

# PRACTICAL CALCULATIONS OF RADIATION-REACTION FORCES

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I'll present calculations done with Ori's prescription for the calculation of the self force, for simple (but non-trivial) cases:

- \* Static scalar charge in Schwarzschild
- \* Static electric charge in Schwarzschild
  - For these cases the final results are known from independent approaches
- \* Scalar charge in circular motion around Schwarzschild:
  - fully relativistic, i.e., no slow-motion or far-field approximations
  - As yet undervived by different approaches

Ori presented his prescription in detail. The main idea is:

where  $F_\mu^{\text{tail}} = \sum_l^{\text{bare}} (F_\mu^l - h_\mu^l) - \lim_{\epsilon \rightarrow 0} \sum_l (\delta_\epsilon F_\mu^l - h_\mu^l)$

$$h_\mu^l = a_\mu l + b_\mu + C_\mu l^{-1} \quad d_\mu$$

The coefficients  $a_\mu, b_\mu, C_\mu, d_\mu$  are susceptible of an analytical computation. E.g., for static scalar charge in Schwarzschild

$$a_\mu = C_\mu = d_\mu = 0$$
$$b_\mu = -\frac{q^2}{ar^2} \frac{1 - \frac{M}{r}}{1 - \frac{2M}{r}} S_\mu^+$$

# STATIC SCALAR CHARGE IN SCHWARZSCHILD

Consider point like scalar test charge held fixed in Schwarzschild spacetime by an external force.

The linearized field equation:  $\nabla_\mu \nabla^\mu \Phi(x^\alpha) = -4\pi\rho(x^\alpha)$

where the charge density  $\rho = q \int_{-\infty}^{\infty} dt \frac{\delta^4[x^\mu - x_s^\mu(t)]}{\sqrt{-g}}$ .

Decompose the field according to  $\Phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \phi^l(r) Y^{lm}(\theta, \varphi)$

and find the radial equation

$$1 - \frac{2M}{r} \phi_{,rr}^l + \frac{2}{r^2}(r-M)\phi_{,r}^l - \frac{l(l+1)}{r^2}\phi^l + 4\pi q \frac{\delta(r-r_0)}{r^2} \frac{1}{U_{(l)}} Y^{lm} \frac{\partial}{\partial r} Y^{lm}$$

The solution is given by

$$\Phi = \frac{q}{M\sqrt{1-\frac{2M}{r_0}}} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\varphi) \left[ P_l\left(\frac{r_0-M}{M}\right) Q_l\left(\frac{r_0-M}{M}\right) \Theta(r-r_0) + P_l\left(\frac{r-M}{M}\right) Q_l\left(\frac{r-M}{M}\right) \Theta(r-r_0) \right]$$

where  $\cos\varphi = \cos\theta \cos\theta + \sin\theta \sin\theta \cos(\varphi - \psi)$

The force which a scalar field  $\Psi$  exerts on a scalar charge  $e$  is  $f_\alpha = e(\Psi_{,\alpha} + u_\beta u^\beta \Psi_{,\beta})$ , where  $u^\alpha$  is the 4-velocity of  $e$

We take  $\Psi, e$  to be those of the charge  $q$  which created the field. Thereby,

$$f_\alpha = q(\Phi_{,\alpha} + u_\alpha u^\beta \Phi_{,\beta})$$

find

$$\text{bare } f_r^l = \frac{1}{2} \frac{q^2}{M^2} \sqrt{1 - \frac{2M}{r_0}} (2l+1) \left\{ \frac{Q\left(\frac{r_0 M}{l}\right)}{l} Q'\left(\frac{r_0 M}{l}\right) + \frac{Q'\left(\frac{r_0 M}{l}\right)}{l} Q\left(\frac{r_0 M}{l}\right) \right\}, \text{ and}$$

$$\text{bare } f_r = \sum_{l=0}^{\infty} f_r^l$$

where a prime is derivative with respect to the argument.

We first find that  $\overset{\text{bare } l}{f_r} \xrightarrow{l \rightarrow \infty} \text{const}$

i.e.,  $a_r = 0$ . Next,  $\frac{df_r^l}{dl} \sim l^{-3}$  for large  $l$ , i.e.,  $c_r = 0$ .

(The large  $l$  behavior of  $\overset{\text{bare } l}{f_r}$  is similar to  $h_r^l$ .)

We can't find with  $\overset{\text{bare } l}{f_r}$  the value of  $d_r$ !!!

$\overset{\text{bare } l}{f_r} \xrightarrow{l \rightarrow \infty} b_r = -\frac{q^2}{2r_0^2} \frac{1-M/r}{1-2M/r}$ , which we indeed find.

shall demonstrate this explicitly in the sequel.)

next regularize the bare force by

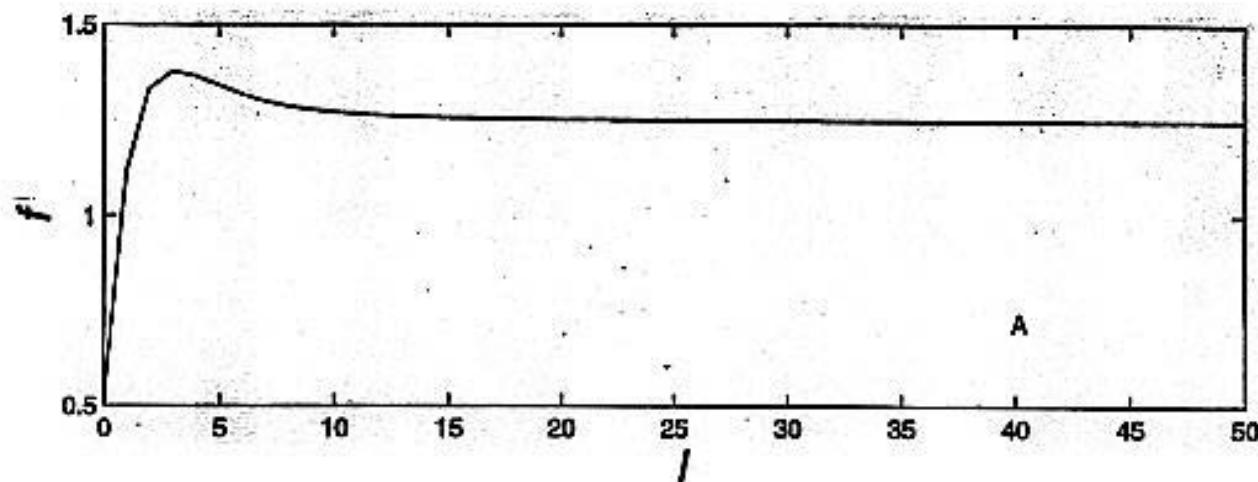
$$\text{tail } f_r = \sum_{l=0}^{\infty} (\overset{\text{bare } l}{f_r} - b_r)$$

