y_3)(x^2 - y_4) y_4^{1+n} (x^2 + y_3 + y_4) \\
& \qquad\qquad + (1 - x^2 y_3)(-1 + x^2 y_4)(y_3 + y_4 + x^2 y_3 y_4)\Big) \\
& + y_2^2 (y_3 - y_4)\Big(1 - y_3^{3+n} y_4^{3+n} - x^6 y_3 y_4 (y_3 + y_4)(-1 + y_3^n y_4^n) \\
& \qquad\qquad + x^4 (y_3^2 + y_3 y_4 + y_4^2)(-1 + y_3^{1+n} y_4^{1+n})\Big) \\
& + y_2^{3+n} (-1 + y_3 y_4)\Big((x^2 - y_3) y_3^{1+n} (-1 + x^2 y_4)(1 + (x^2 + y_3) y_4) \\
& \qquad\qquad - (-1 + x^2 y_3)(x^2 - y_4) y_4^{1+n} (1 + y_3 (x^2 + y_4))\Big) \\
& + y_2^{2+n} (1 - y_3 y_4)\Big(y_3^{3+n} - y_4^{3+n} + x^6 y_3 y_4 (1 + y_3 y_4)(y_3^n - y_4^n) \\
& \qquad\qquad - x^4 (y_3^{1+n} - y_4^{1+n})(1 + y_3 y_4 (1 + y_3 y_4))\Big)\Bigg],
\end{aligned} \tag{D.14}$$

and the denominator is

$$\begin{aligned}
\text{(denominator)} \quad = \quad & \left(-1 + x^4\right)(y_2 - y_3)(y_2 - y_4)(y_3 - y_4) \times \\
& (-1 + y_2 y_3)(-1 + y_2 y_4)(-1 + y_3 y_4).
\end{aligned} \tag{D.15}$$

Notice that the index is independent of b. We can also obtain this result by summing up the contribution from the $\Delta = 0$ states in Table D.2. Note that we can show that the characters and indices both for $(8, B, -)_n$ and $(8, B, +)_n$ coincide.

The KK spectrum of single supergraviton in $AdS_4 \times \mathbf{S}^7$ was investigated in [141]. Their result is that in the spectrum, denoted by $G_{AdS_4\times\mathbf{S}^7}$, the 1/2 BPS representation $(8, B, -)_n$ appears only once for all n:

$$G_{AdS_4\times\mathbf{S}^7} = \bigoplus_{n=1,2\ldots} (8, B, -)_n. \tag{D.16}$$

For short, we denote $G_{AdS_4\times\mathbf{S}^7}$ as $G_{\mathbf{S}^7}$. Therefore, the character for single supergraviton in $AdS_4 \times \mathbf{S}^7$ is given by

$$\chi_{G_{\mathbf{S}^7}} = \sum_{n=1,2\ldots} \chi_{(8,B,-)_n}, \tag{D.17}$$

and the index for it is

$$I_{G_{\mathbf{S}^7}} = \sum_{n=1,2\ldots} I_{(8,B,-)_n}, \tag{D.18}$$

which is nothing but the index given in (4.156).

Let us move on to 1/2 BPS representations of $OSp(4|4)$. To describe highest weights of irreducible representations, we use D and j, T_3 for $SU(2)_R$, and T_3' for $SU(2)_R'$. After taking orbifold by the discrete actions (2.52), we are left with eight supersymmetries in Table D.3. They are related charges in the former convention

Table D.3: Eight supercharges preserved after the orbifolding.

