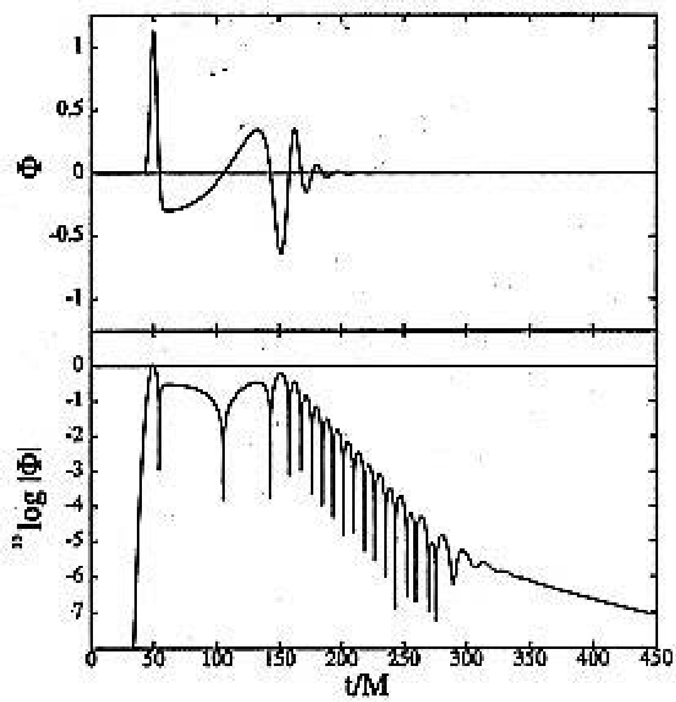
POOR MANS NUMERICAL RELATIVITY

The perturbation equations were used to investigate the dynamics of black holes in the 1970s:

- wave packets scattered by BH (Vishveshwara, Press)
- small bodies falling into, or passing close by, BH (Zerilli, DRPP)
- gravitational collapse to form BH (Cunningham, Price, Moncrief)

The emerging radiation shows similar features in all cases, and QNM ringing dominate most BH signals (cf collisions).



The response of a Schwarzschild black hole as a Gaussian wavepacket of scalar waves impinges upon it. An initial broadband burst is followed by QNM oscillations. At very late times the field is dominated by a power-law fall-off with time.

BLACK HOLE FINGERPRINTS

The effective potential V is of short range, and corresponds to a single potential barrier => the BH problem is in many ways similar to one of potential scattering in quantum mechanics

Assuming a time-dependence $e^{-i\omega t}$, a general solution is

$$\psi \sim \begin{cases} e^{-i\omega r_*} & \text{as } r \to 2M \,, \\ A_{out} e^{i\omega r_*} + A_{in} e^{-i\omega r_*} & \text{as } r \to +\infty \,. \end{cases}$$

QNM correspond to $A_{in} = 0$, i.e. are analogous to scattering resonances.

WKB approximation (Schutz, Will) =>

$$\text{Re } \omega_0 \approx \frac{1}{3\sqrt{3}M}\left(l + \frac{1}{2}\right)$$

$$\text{Im } \omega_0 \approx -\frac{\sqrt{3}}{18M}$$

or (in human units)

$$f \approx 12\text{kHz}\left(\frac{M_\odot}{M}\right) \qquad \tau \approx 0.05\text{ms}\left(\frac{M}{M_\odot}\right)$$

This means that a nonrotating black hole is a very poor oscillator. With

$$Q \approx \frac{1}{2}\left|\frac{\text{Re } \omega_0}{\text{Im } \omega_0}\right|$$

we find $Q \approx \ell$. The f-mode of a neutron star leads to $Q \sim 1000$ and a typical value for an atom is $Q \sim 10^6$.

