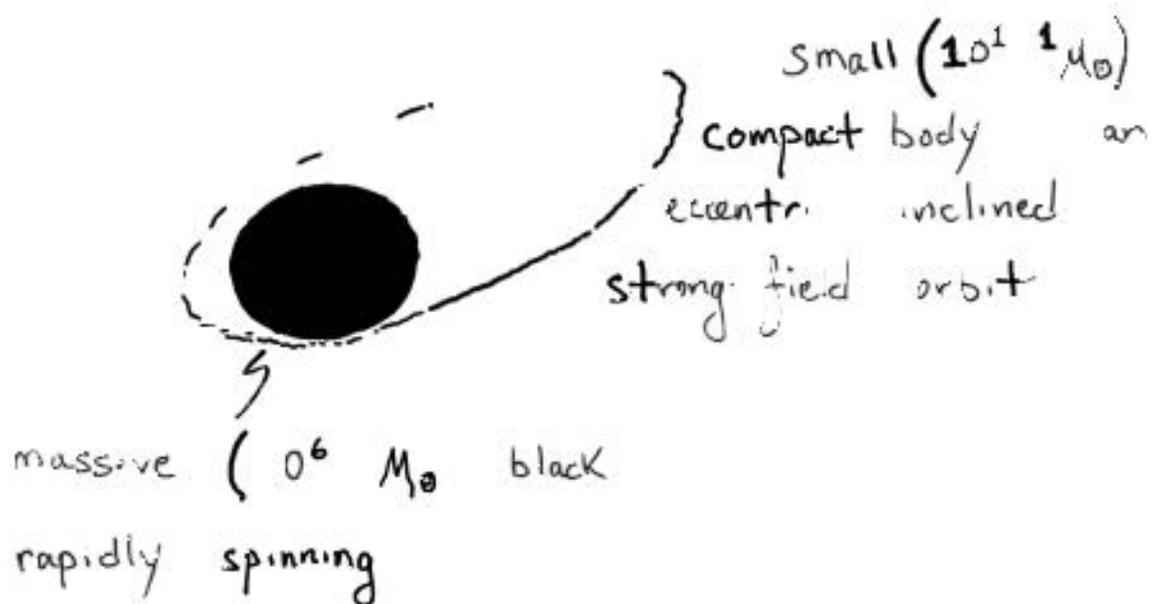


Poor man's radiation reaction

Progress and recent results with
the Teukolsky formalism



- 1 Why is this interesting
- 2 What is needed to understand radiation reaction
- 3 What is being done now
- 4 What should we do next?

Details

Solution of the relativistic two-body problem in the limit of one body being much more massive than the other:

• μ


 M, a

$$g_{ab} = g_{ab}^{\text{Kerr}}(M, a) + h_{ab}(\mu)$$

$$\frac{\|h\|}{\|g^{\text{Kerr}}\|} \sim \frac{\mu}{M}$$

If μ/M is small enough, can use perturbation theory + study the system

The interest: sources of gravitational radiation
for space-based detectors (LISA)

Consider final inspiral of a $1 M_{\odot}$ compact
body into a $10^6 M_{\odot}$ black hole



The compact body takes
 ~ 1 year to spiral from
 $r \sim 4M$ to the ISCO

During this final year, the emitted gravitational
waves sweep through the frequency band

$7 \times 10^{-3} \text{ Hz} \leq f \leq 3 \times 10^{-2} \text{ Hz}$
emitting roughly 4×10^5 cycles of radiation

➡ Frequencies are right in LISA's band

➡ Large number of cycles means very good
ability to determine system's parameters

Measurement in the strong field is a great probe of general relativity

The waveforms emitted in this regime will be strongly imprinted with the characteristics of the Kerr spacetime.

Consider purely geodesic motion: an eccentric, inclined orbit resides on a plane passing through the black hole

1. The particle cycles around the ellipse with some frequency Ω_ϕ
2. The ellipse rotates within the plane with some frequency Ω_{pp}
3. The plane rotates about the spin axis with some frequency Ω_{LT}

All three frequencies and their harmonics will be present in the waveform!

Interesting waveform and potential for interesting science but are there enough inspirals for this to be an interesting source?

Rees and Sigurdsson examined the distribution of massive black holes and stellar populations at galactic centers

Conservative parameter choice $\rightarrow R \sim \frac{1 \text{ event}}{\text{yr [Gpc]}^3}$

Some uncertainties:

Distribution of $10^{6 \pm 1} M_{\odot}$ black holes is not well-developed (poor statistics beyond local group)

Mechanism for replenishing compact bodies which have been fed to the BH is not well understood

Rate could be a lot higher!

$$R \sim 1 / [\text{month Gpc}^3]$$

$$\sim 1 / [\text{week Gpc}^3]$$

External Influences: are mostly negligible

Most important such influence is likely to be drag from accreting matter. Timescale for drag to change the orbital angular momentum has been estimated by Ramesh Narayan

$$t_d = J/\dot{J}$$

Key issue need drag to change the orbital phase by a fraction less than 10^{-7} (one radian over a full year)

For 1 year of observation we hope for

$$t_d > 10^7 \text{ years}$$

Nature of drag follows from nature of accretion flow

ADAF



Look at Bondi-Hoyle accretion onto the compact body as it orbits - calculate how rapidly it loses orbital velocity



t_d

WD

0^2

0^{16}

years

BH

Thin Disk



Orbiting body takes a whack each time it passes through the disk

Worst case (orbit in disk)

t_d 10^6 years

Evidence suggests

NADAF
NTD

100 1000

→ In a most a cases the idealized two body description will be sufficient to model these systems