	D	j	H_1	H_2	T_3	T_3'
Q^1	$\frac{1}{2}$	$\pm\frac{1}{2}$	$+1$	0	$+\frac{1}{2}$	$+\frac{1}{2}$
$\overline{Q}^1$	$\frac{1}{2}$	$\pm\frac{1}{2}$	-1	0	$-\frac{1}{2}$	$-\frac{1}{2}$
Q^2	$\frac{1}{2}$	$\pm\frac{1}{2}$	0	$+1$	$-\frac{1}{2}$	$+\frac{1}{2}$
$\overline{Q}^2$	$\frac{1}{2}$	$\pm\frac{1}{2}$	0	-1	$+\frac{1}{2}$	$-\frac{1}{2}$

by

$$H_1 = -h_1, \quad H_2 = -h_3; \quad Q^1 = \bar{q}^1, \quad Q^2 = \bar{q}^3, \quad \bar{Q}^1 = q^1, \quad \bar{Q}^2 = q^3, \tag{D.19}$$

$$T_3 = \frac{1}{2}(-h_1 + h_3), \quad T_3' = -\frac{1}{2}(h_1 + h_3), \quad P = h_2 + h_4, \quad P' = h_2 - h_4. \tag{D.20}$$

As usual, we use highest weights to specify representations. The highest weights T_3 and T_3' of an $SO(4)_R$ representation are non-negative, and there is the following bound for H_1 and H_2.

$$H_1 \geq |H_2|. \tag{D.21}$$

The highest weight states of 1/2 BPS representations saturate this bound. A representation in each series is specified by one integer n, and the components of its highest weight are

$$(D, j, H_1, H_2) = (\frac{n}{2}, 0, \frac{n}{2}, \pm\frac{n}{2}), \quad n = 1, 2, \ldots \tag{D.22}$$

The last component H_2 is positive for $(4, B, +)$, and negative for $(4, B, -)$. We denote the 1/2 BPS representation with the highest weight (D.22) by $(4, B, \pm)_n$. The highest weight states $|0\rangle_{(4,B,\pm)}$ satisfy

$$Q^1|0\rangle_{(4,B,+)} = Q^2|0\rangle_{(4,B,+)} = 0, \quad Q^1|0\rangle_{(4,B,-)} = \overline{Q}^2|0\rangle_{(4,B,-)} = 0. \tag{D.23}$$

The spectra of $(4, B, \pm)_n$ representations is decomposed into six irreducible representations of the bosonic subgroup $Sp(4, \mathbf{R}) \times SO(4)$. The decomposition of $(4, B, +)_n$ is shown in Table D.4. That of $(4, B, -)_n$ is obtained from this by exchanging T_3 and T_3' and flipping the sign of H_2.

Corresponding to the two fixed loci $\mathcal{S}_U$ and $\mathcal{S}_U{}'$, both $(4, B, +)$ and $(4, B, -)$ arise from the twisted sectors. Because T_3 does not move $\mathcal{S}_U$, it is an internal charge in the context of the field theory in $\mathcal{S}_U$, while T_3' is an orbital angular momentum. This fact implies that the Kaluza-Klein modes in $\mathcal{S}_U$ should be identified with $(4, B, +)$, which can take an arbitrarily large T_3'. Contrary, the Kaluza-Klein modes in the other locus $\mathcal{S}_U{}'$ belong to the other series of representations $(4, B, -)$.

Table D.4: The spectrum of the 1/2 BPS representation $(4, B, +)_n$. The representation is decomposed into six irreducible representations of the bosonic subgroup $Sp(4, \mathbf{R}) \times SO(4)_R$. The highest weights of these representations are given. Δ is defined later in (D.28). The spectrum of $(4, B, -)_n$ is obtained by exchanging T_3 and T_3' and flipping the sign of H_2.