BLACK HOLE SPECTROSCOPY

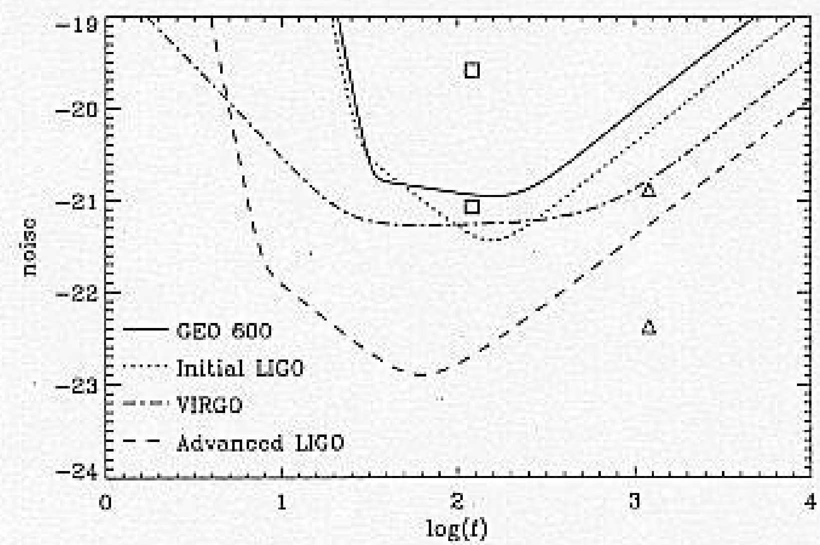
Estimate the strength of the gravitational waves associated with a certain QNM using

$$F = \frac{c^3}{16\pi G}|\dot{h}|^2 = \frac{1}{4\pi r^2}\frac{dE}{dt} \approx \frac{1}{4\pi r^2}\frac{E}{2\tau}$$

where τ is the e-folding time of the QNM. Assuming a monochromatic wave (of frequency f) such that $\dot{h} = 2\pi f h$ (not really justified for rapidly damped QNMs), we estimate the effective amplitude achievable after matched filtering as

$$h_{\rm eff} \approx h\sqrt{n} \approx h\sqrt{f\tau} \approx 4.2\times 10^{-24}\left(\frac{\delta}{10^{-6}}\right)^{1/2}\left(\frac{M_{BH}}{M_\odot}\right)\left(\frac{15\mathrm{Mpc}}{r}\right)$$

where δ is the radiated energy as a fraction of the BH mass (cf. more detailed studies by Echeverria, Finn, Flanagan, Hughes)



Illustrating the "detectability" of QNM ringing for a $10M_\odot$ (squares) and a $100M_\odot$ (triangles) BH. The upper cases correspond to $\delta = 10^{-3}$ while the lower ones are for $\delta = 10^{-6}$.

WHY IS KERR DIFFERENT?

- What effect does rotation have on QNMs and the tail?
- Are there any new physical effects?

For Kerr BH the QNM are no longer symmetrically placed relative to the Im ω axis. Instead, if ω is a QNM corresponding to l and m the complex conjugate $-\omega^*$ will be a QNM for l and $-m$.

Co-rotating modes tend to become longer lived as a increases, while counter-rotating ones remain essentially unchanged.

In particular, some QNM become very slowly damped as $a \to M$. In the limit these modes can be approximated by (Teukolsky, Press, Detweiler)

$$\omega_n M \approx \frac{m}{2} - \frac{1}{4m} e^{(\theta - 2n\pi)/2\delta} \cos\beta - \frac{i}{4m} e^{(\theta-2n\pi)/2\delta} \sin\beta$$