The analytical calculation which shows that  $d_r = 0$  ensures that this expression for  $\overset{\text{tail}}{f_r}$  yields the correct physical result.

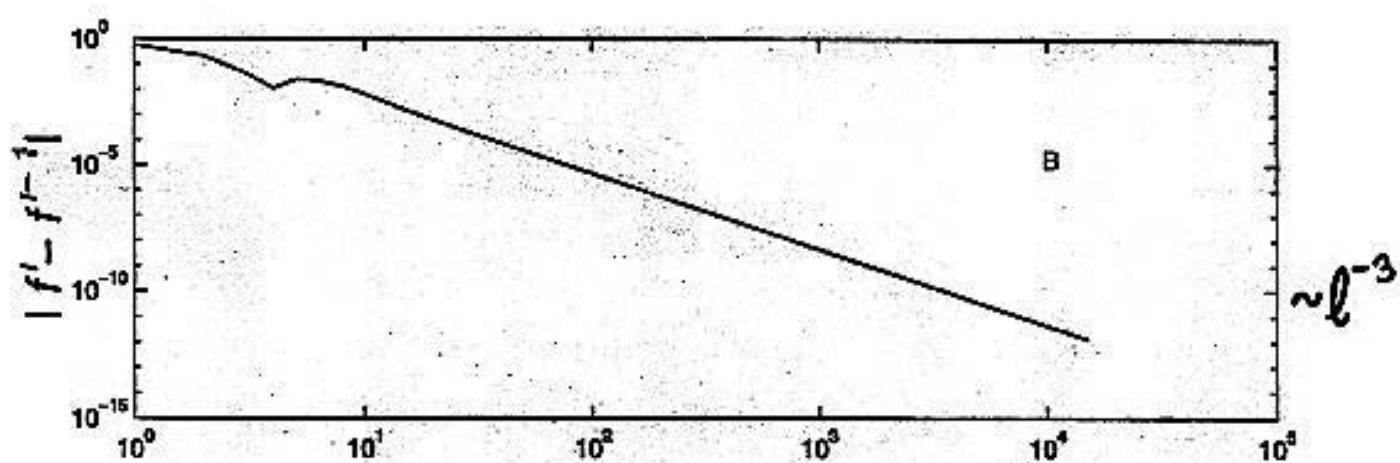
We find that  $\overset{\text{tail}}{f_r} = 0$ , in agreement with Zelnikov & Frolov (1982), Wiseman (1998), Mayo (1999).