Radiation Reaction

The small body which orbits the black hole moves on a geodesic of the g_{ab} spacetime metric

$$g_{ab} = g_{ab} + h_{ab}$$

metric of a Kerr black hole perturbation due to small body

It is convenient to regard the body as moving on a geodesic of the Kerr hole plus a "radiation reaction" correction

$$\vec{u} = \vec{u}_0 + \int \vec{f}_{RR}$$

linear \sim perturbation

Major task compute the radiation reaction force \vec{f}_{RR}

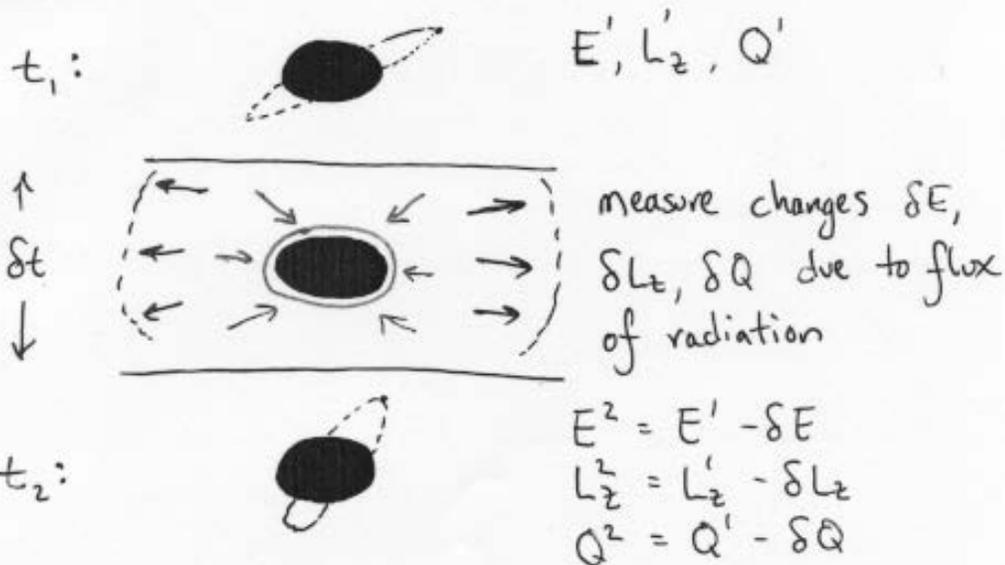
Adiabaticity

When $\mu/M \ll 1$ the radiation reaction timescale τ_{RR} is much longer than an orbital period T_{orb} over a large fraction of the system's evolution

Suggests an approach to radiation reaction without requiring a radiation reaction force:

"Poor Man's Radiation Reaction"

In this approximation, the system evolves from one geodesic, characterized by (E', L_z', Q') to another, (E^2, L_z^2, Q^2) , in a quasi-equilibrium manner:



THIS APPROACH DOESN'T WORK IN GENERAL

Although one can read the charges in the energy and the angular momentum out of the radiation fluxes, one cannot read off the change in the Carter constant.

Consider: particle of momentum p_1^μ emits radiation of momentum Δp^μ , goes to new momentum p_2^μ :

$$E_1 = T_{\mu} p_1^\mu \quad E_2 = T_{\mu} p_2^\mu$$

(time Killing vector)

$$\Delta E = E_1 - E_2 = \underbrace{T_{\mu} \Delta p^\mu}_{\text{Easily read out of the radiation!}}$$

Easily read out of the radiation!

$$Q = K_{\mu\nu} p_1^\mu p_1^\nu \quad Q_2 = K_{\mu\nu} p_2^\mu p_2^\nu$$

(Killing tensor!)

$$\Delta Q = Q - Q_2 = -2 K_{\mu\nu} p_1^\mu \Delta p^\nu + \mathcal{O}(\Delta p^2)$$

NOT easily read out of the radiation!

Take limits, find that \dot{Q} needs the local radiation reaction force:

$$\dot{Q} = \lim_{\Delta t \rightarrow 0} \frac{\Delta Q}{\Delta t} = -2 K_{\mu\nu} p^\mu \dot{f}^\nu$$

Poor man's radiation reaction fails
in general: need radiation reaction force even for adiabatically evolving systems!

Without radiation reaction force, we're restricted to systems whose motion is somewhat constrained:

Equatorial Orbits: $Q = 0 = \dot{Q}$, so we only need to worry about \dot{E} and \dot{L}_z

→ Since Schwarzschild holes have no preferred orientation, any Schw. orbit can be treated as equatorial.

Circular Orbits: it has been shown that adiabatically evolving circular orbits go from one circular configuration to another. [Proof: Kennedick and Ori, Ryan, Mino]

→ This sufficiently constrains the system that \dot{Q} can be computed from \dot{E} and \dot{L}_z

∴ IN THIS CASE, \dot{Q} can be determined from the radiation fluxes!

