Radiation Reaction in a Normal Neighbourhood

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August 999

(Work in progress)

Radiation Reaction

- **Small** particle mass μ) moves on a geodesic $z^\alpha(\tau)$ of a curved background spacetime $(M, g_{\alpha\beta})$

$$\mu^2 R(g_{\alpha\beta}) << 1$$

- Particle's gravitation alters metric:

$$g_{\alpha\beta} \longrightarrow g_{\alpha\beta} + \gamma_{\alpha\beta}$$

- linearized gravity:

$$\mathcal{D}\gamma)_{\alpha\beta} = T_{\alpha\beta},$$

where

$$\mathcal{D}\gamma)_{\alpha\beta} = \Box\gamma_{\alpha\beta} - 2R^\mu{}_{\alpha\beta\nu}\gamma_{\mu\nu},$$
$$T_{\alpha\beta} = 16\pi\mu\int_{-\infty}^{\infty}\delta(x - z(\tau))u_\alpha(\tau)u_\beta(\tau)d\tau$$

and $u^\alpha = \partial_\tau z^\alpha(\tau)$

Notation:

$$E_{ij} \equiv C_{i\alpha j\beta} u^\alpha u^\beta, \quad B_{ij} \equiv C_{\alpha\beta i\gamma} \varepsilon^{\alpha\beta}{}_j{}^\delta u^\gamma u_\delta$$

$$\varepsilon^{ijk} \equiv \varepsilon^{ijk\alpha} u_\alpha, \quad \dot{f} \equiv u^\alpha \nabla_\alpha f, \quad f_{|j} \equiv (\delta^\alpha_j + u^\alpha u_j) \nabla_\alpha f$$

Survivors at λ^3:

$$a_1^l \equiv \alpha_1 \mu E_i{}^j \dot{E}_{jk} \varepsilon^{ikl} \lambda^3, \quad a_2^l \equiv \alpha_2 \mu E_i{}^j \dot{B}_{jk} \varepsilon^{ikl} \lambda^3$$

$$\underline{a_3^l \equiv \alpha_3 \mu B_i{}^j \dot{E}_{jk} \varepsilon^{ikl} \lambda^3}, \quad a_4^l \equiv \alpha_4 \mu B_i{}^j \dot{B}_{jk} \varepsilon^{ikl} \lambda^3$$

$$\underline{a_5^l \equiv \alpha_5 \mu E^{ij} E_{ij}{}^{|l} \lambda^3}, \quad a_6^l \equiv \alpha_6 \mu E^{ij} B_{ij}{}^{|l} \lambda^3$$

$$a_7^l \equiv \alpha_7 \mu B^{ij} E_{ij}{}^{|l} \lambda^3, \quad \underline{a_8^l \equiv \alpha_8 \mu B^{ij} B_{ij}{}^{|l} \lambda^3}$$

Leading PN contribution: Schwarzschild, Harmonic, Equatori

$$a_3: \quad r'' \sim \left(\frac{\mu}{M}\right) r (\phi')^2 \left(\frac{M}{r}\right)^3, \quad \phi'' \sim \left(\frac{\mu}{M}\right) \frac{\phi' r'}{r} \left(\frac{M}{r}\right)^3$$

$$a_5: \quad r'' \sim \left(\frac{\mu}{M}\right) \frac{M}{r^2} \left(\frac{M}{r}\right)^2, \quad \phi'' \sim \left(\frac{\mu}{M}\right) \frac{\phi' r'}{r} \left(\frac{M}{r}\right)^3$$

$$a_8: \quad r'' \sim \left(\frac{\mu}{M}\right) r (\phi')^2 \left(\frac{M}{r}\right)^3, \quad \phi'' \sim \left(\frac{\mu}{M}\right) \frac{\phi' r'}{r} \left(\frac{M}{r}\right)^3$$

$$(r' \equiv \partial_t r)$$

$$\ddot{r} \sim \frac{\mu}{M}\left(r\dot{\phi}^2 - \frac{M}{r^2}\right)$$

$$\ddot{\phi} \sim \frac{\mu}{M}\left(-2\frac{\dot{r}\dot{\phi}}{r}\right)$$

$$\partial_t^2 r \sim r(\partial_t\phi)^2 - \frac{M}{r^2}$$

$$(\partial_t\phi)^2 \sim -4\frac{\dot{r}\,\partial_t\phi}{r}$$

$$G_2: \partial_t^2 r \sim \frac{\mu}{M}\left(\dot{\phi}^2\right)\left(\frac{M}{R}\right)^3$$

$$G_3: \partial_t^2\phi \sim \left(\frac{\mu}{M}\right)\frac{\dot{\phi}\dot{R}}{R}\left(\frac{M}{R}\right)^3$$

$$G_4: \partial_t^2 r \sim \frac{\mu}{M}\,\frac{M}{r^2}\left(\frac{M}{r}\right)^3$$

$$\partial_t^2\phi \sim \frac{\mu}{M}\,\frac{\dot{\phi}\dot{r}}{r}\left(\frac{M}{r}\right)^3$$

$$G_5: \partial_t^2 r \sim \frac{\mu}{M}\,r\dot{\phi}^2\left(\frac{M}{r}\right)^3$$

$$\partial_t^2\phi \sim \frac{\mu}{M}\,\frac{\dot{r}\dot{\phi}}{r}\left(\frac{M}{r}\right)^3$$