Static scalar charge in Schwarzschild

The charge is at  $r_0 = 2.1M$



A



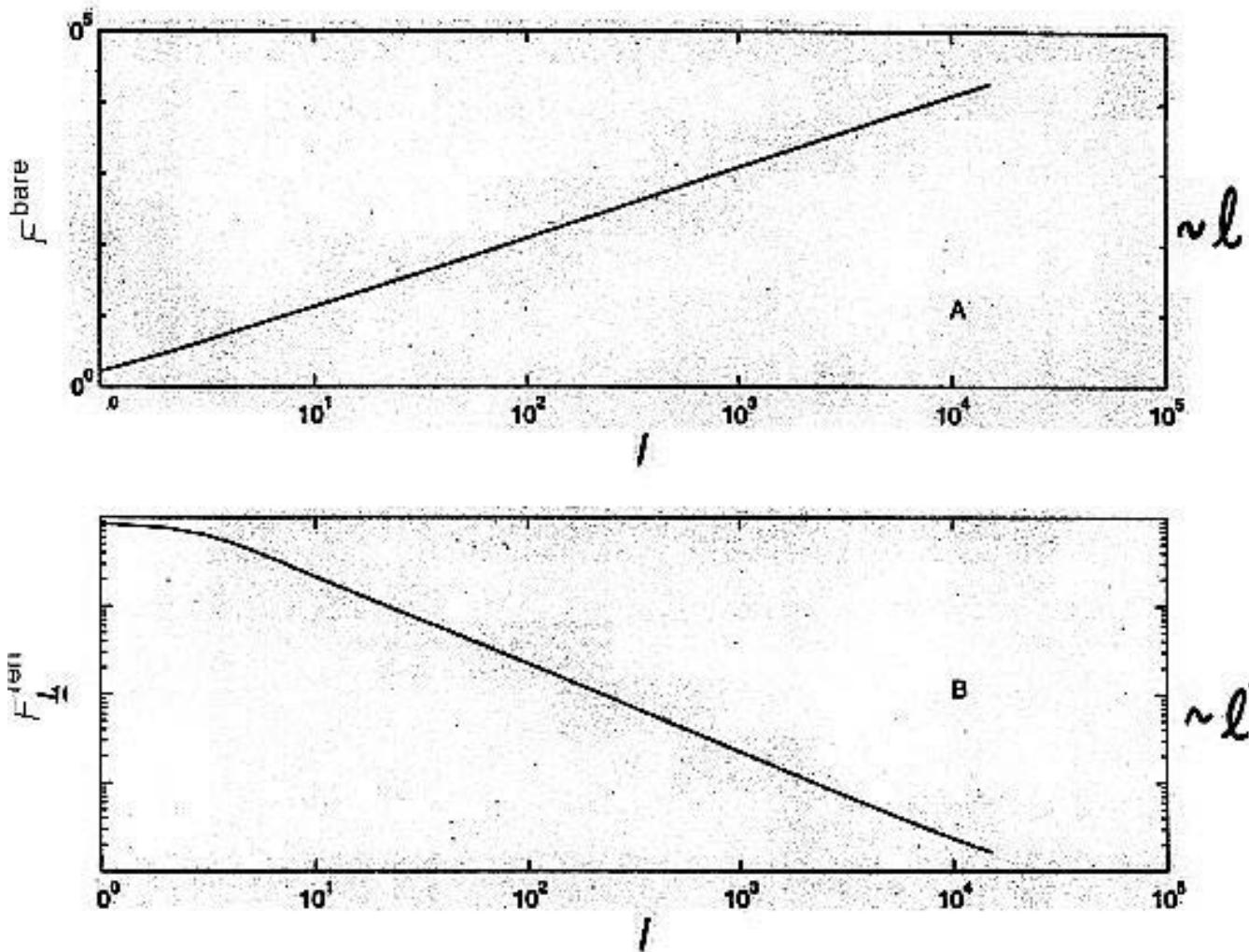
B

$\sim l^{-3}$

$$\frac{1}{2r_0^2} \left| \frac{1 - \frac{M}{r_0}}{1 - \frac{2M}{r_0}} \right|_{r_0 = 2.1M} = 1.247655$$

## Static scalar charge in Schwarzschild

The charge is at  $r_0 = 2.1M$



These are the sums of the bare and regularized radial forces up to a certain  $l$  as functions of  $l$ .

## STATIC ELECTRIC CHARGE IN SCHWARZSCHILD

Consider a point-like electric test charge, held fixed in Schwarzschild spacetime by an external force.

The linearized field equation:  $\nabla_\nu F^{\mu\nu} = 4\pi j^\mu$ , where  $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ ,  $j^\mu = \rho u^\mu$ .

Because of staticity,  $\frac{1}{\sqrt{-g}} (\sqrt{-g} g^{\alpha\beta} g^{\gamma\delta} A_{t,\alpha})_{,\beta} = -4\pi j^t$ , or

$$(r^2 A_{t,r})_{,r} + \frac{1}{1-2M/r} \left[ \frac{1}{\sin\theta} (\sin\theta A_{\theta,\theta})_{,\theta} + \frac{1}{\sin^2\theta} A_{\phi,\phi\phi} \right] = 4\pi r^2 j^t$$

We decompose  $A_t(r, \theta, \phi)$  according to  $A_t = \sum_{l=0}^{\infty} \sum_{m=-l}^l R_l^l(r) Y_l^m(\theta, \phi)$ , and find the radial equation

$$\frac{d}{dr} \left[ r^2 \frac{dR_l^l}{dr} \right] - l(l+1) \frac{1}{1-2M/r} R_l^l = 4\pi \delta(r-r_0) Y_l^m(\frac{\pi}{2}, 0).$$

The solution for  $A_t^l$  is given by:  $A_t^{l=0} = -\frac{q}{r} \Theta(r-r_0) - \frac{q}{r_0} \Theta(r_0-r)$ ,

$$A_t^{l \neq 0} = \frac{q}{M^3} \frac{2l+1}{l(l+1)} (r-2M)(r_0-2M) P_l(\cos\theta) \times$$

$$\left\{ P_l' \left( \frac{r-M}{M} \right) Q_l' \left( \frac{r_0-M}{M} \right) \Theta(r_0-r) + P_l' \left( \frac{r_0-M}{M} \right) Q_l' \left( \frac{r-M}{M} \right) \Theta(r-r_0) \right\},$$

where a prime denotes derivative with respect to the argument.

For the bare force we find  
are  $f_r = \frac{q^2}{\sqrt{1-2M/r_0}} \left[ -\frac{1}{r_0^2} + \frac{(r_0-2M)^2}{r_0 M^4} \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} P_l' \left( \frac{r_0-M}{M} \right) Q_l' \left( \frac{r_0-M}{M} \right) \right]$ .

We find that  $f_r^{\text{bare } l} \xrightarrow{l \rightarrow \infty} \text{const}$  from which we infer that  $a_r = 0$ .

Next, for large values of  $l$ ,  $\frac{d f_r^b}{dl} \sim l^{-3}$ , from which we infer that  $C_r = 0$ .

The asymptotic constant should equal  $b_r$ , but in this case we do not have an analytical value to compare with.

Most importantly, we do not know the value of  $d_r$ . With the knowledge of the final result for the regularized force, we shall show *a posteriori* that  $d_r = 0$ .