Specialize to circular orbits

Radial motion in the Kerr metric is governed by the equation

$$\Sigma^2 \left(\frac{dr}{dt} \right)^2 = [E(r^2 + a^2) - aL_z]^2 - \Delta [r^2 + (L_z - aE)^2 + Q]$$

$$\equiv R$$

$$\left[\text{Note: } \Sigma = r^2 + a^2 \cos^2 \theta \right.$$

$$\Delta = r^2 - 2Mr + a^2 \left. \right]$$

Circular motion requires

$$R = 0, \quad R' = \frac{dR}{dr} = 0$$

Circular adiabatically evolves to circular:

$$R(t) = R(t + dt) = 0$$

$$R'(t) = R'(t + dt) = 0$$

$$\rightarrow \dot{R} = 0, \quad \dot{R}' = 0$$

Simultaneous solution of these equations yields rates at which Q and r change (given \dot{E} and \dot{L}_z).

The Teukolsky equation

Governs the evolution of the curvature perturbation

$$\Psi_4 = -C_{abcd} n^a m^b \bar{n}^c m^d$$

Weyl curvature \nearrow members of the Newman Penrose null tetrad $\nwarrow \nearrow \nearrow$

Useful because Ψ_4 contains information about radiation flux at infinity

$$\Psi_4(r \rightarrow \infty) \sim 2(h_+ - h_x)$$

\rightarrow easy to extract E^0 L_2^∞ from this

With effort can show that this also provides fluxes down the horizon

\Rightarrow ALL of E, L_2 can be extracted from Ψ_4

Computational details

The equation for ψ_4 is separated via a multipolar decomposition:

$$\psi_4 = \frac{1}{(r - ia \cos \theta)^4} \int_{-\infty}^{\infty} d\omega \sum_{lm} R_{lm\omega}(r) {}_{-2}S_{lm}^{\omega}(\theta) \underbrace{e^{im\phi}}_{\text{trivial}} e^{-i\omega t}$$

The "Teukolsky function": HARD!

spin-weighted
spheroidal harmonic:
easy

The radial part of the solution obeys a simple looking ODE:

$$\Delta^2 \frac{d}{dr} \left(\frac{1}{\Delta} \frac{dR_{lm\omega}}{dr} \right) - V(r) R_{lm\omega}(r) = -T_{lm\omega}(r)$$

$$V(r) = - \frac{K^2 + 4i(r-M)K}{\Delta^2} + 8i\omega r + \lambda$$

$$K(r) = (r^2 + a^2)\omega - ma$$

Sanity checks believe No numerical calculation
unless + can reproduce known behavior!