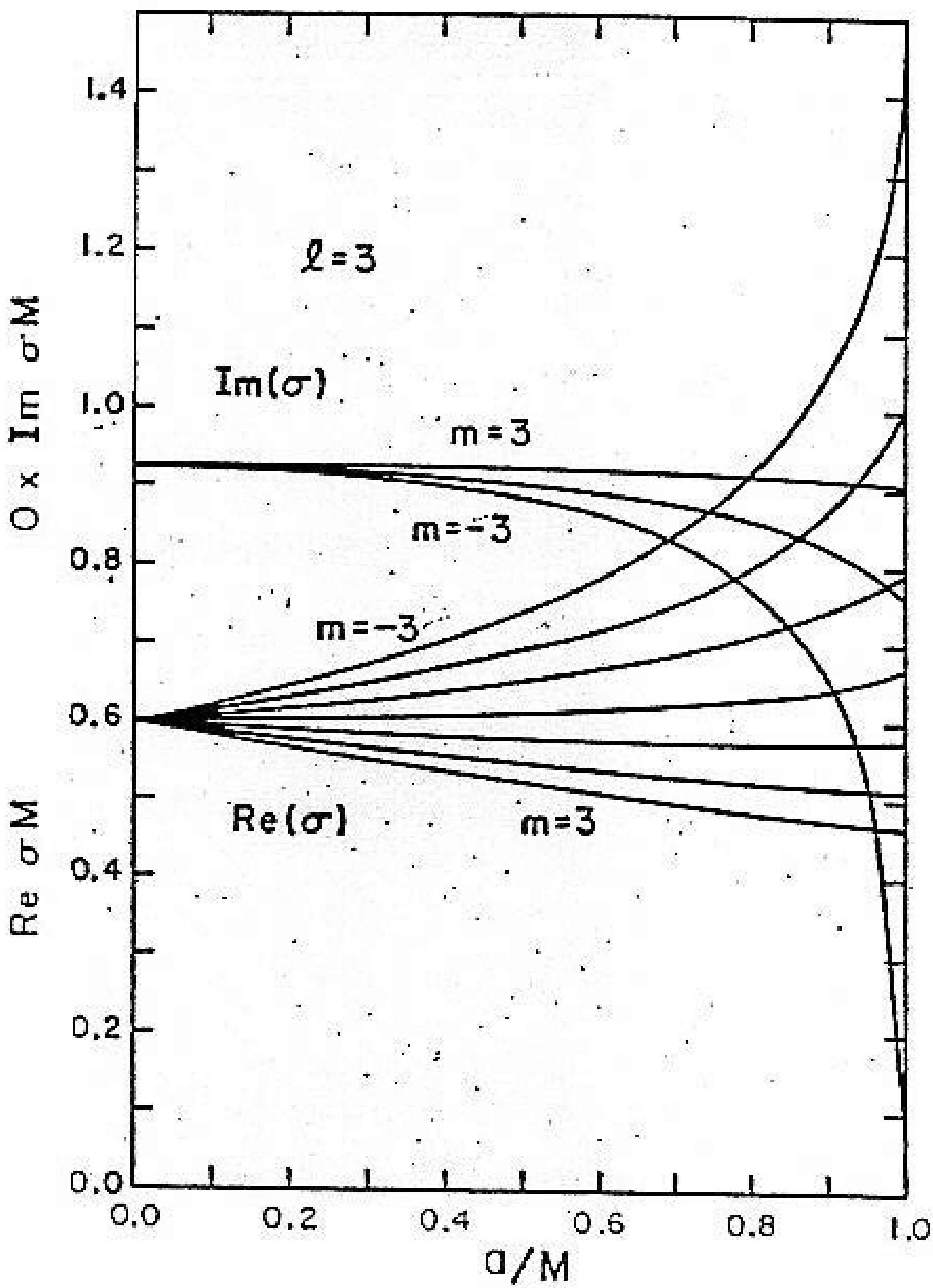
where n is an integer labelling the modes, θ, β and δ are constants.

Alternatively, the slowest damped GW mode is well fitted by (Echeverria)

$$\omega_0 \approx \frac{1}{M}\left[1 - \frac{63}{100}(1-a/M)^{3/10}\right]\left[1 - \frac{i}{4M}(1-a/M)^{9/10}\right]$$

Physical explanation: The long lived modes co-rotate with the BH and can be viewed as a non-radiating additional part of the quadrupole (etc) moment.

DETWEILER ApJ 239 292 295 (1980)



SUPERRADIANT SCATTERING

A physically acceptable scalar-field solution to the radial Teukolsky equation is

$$R_{lm}^{in} \sim \begin{cases} e^{-i\tilde{\omega}r_*} & \text{as } r \to r_+ , \\ A_{out}e^{i\omega r_*} + A_{in}e^{-i\omega r_*} & \text{as } r \to +\infty \end{cases}$$

where

$$\tilde{\omega} = \omega - \frac{ma}{2Mr_+} = \omega - m\Omega_H ,$$

and Ω_H is the angular velocity of the event horizon (r_+).

Using this solution, and its complex conjugate, we find

$$\left(1 - \frac{m\Omega_H}{\omega}\right) T = 1 - R$$

where T and R are the transmission and reflection coefficients. We have superradiance ($R > 1$) if

$$0 < \omega < m\Omega_H = \frac{ma}{2Mr_+}$$

The maximum amplification of an incoming monochromatic wave is 0.3% for scalar waves, 4.4% for electromagnetic waves and 138% for gravitational waves (Press, Teukolsky).

THE ROLE OF SLOWLY DAMPED QNMs

Cherished belief: Some QNMs become long lived as $a \to M$ which significantly enhances the detectability the associated gravitational waves. But that this really is the case is far from clear...

Represent the mode signal by

$$h \approx \frac{Ae^{-t/\tau}}{r} \sin(2\pi f t)$$

Then

$$h_{\text{eff}} \approx \frac{\sqrt{f\tau}}{2} A \approx \left(1 - \frac{a}{M}\right)^{-3/10} \frac{A}{\sqrt{2\pi}}$$

Clearly, a decrease in A may easily compensate for the increase in τ as the black hole spins up. To correctly discuss the detectability of the Kerr QNMs one must investigate this balance.