We try to regularize according to

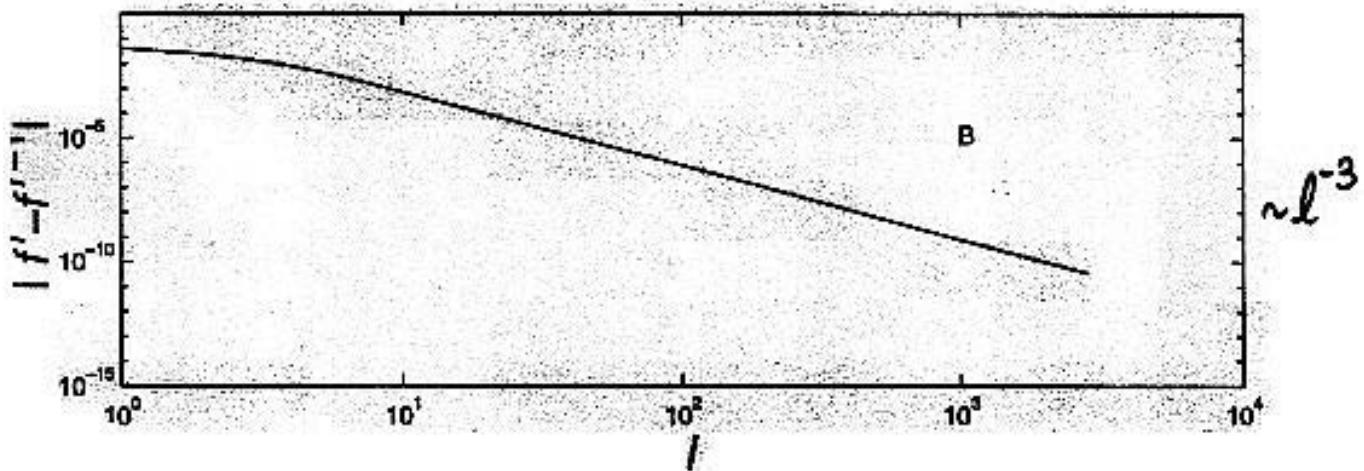
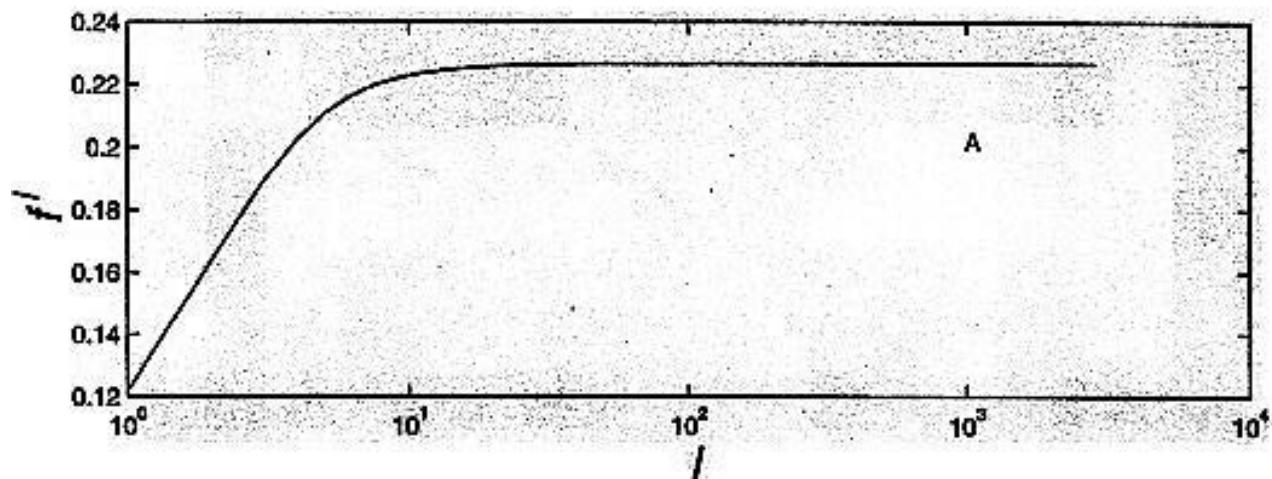
$$\text{tail } f_r = \sum_{l=0}^{\infty} (\text{bare } f_r^l - b_r), \text{ with } b_r = \lim_{l \rightarrow \infty} \text{bare } f_r^l$$

The result is a finite force, which we compare with Smith & Will (1976) and Frolov & Zel'mikor (1980), who give  $\text{tail } f_r = M \frac{q_2}{r^3} \frac{1}{\sqrt{1 - 2M/r}}$

The sum up to  $l=1000$  agrees with this result to  $4 \times 10^{-4}$  for  $r_o = 2.1M$ .