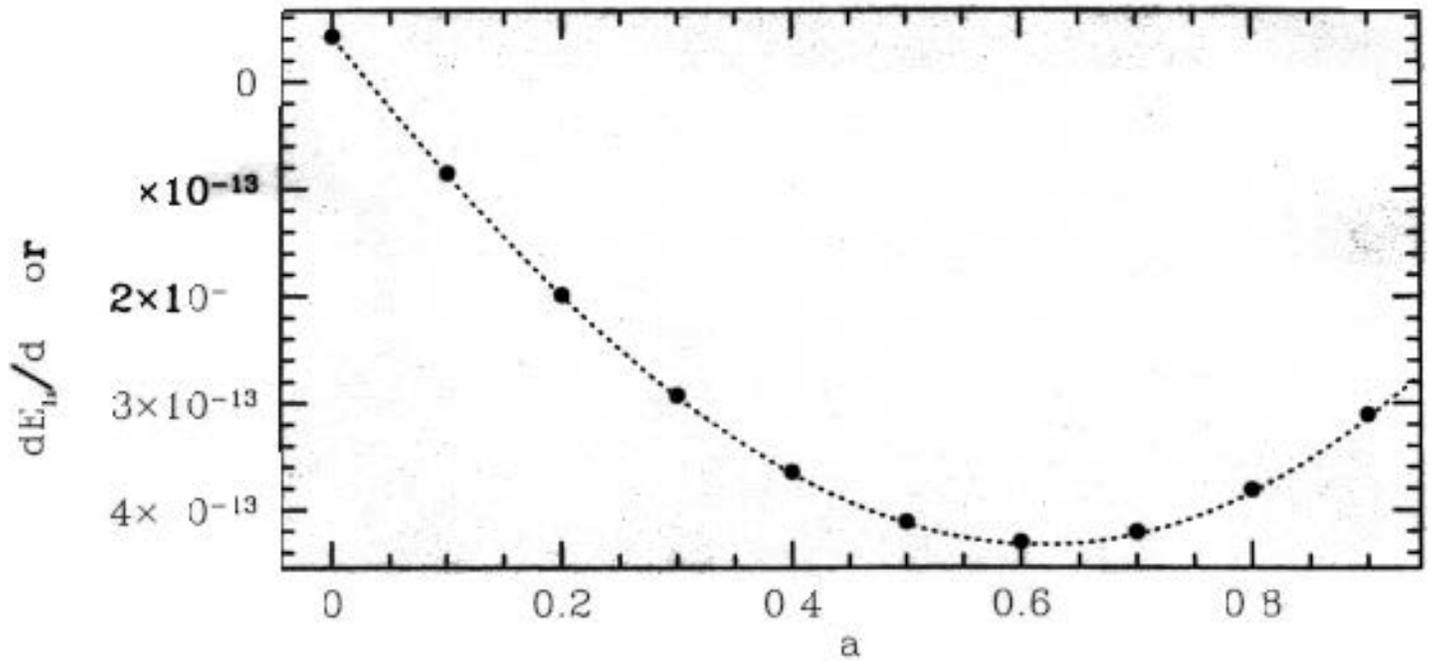
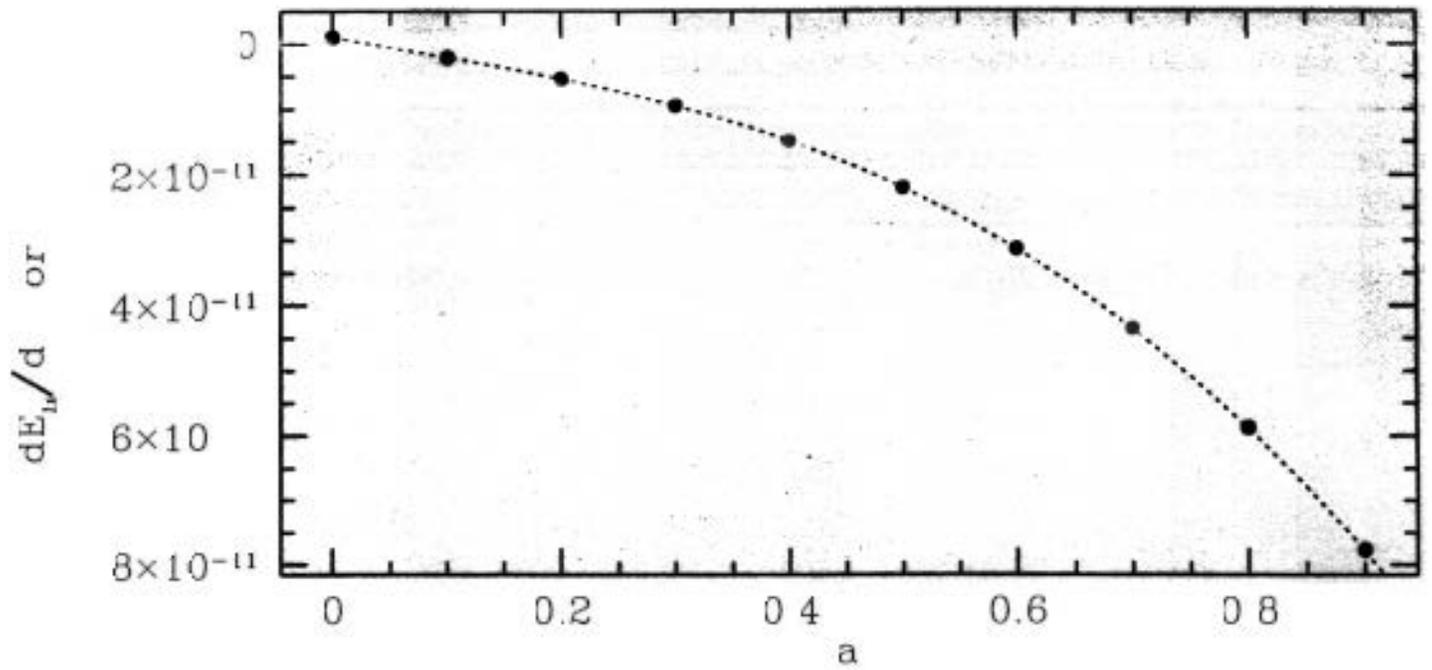
Weak field limit extensive work by