Lowest Order Survivor

- The lowest order survivor is $a_2^\alpha = C^{\alpha\beta\gamma\delta}\, C_{\beta\gamma\delta\mu} u^\mu$:
- Corrections parallel to u^α are just corrections to geodesic's parameterization. Radiation reaction is the orthogonal projection:

$$a^\alpha = \alpha\ \mu\ (\delta^\alpha_\nu + u^\alpha u_\nu)\ C^{\nu\beta\gamma\delta}\ C_{\beta\gamma\delta\mu}\ u^\mu\ \lambda^2 + \mathcal{O}(\lambda^3)$$

- α is a numerical coefficient that we must find. How?
- Back to perturbation theory!

Special Case: Kerr

- Recall that Kerr is a Petrov type D spacetime.
- Recall that for Petrov type D Ψ_2 is the only nonvanishing component of Weyl.
- Recall that in terms of the Newman-Penrose basis $\Psi_2 = -C_{0123} \rightarrow C^{\mu\beta\gamma\delta} C_{\beta\gamma\delta\nu} = |\Psi_2|^2 \delta^\mu_\nu$.

Conclusion

Thus radiation reaction from normal neighbourhood vanishes to lowest order in Kerr spacetime.

Question

Does it vanish to all orders?

Answer

No, some λ^3 terms survive in Schwarzschild.

A Shortcut

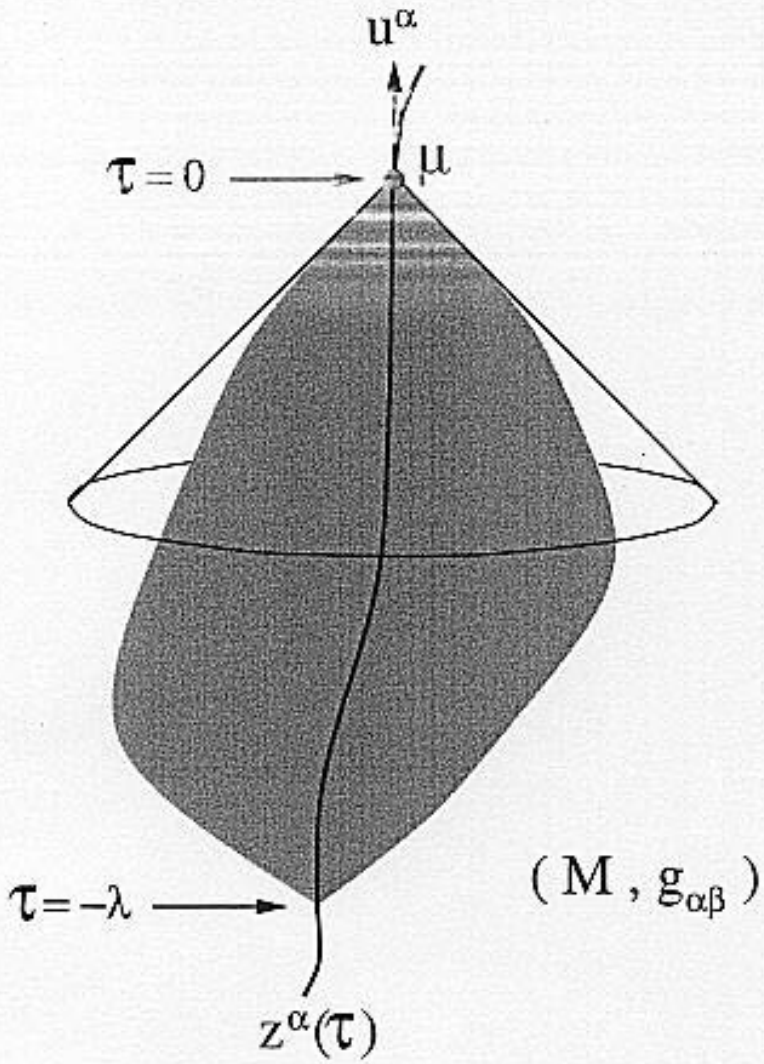
- Notice that at the end of the day, the acceleration is expressed as a power series in λ,

$$a^\alpha = \mu(a_0^\alpha + a_1^\alpha \lambda + a_2^\alpha \lambda^2 + \mathcal{O}(\lambda^3))$$

- Notice that the coefficents, a_n^α, will be built out of geometric quantities associate with the spacetime or the particle.
- The only geometric quantity carried by the particle is its 4-velocity, u^α.
- In a vacuum background, the only geometric quantities describing the spacetime are the $g_{\alpha\beta}$, the curvature $C_{\alpha\beta\gamma\delta}$ and its derivatives, and the Levi-Civita symbol $\varepsilon_{\alpha\beta\gamma\delta}$.
- There are only a finite number of forms that each a_n^α can take! We can see what they are in advance.