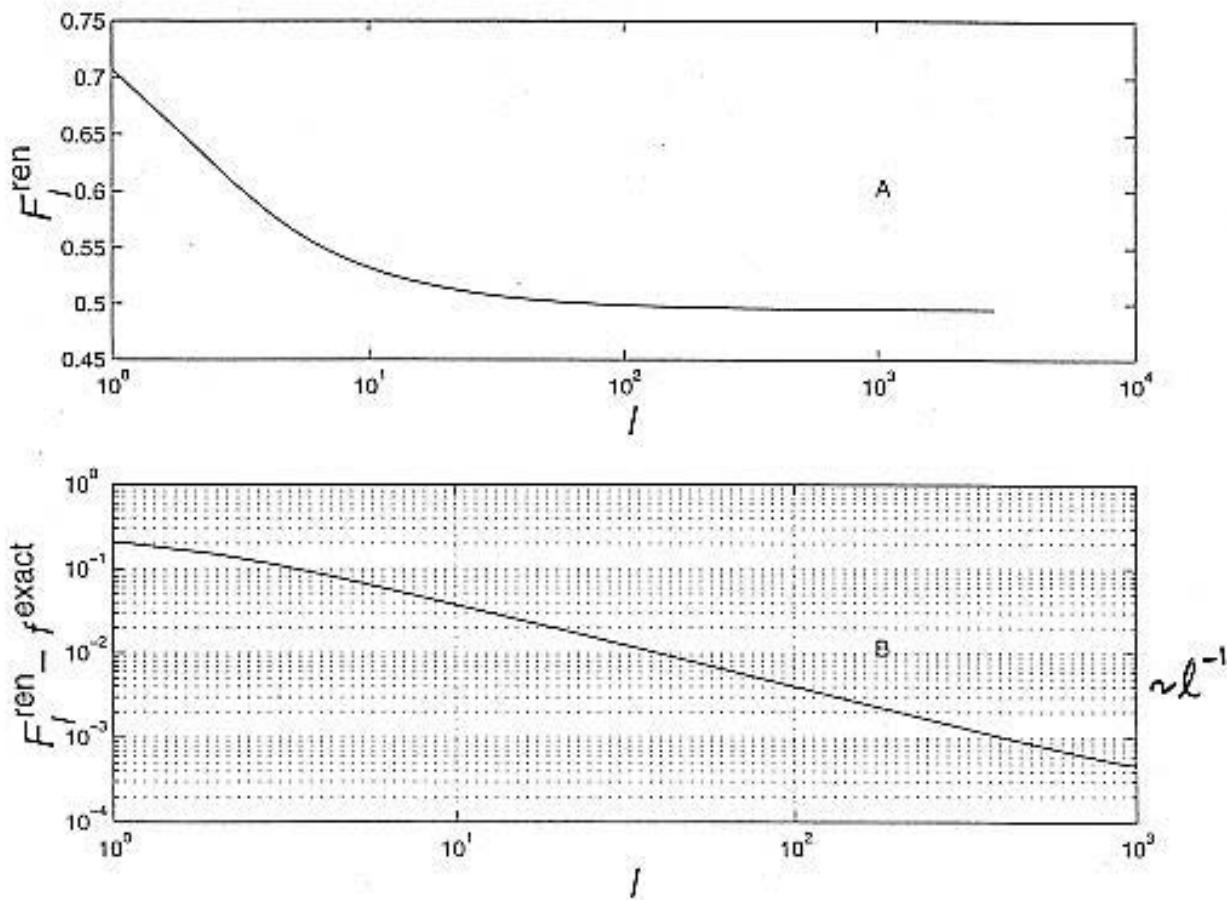
Static electric charge in Schwarzschild

charge is at  $r_0 = 2.1 \text{ M}$



Static electric charge in Schwarzschild

The charge is at  $r_0 = 2.1 M$



## SCALAR CHARGE IN CIRCULAR MOTION AROUND SCHWARZSCHILD

Our analysis is fully relativistic, i.e., we do not introduce any simplifying assumptions such as far-field or slow motion. Therefore, our solution is numerical.

It is convenient to use Regge-Wheeler coordinates, where

$$ds^2 = \left(1 - \frac{2M}{r}\right) (-dt^2 + dr_*^2) + r(r_*) (d\theta^2 + \sin^2\theta d\varphi^2).$$

We take the charge to be in uniform circular motion in the equatorial plane  $\theta = \frac{\pi}{2}$ , and  $\frac{d\varphi}{dt} = \Omega$ .

We decompose the field  $\Phi(t, r_*, \theta, \varphi)$  according to

$$\Phi = \int_{-\infty}^{\infty} dw \sum_{lm}^{\infty} e^{-iwt} \frac{i}{r(r_*)} \psi_{lm}^{(r_*)} Y^{lm}(\theta, \varphi),$$

and the radial equation is

$$\frac{d^2 \psi_{lm}}{dr_*^2} + \left\{ \omega^2 - V_l[r(r_*)] \right\} \psi_{lm} = -4\pi \frac{g}{r_*} \frac{\delta(r_* - r_*)}{r_*^m} \delta(\omega - m\Omega) Y^{lm}\left(\frac{\pi}{2}, \varphi\right) e^{-im\varphi}.$$

This equation should be solved for each mode  $lm$  with appropriate boundary conditions: ingoing waves at the horizon ( $r_* \rightarrow -\infty$ ) and outgoing waves at infinity ( $r_* \rightarrow +\infty$ ).

The potential  $V_l(r) = (1 - \frac{2M}{r}) \left[ \frac{2M}{r^3} + \frac{l(l+1)}{r^2} \right]$ , and  $g = (1 - \frac{2M}{r} - r^2\Omega^2)^{-1/2}$ .

We generate the solution to the inhomogeneous radial equation with the solutions to the corresponding homogeneous equation.

These solutions are well-known from the theory of quasi-normal ringing of black holes.

The homogeneous solutions are  $\psi_{lm}^{\pm}$ .

We know that at infinity  $\psi_{lm}^{+}(r_*) \sim e^{i\omega r_*}$   
and at the event horizon  $\psi_{lm}^{-}(r_*) \sim e^{-i\omega r_*}$ .

In order to avoid the problem of phase determination for the initial conditions for the radial equations, we define