Mino Nakamura Sasaki Shibata

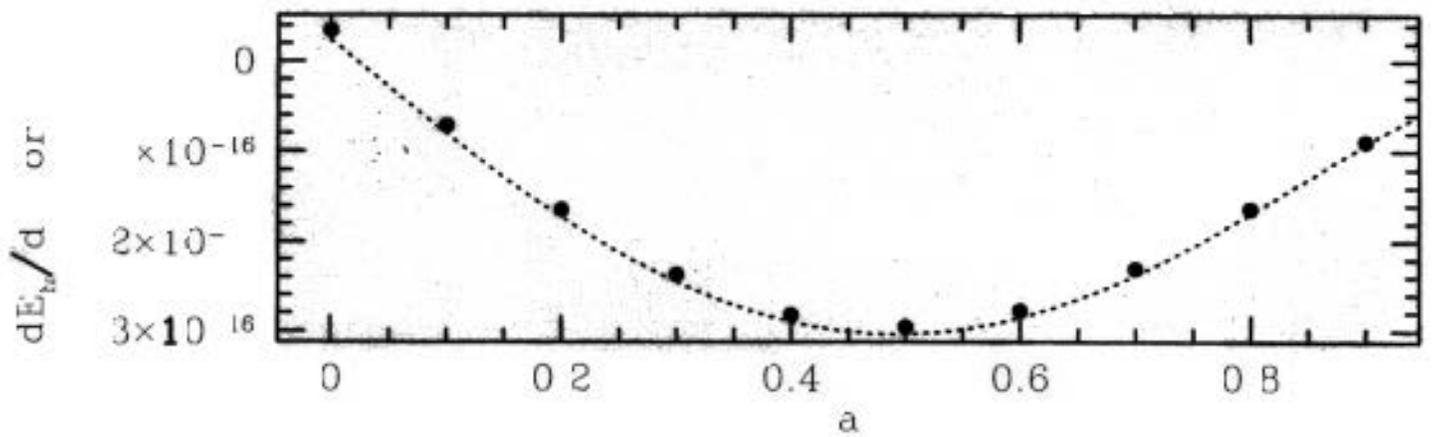
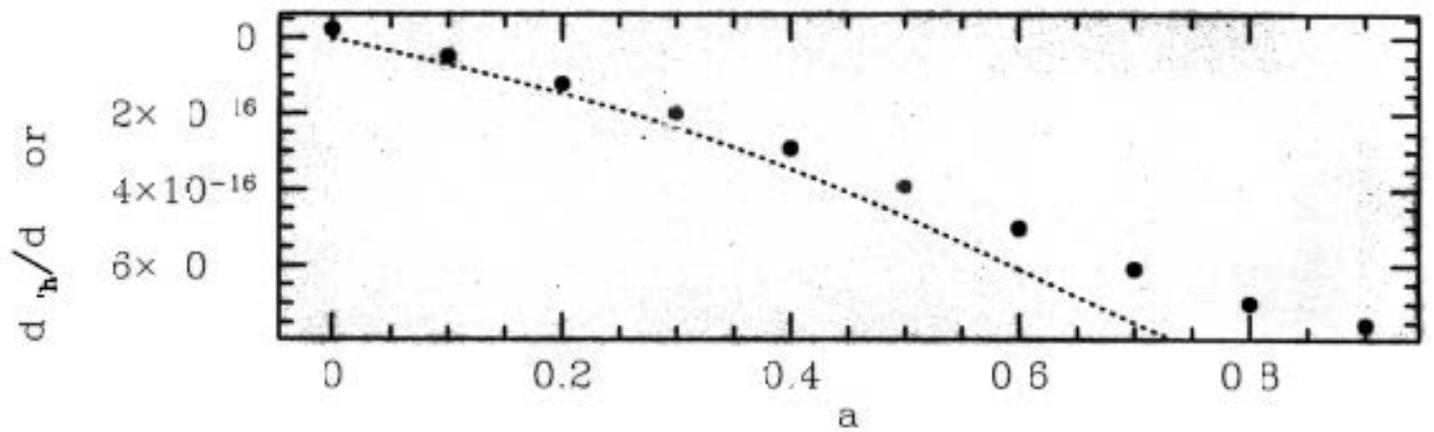
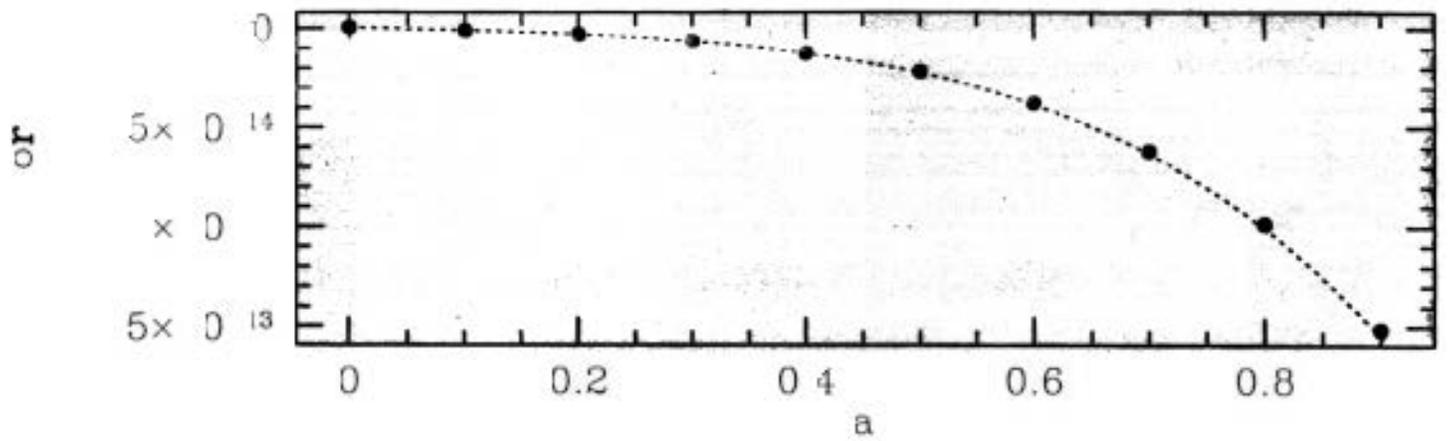
Tagosh Tanaka gives post-Newtonian