Turns out to be rather delicate!

Contradictory (?) statements in the literature

- the amplitude of each individual long-lived QNM must vanish as $a \to M$ (Ferrari, Mashhoon)
- the integrated GW flux due to <u>all</u> long-lived modes diverges as $u \to \infty$ (Sasaki, Nakamura)

INITIAL VALUE PROBLEMS

Consider a massless scalar field Φ in the Schwarzschild geometry. With

$$\Phi = \sum_{l,m} \frac{u_\ell(r_*, t)}{r} Y_{\ell m}(\theta, \varphi)$$

the function $u_\ell(r_*, t)$ solves the Regge-Wheeler equation (with $s = 0$), and the future evolution of a field given at some initial time ($t = 0$) follows from

$$u_\ell(r_*, t) = \int G(r_*, y, t)\partial_t u_\ell(y, 0)dy + \int \partial_t G(r_*, y, t) u_\ell(y, 0)dy$$

Here G is the appropriate (retarded) Green's function, and $G(r_*, y, t) = 0$ for $t \leq 0$.

Introducing the "asymptotic approximation":

- the observer is situated far away from the black hole
- the initial data has considerable support only far away from the black hole
- the initial data has no support outside the observer

and using the residue theorem, we get the mode-contribution:

$$G^{QNM}(r_*, y, t) = \mathrm{Re}\left[\sum_{n=0}^{\infty} \frac{A_{\mathrm{out}}(\omega_n)}{\omega_n \alpha_n} e^{-i\omega_n(t - r_* - y)}\right]$$

where

$$A_{\mathrm{in}}(\omega) \approx (\omega - \omega_n)\alpha_n$$

and the sum is over all QNM in the fourth quadrant.

ESTIMATED MODE-EXCITATION

A useful measure (independent of the initial data) of QNM excitation/detectability is the coefficient

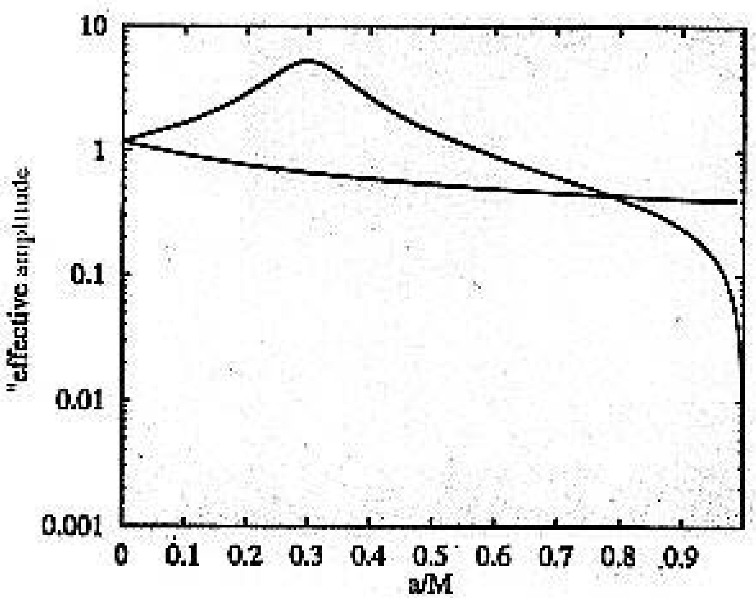
$$\sqrt{\frac{\mathrm{Re}\,\omega_n}{\mathrm{Im}\,\omega_n}\frac{A_{\mathrm{out}}}{\alpha_n}}$$

Recent numerical calculations (for scalar field) indicates that the long-lived are not strongly excited in the limit $a \to M$.

The numerical results are supported by analytic estimates using the approximate modes

$$\sqrt{\frac{\mathrm{Re}\,\omega_n}{\mathrm{Im}\,\omega_n}\frac{A_{out}}{\alpha_n}} \sim e^{-n\pi/2\delta}$$

(δ is almost real for $l = m$). Recall that the slowest damped modes correspond to $n \to \infty$.



An assessment of the "detectability" of the QNMs as $a \to M$. We compare the slowest damped mode in the right half of the ωM plane to the slowest damped mode in the left half-plane. The mode that becomes very long lived has a much smaller excitation coefficient as $a \to M$.