$$\psi_{lm}^{\pm}(r_*) = e^{\pm i\omega r_*} Z^{\pm lm}(r_*)$$

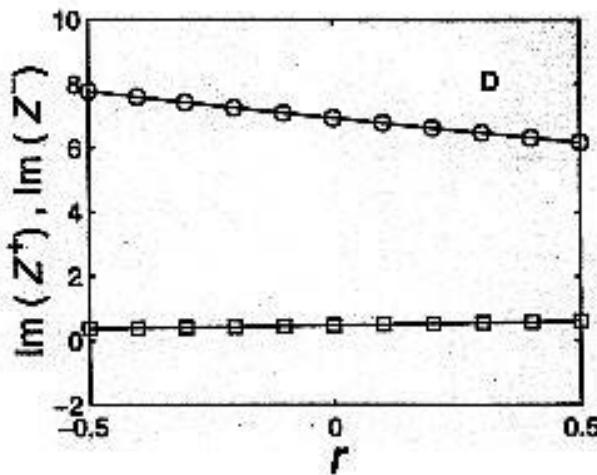
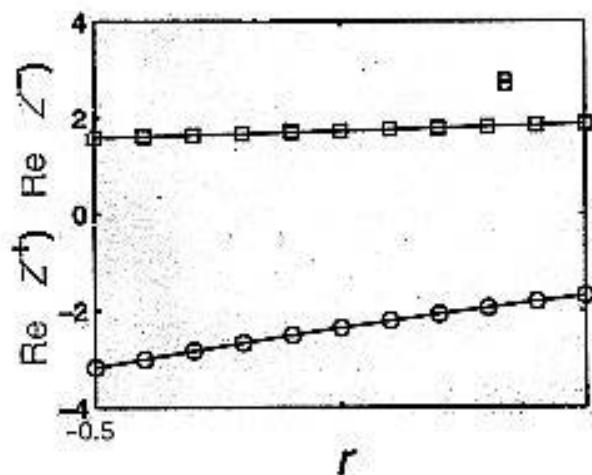
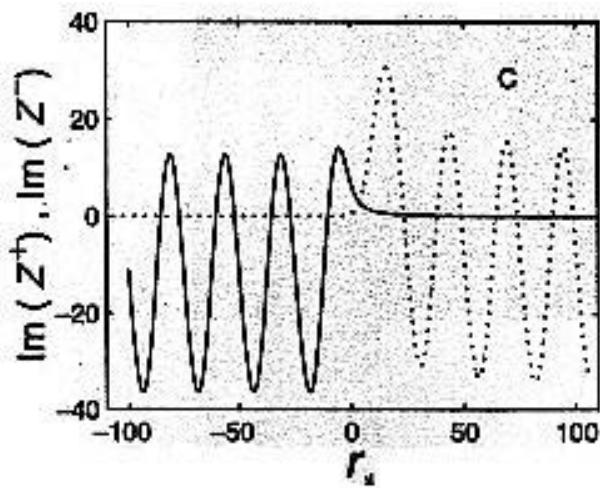
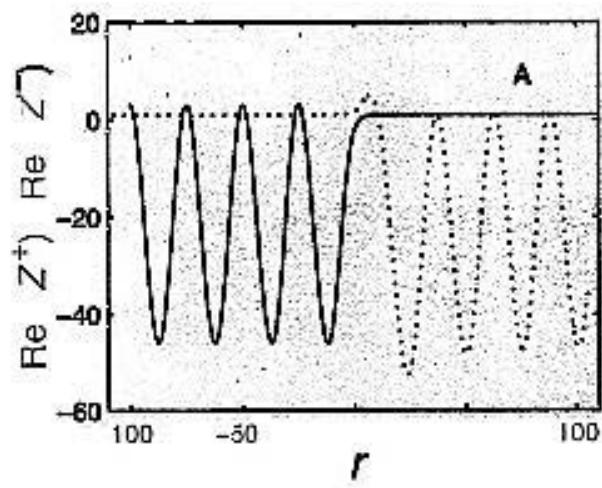
$$\frac{d^2 Z^{\pm lm}}{dr_*^2} \mp 2i\omega \frac{dZ^{\pm lm}}{dr_*} - V_l[r(r_*)] Z^{\pm lm} = 0$$

with boundary conditions  $Z^{\pm lm}(r_*) \xrightarrow[r_* \rightarrow \infty]{} 1$ .

We integrate  $Z^{+lm}$  from  $r_*^b = 1.5 \times 10^3 M$  inwards and  $Z^{-lm}$  from  $r_*^b = -1.5 \times 10^3 M$  outwards, and approximate the values of  $Z^{\pm lm}$  there by their asymptotic values. [These are good approximations for  $|\omega r_*| \gg 1$ , which we satisfy.]

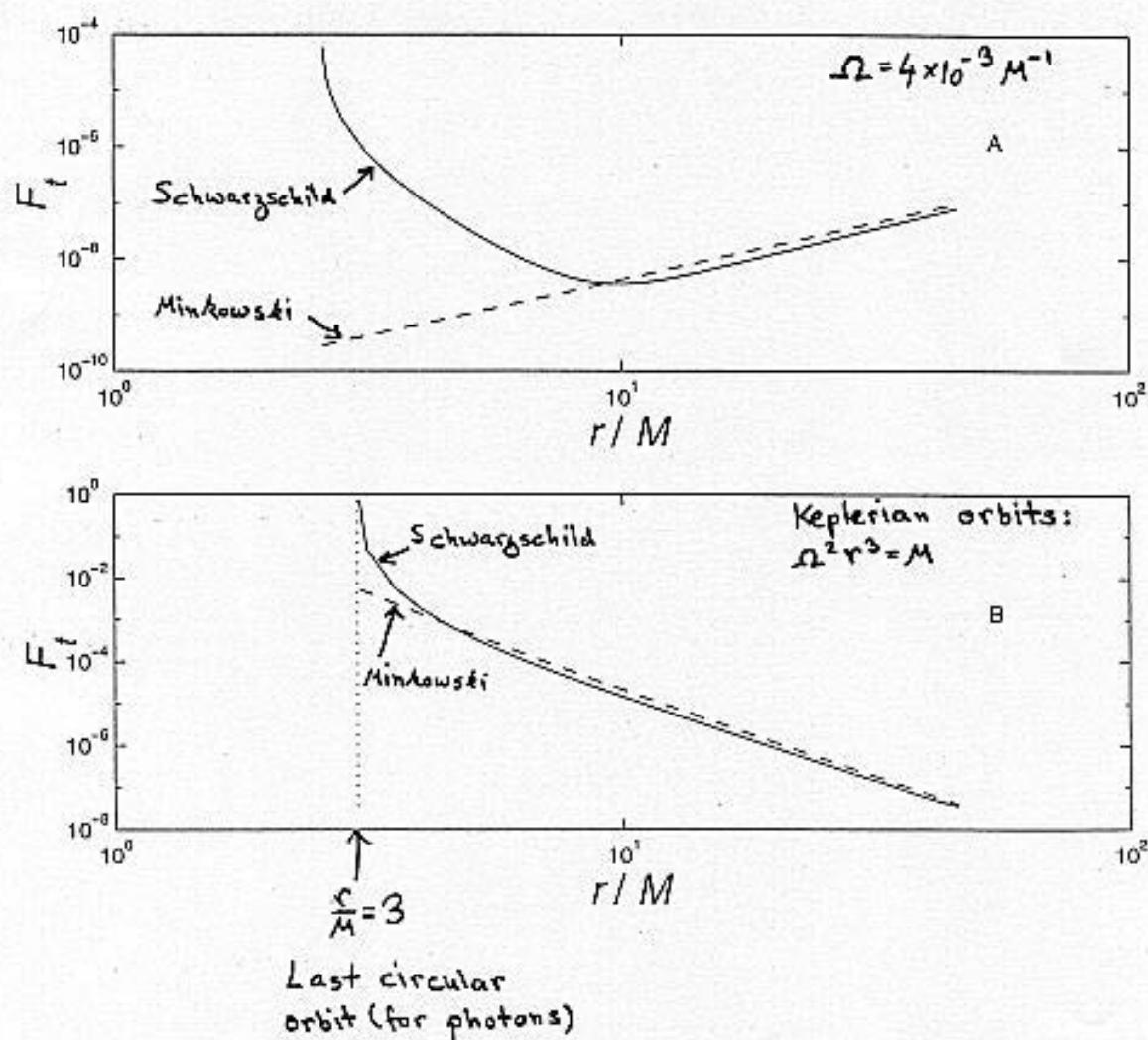
Scalar charge in circular motion around  
Schwarzschild