expansions for many relevant quantities

Useful check for larger



$r = 25 M$



r 2SM

Schwarzschild limit: in this limit,

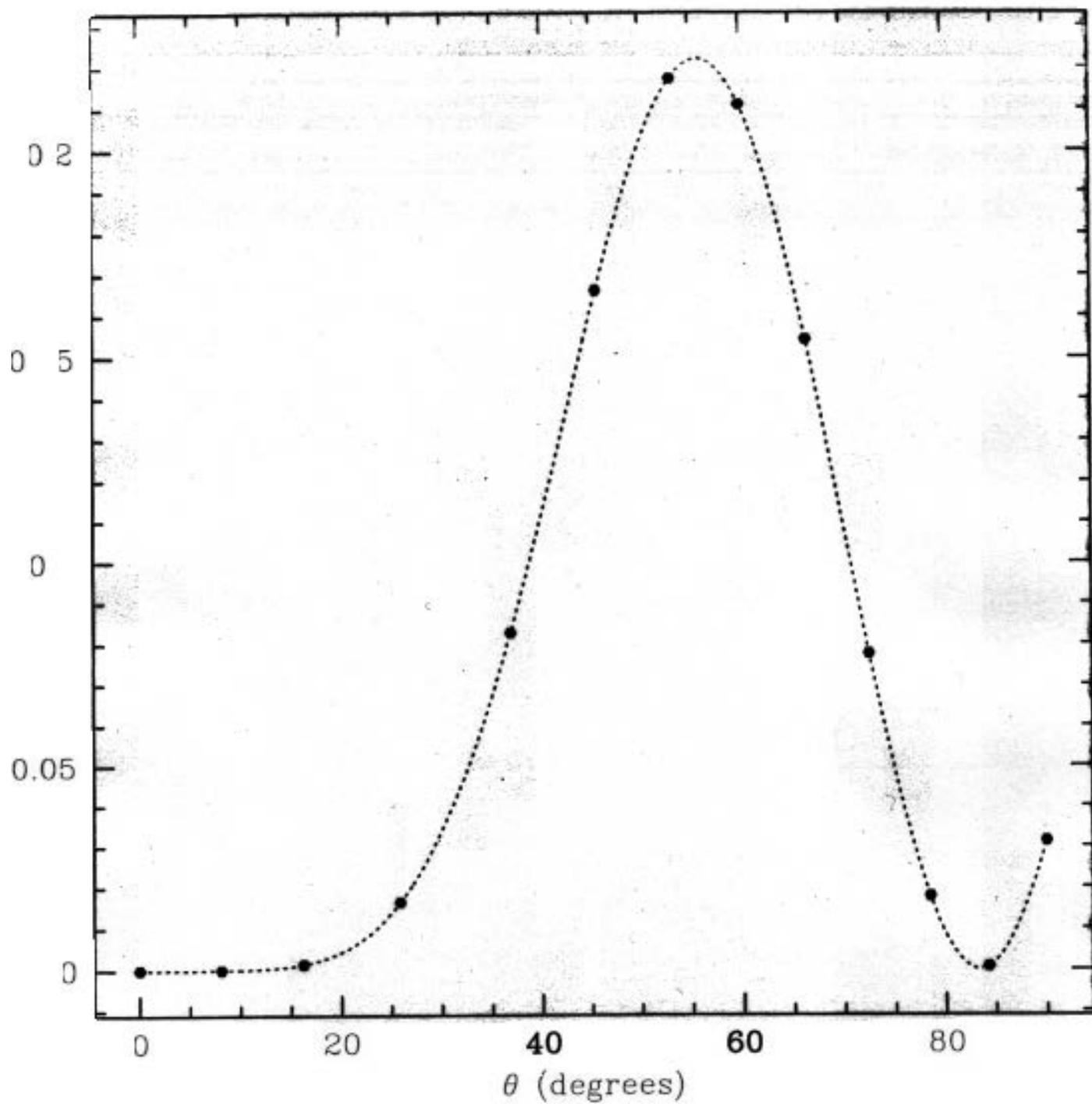
$$\Omega_0 = \Omega_\phi$$

Equatorial and off equator orbits must radiate the same amount - the distribution of energy among the harmonics changes.