D	j	T_3	T_3'	H_1	H_2	Δ	range
$\frac{n}{2}$	0	0	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	0	$(n \geq 1)$
$\frac{n+1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{n-1}{2}$	$\frac{n}{2}$	$\frac{n-2}{2}$	0	$(n \geq 1)$
$\frac{n+2}{2}$	1	0	$\frac{n-2}{2}$	$\frac{n-2}{2}$	$\frac{n-2}{2}$	1	$(n \geq 2)$
$\frac{n+2}{2}$	0	1	$\frac{n-2}{2}$	$\frac{n}{2}$	$\frac{n-4}{2}$	1	$(n \geq 2)$
$\frac{n+3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{n-3}{2}$	$\frac{n-2}{2}$	$\frac{n-4}{2}$	2	$(n \geq 3)$
$\frac{n+4}{2}$	0	0	$\frac{n-4}{2}$	$\frac{n-4}{2}$	$\frac{n-4}{2}$	4	$(n \geq 4)$

We define the superconformal character for a representation R by

$$\chi_R = \mathrm{Tr}_R\left(s^{2D}x^{2j}y^{T_3}y'^{T_3'}\right) = \mathrm{Tr}_R\left(s^{2D}x^{2j}Y_1^{H_1}Y_2^{H_2}\right), \tag{D.24}$$

where Tr_R means the trace over the representation R. We used Cartan generators (D, j, T_3, T_3') for the middle expression and (D, j, H_1, H_2) for the last one. These two choices of the Cartan generators for $SO(4)_R$ are related by (4.145), and correspondingly, the two sets of variables (y, y') and (Y_1, Y_2) are related by

$$y = \frac{Y_1}{Y_2}, \quad y' = Y_1Y_2. \tag{D.25}$$

As is shown in Table D.4, $(4, B, +)_n$ is decomposed into six irreducible representations of the bosonic subgroup $Sp(4, \mathbf{R}) \times SO(4)_R$. The character is obtained by summing up those for the six representations.

$$\begin{aligned}\chi_{(4,B,+)_n} \;=\;& \chi^{\mathrm{conf}}_{(\frac{n}{2},0)}(s^2, x^2)\chi_{\frac{n}{2}}(y') + \chi^{\mathrm{conf}}_{(\frac{n+1}{2},\frac{1}{2})}(s^2, x^2)\chi_{\frac{1}{2}}(y)\chi_{\frac{n-1}{2}}(y') \\ &+\chi^{\mathrm{conf}}_{(\frac{n+2}{2},1)}(s^2, x^2)\chi_{\frac{n-2}{2}}(y') + \chi^{\mathrm{conf}}_{(\frac{n+2}{2},0)}(s^2, x^2)\chi_1(y)\chi_{\frac{n-2}{2}}(y') \\ &+\chi^{\mathrm{conf}}_{(\frac{n+3}{2},\frac{1}{2})}(s^2, x^2)\chi_{\frac{1}{2}}(y)\,\chi_{\frac{n-3}{2}}(y') + \chi^{\mathrm{conf}}_{(\frac{n+4}{2},0)}(s^2, x^2)\chi_{\frac{n-4}{2}}(y').\end{aligned} \tag{D.26}$$

Let us choose $Q = Q^1_{\downarrow}$. In this case, $OSp(4|4)$ algebra tells us that two Cartan generators $D + j$ and H_2 commute with Q. The index is given by

$$I_R \;=\; \mathrm{Tr}_R\left((-)^F e^{-\beta'\Delta}x^{2(D+j)}Y_2^{H_2}\right), \tag{D.27}$$

where F is the fermion number operator, and Δ is defined by

$$\Delta = \{Q, S\} = D - (j + H_1). \tag{D.28}$$

This index does not depend on β'. By substituting (D.26) into the relation (D.12), we obtain the superconformal index for a half BPS representation $(4, B, +)_n$

$$I_{(4,B,+)_n} \;=\; \frac{x^nY_2^{\frac{n}{2}}(1 - x^2Y_2^{-1})}{1 - x^4}. \tag{D.29}$$

In the same way, we obtain

$$I_{(4,B,-)_n} \;=\; \frac{x^nY_2^{-\frac{n}{2}}(1 - x^2Y_2)}{1 - x^4}. \tag{D.30}$$

As in the case of $OSp(8|4)$, we can also obtain this result by summing up the contribution from the $\Delta = 0$ states in Table D.4.

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