Data for  $\ell_1 = 1$



- Burmisch-Stoer code
- 4<sup>th</sup> Order Runge-Kutta code
- both with adaptive step-size controls.

# Scalar charge in circular motion around Schwarzschild



Radial component:

- first study the slow motion limit, close to the event horizon (non-geodesic motion).

We find agreement with the analytical value for  $b_r = -\frac{q^2}{2r^2} \frac{1-\frac{M}{r}}{1-2\frac{M}{r}}$  to  $3 \times 10^{-7}$  for  $l \approx 9 \times 10^4$ . This also implies  $a_r = 0 = C_r$ .

For geodesic (Keplerian) orbits we find again that  $\overset{\text{bare}}{F_r} \xrightarrow[l \rightarrow \infty]{} \text{const}$ , which implies  $a_r = 0$ .

In addition  $\frac{d \overset{\text{bare}}{F_r}}{dl} \sim l^{-3}$  for large values of  $l$ , which imply  $C_r = 0$ .

It can be shown analytically (Ori) that  $d_r = 0$ .

The regularized radial force is therefore

$\overset{\text{tail}}{F_r} = \sum_{l=0}^{\infty} (\overset{\text{bare}}{F_r} - b_r)$ , and we approximate  $b_r$  by

$b_r = \lim_{l \rightarrow \infty} \overset{\text{bare}}{F_r} \approx \overset{\text{bare}}{F_r}'$  with  $l' \gg l$  up to which we sum over the modes numerically.

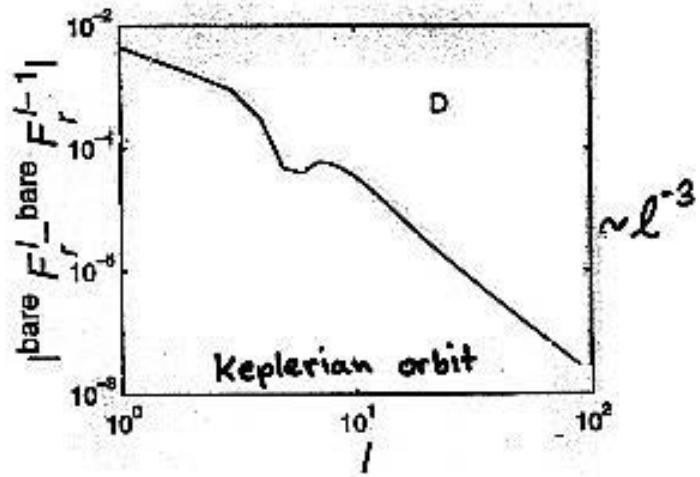
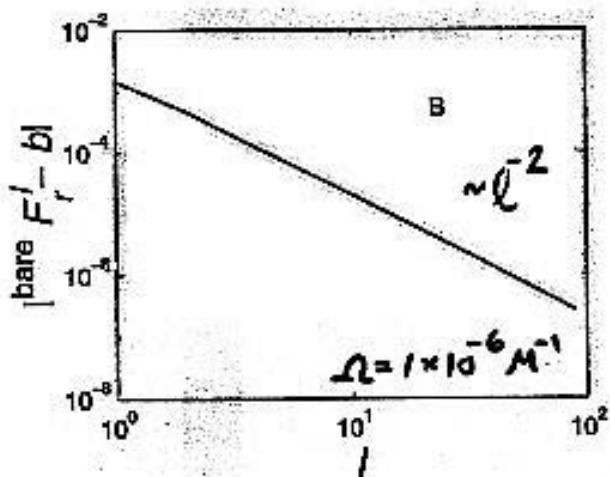
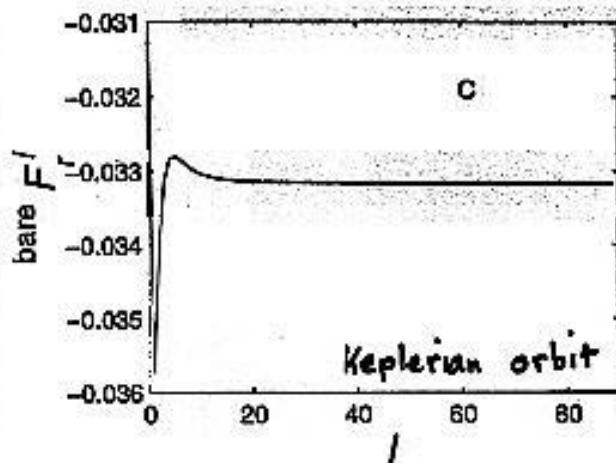
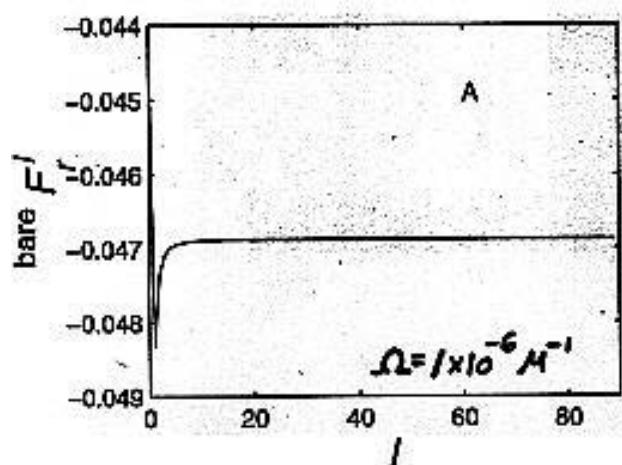
In practice we take  $l \approx 90$  and  $l' = 300$  or  $400$