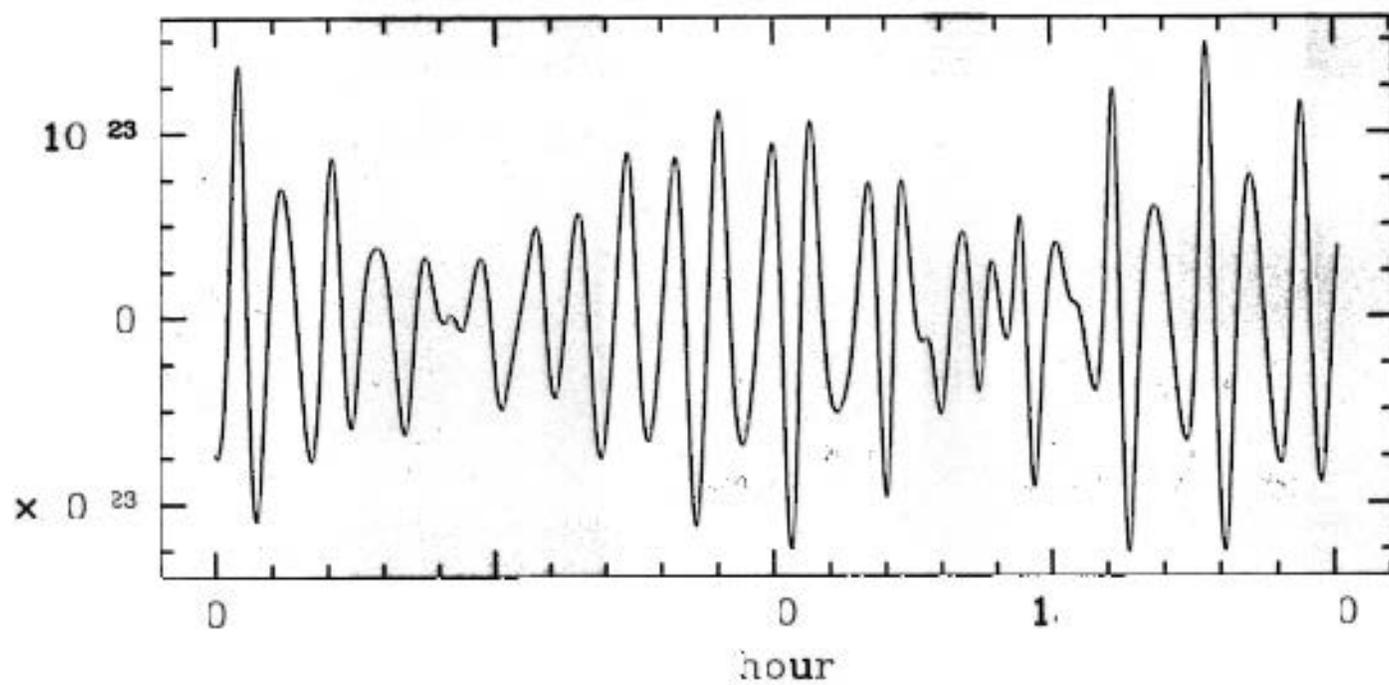
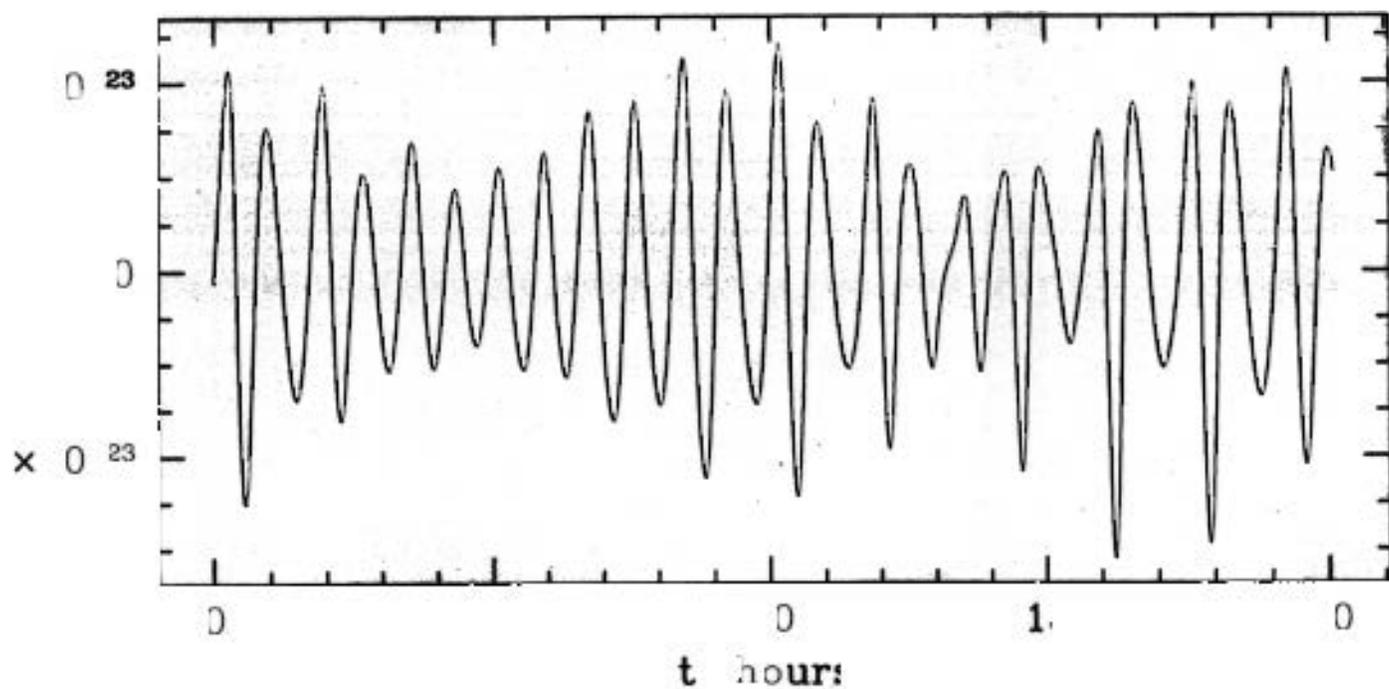
The rule for determining this distribution is given by the rules for rotating spherical harmonics: we get