LATE TIME KERR DYNAMICS

Analytic approximations:

Have concluded that each individual QNM makes an infinitesimal contribution in the limit $a \to M$. This seems like bad news from a "GW detection" point of view.

But... we have a large number of slowly damped modes as $a \to M$. If their contribution is summed we find that (NA, Glambedakis)

$$\Phi \sim \frac{e^{-imt/2M}}{t} \qquad \text{as } t \to \infty \qquad \text{for} \quad a = M$$

according to an observer at fixed r.

This is a very surprising result!

The suggested fall-off is much slower than that predicted for the standard power-law tail. Recent estimates suggest (Barack, Ori)

$$\Phi \sim t^{-l-|m|-3-q}$$

where $q = 0$ for even $l+m$ and 1 for odd $l+m$.

Numerical evolutions:

For $m = 0$ we get results in agreement with the standard tail estimates, but for $m \neq 0$ we find the predicted oscillating, slowly decaying field also when a is considerably smaller than M.

The fall-off is NOT represented by the slowest damped QNM.

All evolutions (so far) support the analytic result.

THE TEUKOLSKY CODE

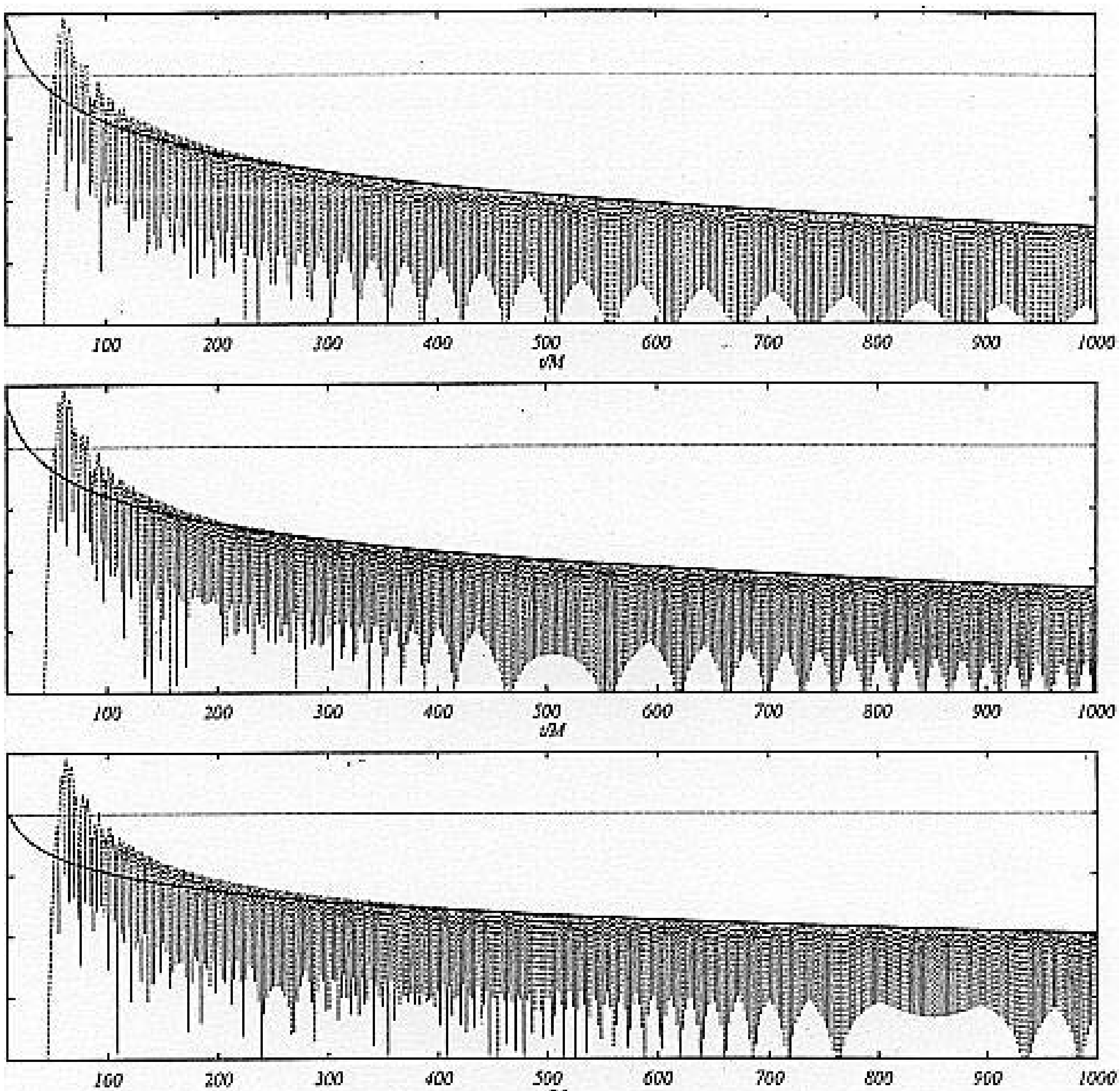
Recent 2D code for evolving the Teukolsky equation (for $s = 0$ and $s = -2$) (Krivan, Laguna, Papadopoulos, NA).