Our results suggest  $\overset{\text{tail}}{F_r} \approx -q^2 M^3 r^{-5}$  for geodesic orbits, or

$$\overset{\text{tail}}{F_r} \approx -q^2 M^2 \frac{\Omega^2}{r^2}$$

for geodesic and non-geodesic orbits.

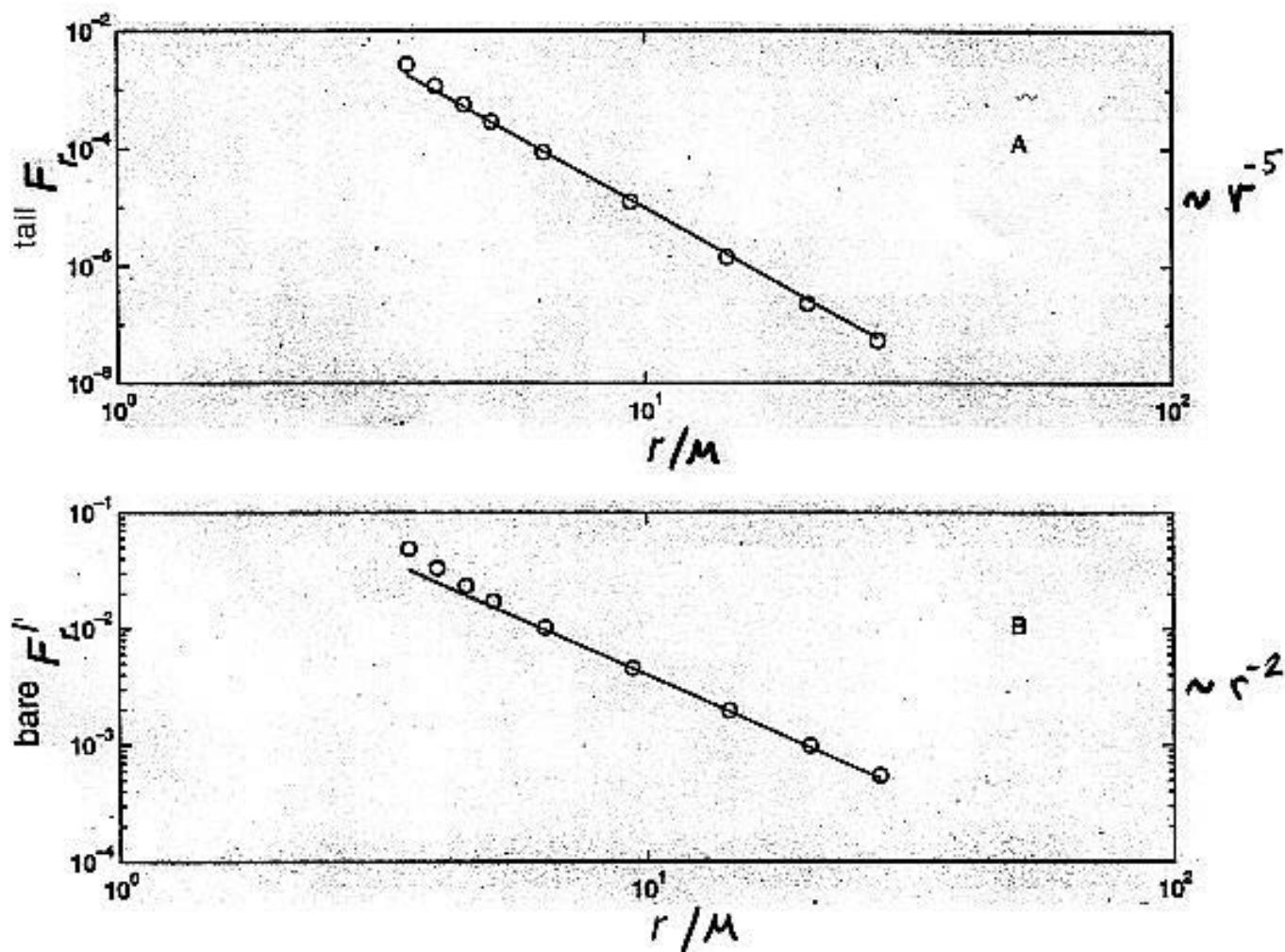
Scalar charge in circular motion around Schwarzschild,  
at  $r = 4M$ .



$$b_r = -\frac{q^2}{2r^2} \frac{1-M/r}{1-2M/r} \text{ in the static limit.}$$

Scalar charge in circular orbit around Schwarzschild

Geodesic (Keplerian) orbits  $\Omega^2 r^3 = M$



## CONCLUSIONS

Oris prescription is useful for applications and is also practical from the calculational viewpoint

More work is needed, in particular for evaluating  $d\mu$ , which cannot be evaluated from the bare force

Our result for non-vanishing radial force for circular orbits suggests non-vanishing force also for elliptical orbits, which imply additional precession of the periastron. Could be observational (Schutz).

Next steps      Elliptical orbits      Schwarzschild  
                  Orbits in Kerr

many  
steps

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- hopefully!  
- Orbital evolution of compact objects around a supermassive black hole  
(for LISA)