An Example: a_0^α

- has dimension $\ell^{-2} \to a_0^\alpha \propto C^\alpha{}_{\beta\gamma\delta}$.
- $C^\alpha{}_{\beta\gamma\delta}$ has too many indices.
  - Can't contract internally, Weyl is traceless.
  - Can't contract with more than 2 u^α's because of antisymmetry on γ and δ.
  - Can't contract on more than 2 indices with epsilon, because $C^\alpha{}_{[\beta\gamma\delta]} = 0$.
- Choices left:
  - $a_0^\alpha \propto C^\alpha{}_{\beta\gamma\delta} u^\beta u^\gamma \varepsilon^{\delta???}$.
  - $a_0^\alpha \propto C^\alpha{}_{\beta\gamma\delta} u^\beta \varepsilon^{\gamma\delta??}$.
  - $a_0^\alpha \propto C^\alpha{}_{\beta\gamma\delta} u^\delta \varepsilon^{\beta\gamma??}$.


u^α
$\tau = 0$
μ
$\tau = -\lambda$
$(M, g_{\alpha\beta})$
$z^\alpha(\tau)$

Solution in a Normal Neighbourhood

- Define Riemann normal coordinate system y^α with origin at the particle position; i.e. for each point x in the normal neighbourhood of $z^\alpha(0)$ define coordinates $y^\alpha = \xi^\alpha \tau$ where:
  - ξ^α is the tangent vector to the geodesic connecting $z^\alpha(0)$ to x at $z^\alpha(0)$,
  - τ is the geodesic distance (affine parameter distance) from x to the origin.
- These coordinates facilitate covariant Taylor series expansion of components of equation of motion, e.g.

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \frac{1}{3} R_{\mu\alpha\beta\nu}(0) y^\mu y^\nu + \mathcal{O}(R^2 y^4)$$

- Expand equation of motion $(\mathcal{D}\gamma) = T$ and collect by order in Ry^2. Note that to 0th order $\mathcal{D} = \Box_{\text{flat}}$.

$$\begin{aligned}
\Box_{\text{flat}} \gamma^{(0)}_{\alpha\beta} &= T^{(0)}_{\alpha\beta}, \\
\Box_{\text{flat}} \gamma^{(1)}_{\alpha\beta} &= T^{(1)}_{\alpha\beta} - (\mathcal{D}^{(1)}\gamma^{(0)})_{\alpha\beta}, \\
\Box_{\text{flat}} \gamma^{(2)}_{\alpha\beta} &= T^{(1)}_{\alpha\beta} - (\mathcal{D}^{(1)}\gamma^{(1)})_{\alpha\beta} - (\mathcal{D}^{(2)}\gamma^{(0)})_{\alpha\beta}, \\
&\vdots
\end{aligned}$$

- Solve order by order using the flat space Green's function, e.g.

$$\gamma^{(0)}_{\alpha\beta}(\tau) = \int_- \lambda^0 G_{\text{flat}}(z(\tau), z(\tau')) T_{\alpha\beta}(\tau') d\tau'$$

The Question

How does $\gamma_{\alpha\beta}$ change particles trajectory?

The Answer

- Find $\gamma_{\alpha\beta}$ using the Green's function $G_{\alpha\beta}{}^{\mu'\nu'}(x,x') = \mathcal{D}^{-1}$:

$$\gamma_{\alpha\beta}(x) = \int_{-\infty}^{0} G_{\alpha\beta}{}^{\mu'\nu'}(x,x')\, T_{\mu'\nu'}(x')\, d\tau.$$

- Express correction to particle path as an acceleration,

$$a^{\alpha} = A^{\alpha\beta\gamma\delta}(g,u)\, \gamma_{\beta\gamma;\delta}.$$

The Problem

- Green's functions for $\mathcal{D}$ difficult to calculate analytically $\rightarrow$ numerical mode sum.
- Green's function distributional on light cone $\rightarrow$ mode sum does not converge well near the cone.

Poisson &
Wiseman's Suggestion

- Do numerical mode sum (good estimate far from particle).
- Do normal neighbourhood analysis near particle.
- Match solutions.