MOTIVATION

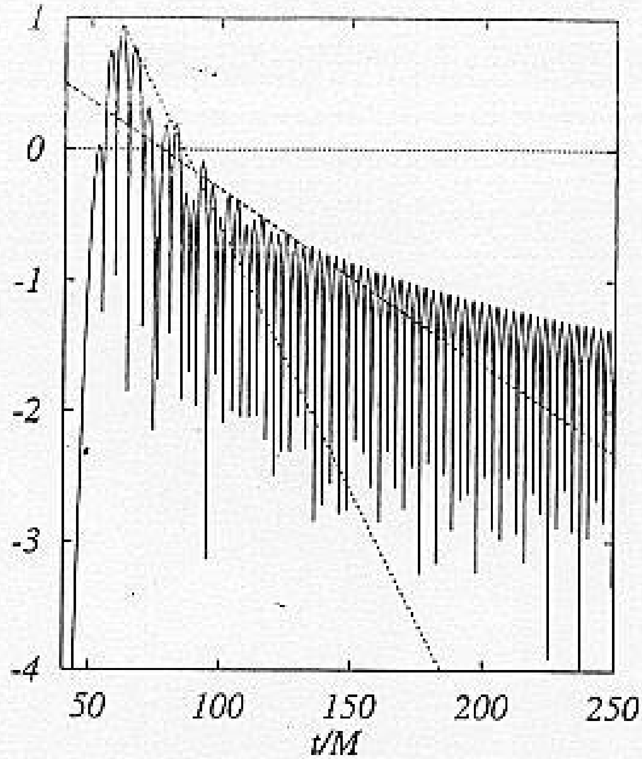
To revisit problems previously approached in the frequency domain, and explore effects due to the rotation of the BH in time-evolutions. To provide a benchmark test for numerical relativity. To contribute to a close-limit approximation for rotating holes.

THE STORY SO FAR

- The late-time tail is similar to Schwarzschild, but different multipoles are mixed due to i) rotational effects, and ii) "imperfect initial data"
- It is sometimes possible to distinguish two distinct regimes of mode-ringing (asymmetries in location of QNMs).
- Slight amplification due to superradiance can be extracted, but the initial data requires considerable fine-tuning.
- In parallell, initial data suitable for the Teukolsky equation have been formulated (Campanelli, Lousto), and the code has provided an independent test of the close-limit approximation for non-rotating BH.
- Framework for 2nd order perturbations (Campanelli, Lousto)
- Also extended to study accretion of fluids (Papadopoulos, Font)

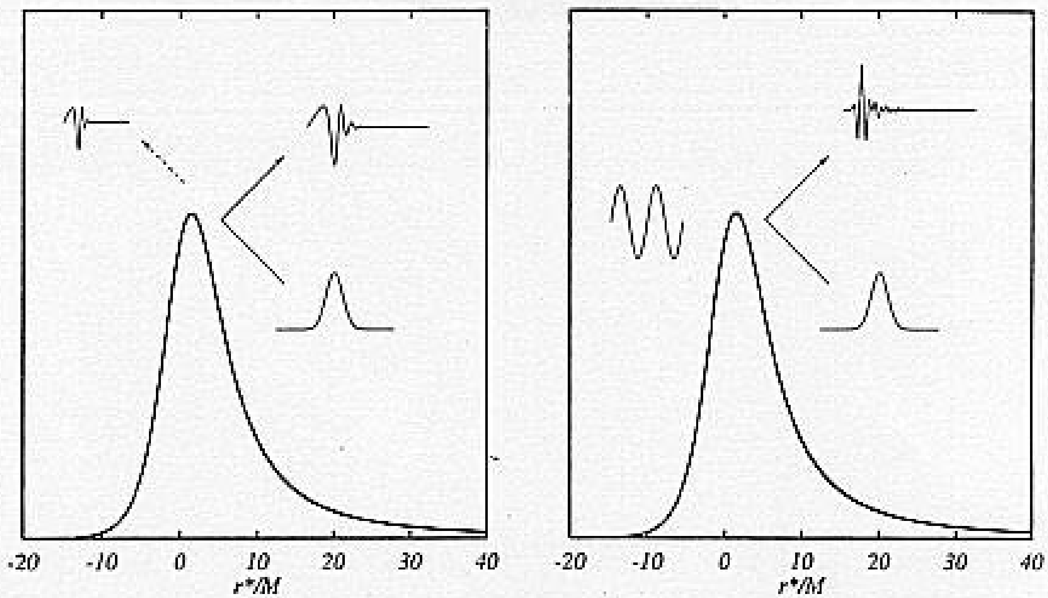


Illustrating th **lowly damped,** *oscillating, 'ta* ***for** a **s**calar fi d outs d a Kerr* **lack hole. The** *frames* ***show*** *a/M 0 999 and and th nveıopes fali off as $/t^{\alpha}$* ***with*** *α* **7** *and 1 respectiv**ely***

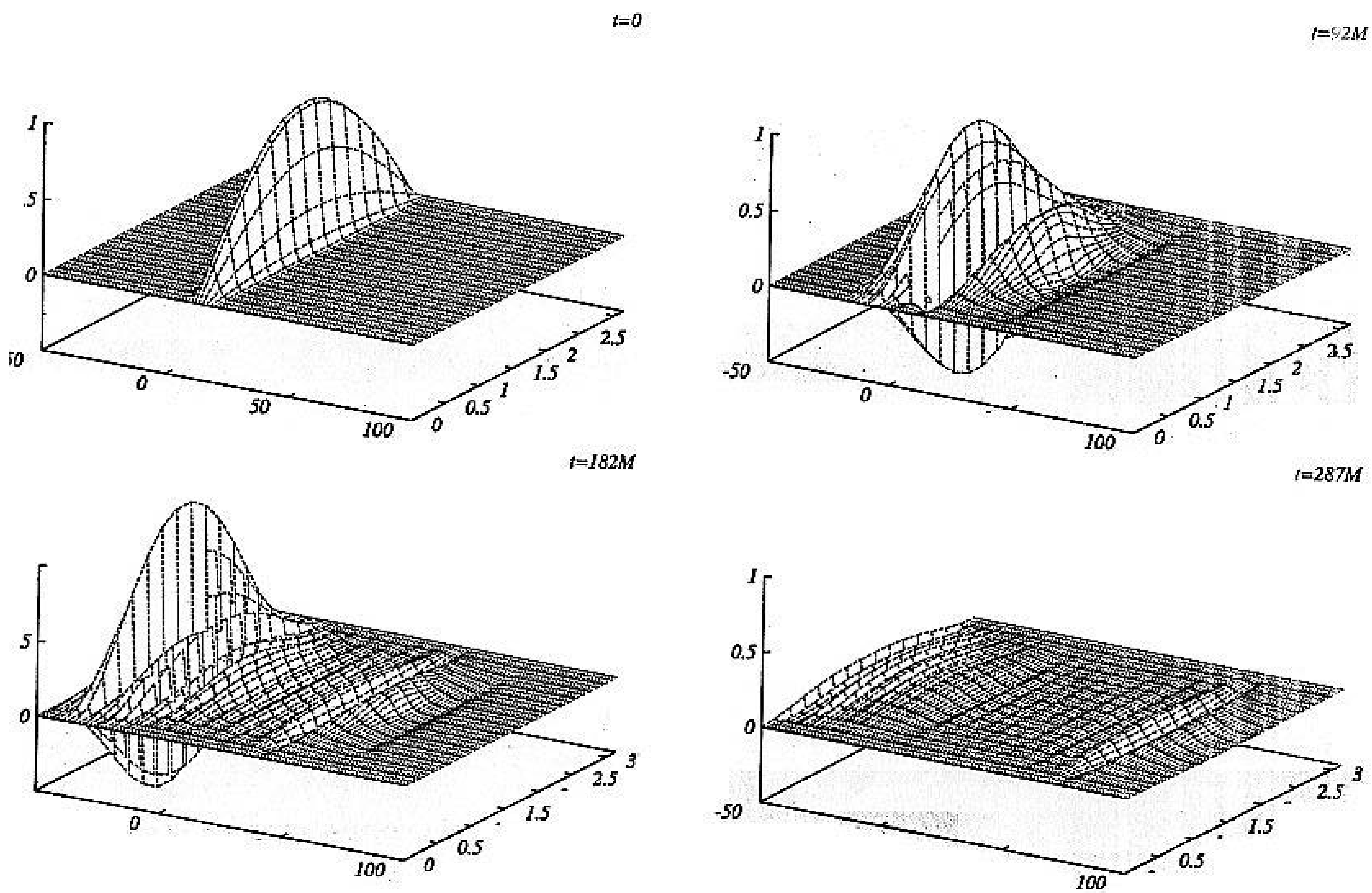


Comparing the late-time behaviour in a typical scalar-field evolution for $m = 2$ and $a = 0.99M$ to the danping rates of the two slowest damped QNMs.

SUPERRADIANT RESONANCE CAVITY



Schematic explanation of the new phenomenon seen in the numerical evolutions of Kerr perturbations. Left panel: The standard scenario: An infalling pulse excites the QNMs that then propagate to infinity and the horizon. At late times, backscattering due to the curvature in the far-zone dominaes. Right panel: For frequencies i) that lie in the superradiant regime, and ii) that experience a "potential peak" in the region $[r_+, \infty]$ *there will be a resonance cavity outside the black hole. At late times, the waves leaking out of this cavity dominate the signal.*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.5
0
50
0
50
100
0
0.5
1
1.5
2
2.5
t=92M
1
0.5
0
-50
0
100
0
0.5
1
1.5
2
3.5
t=182M
5
0
0
100
0
0.5
1.5
2.5
3
t=287M
1
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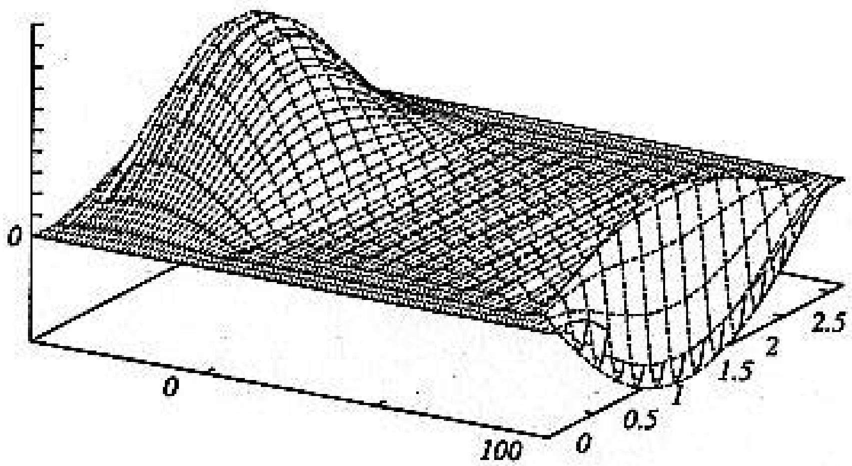
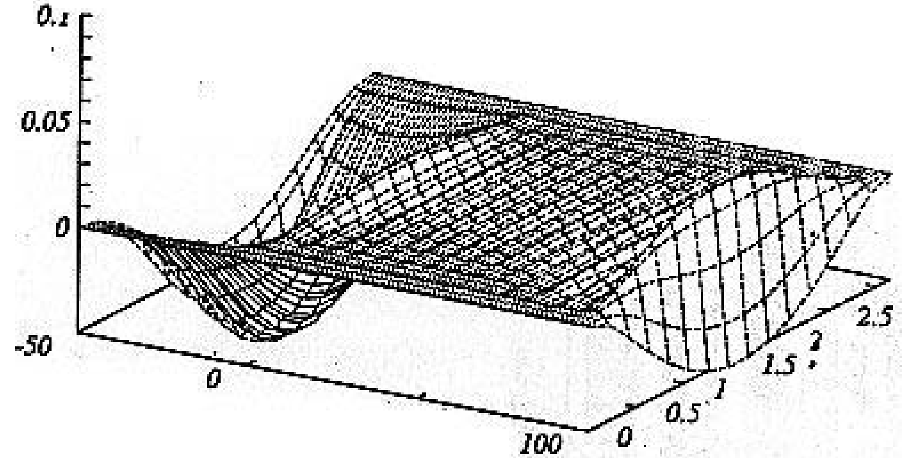
0
-50
100
0.5
1.5
2.5
3

376M

389M





t=403M

t=416M