$$\frac{\dot{E}_{lm}(k)}{\dot{E}_{lm}^{eq}} = |D_{mm'}^l(\theta)|^2$$

Wigner D-function

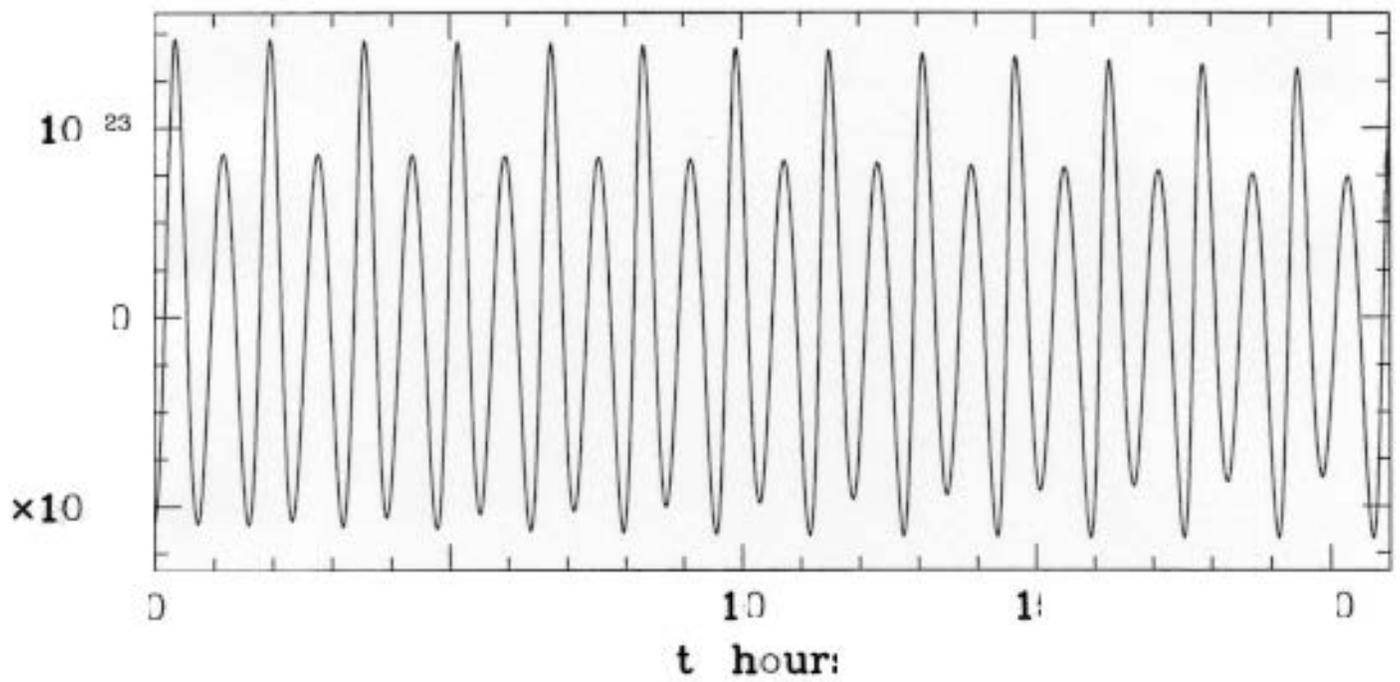
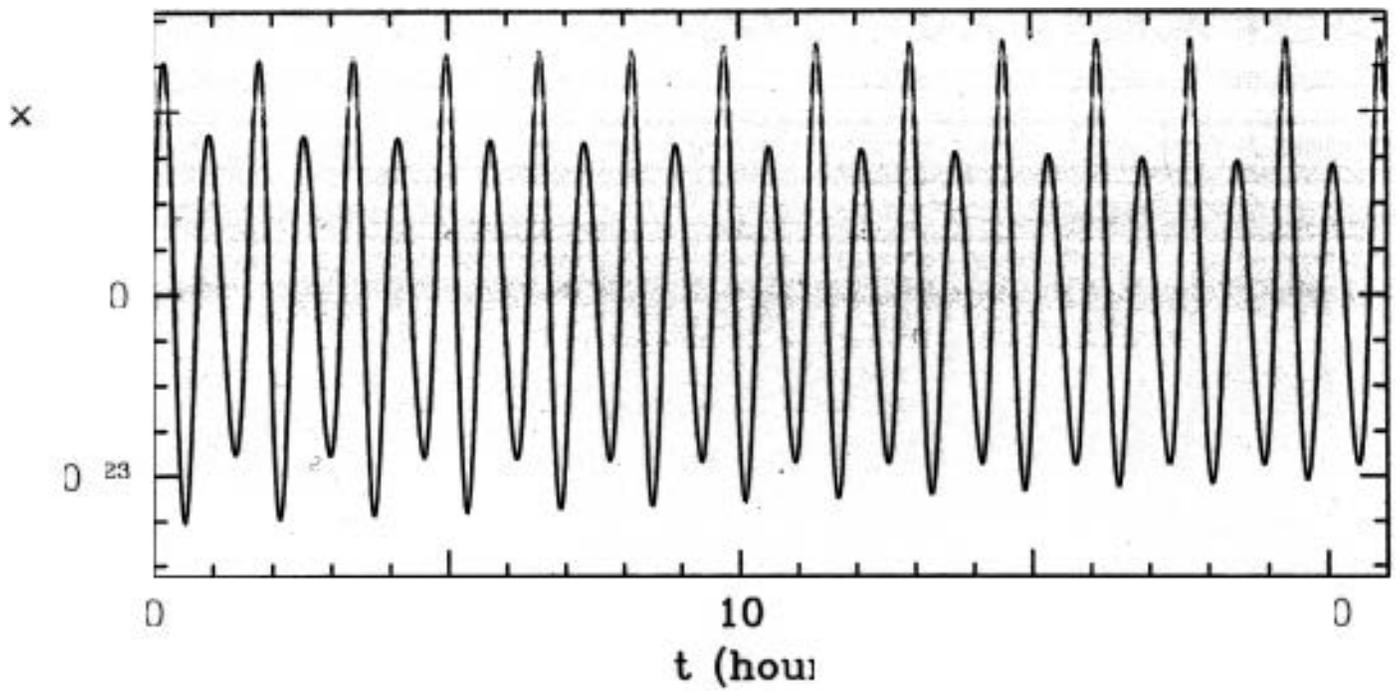


$l = 4 \quad m = 2 \quad m = 1$



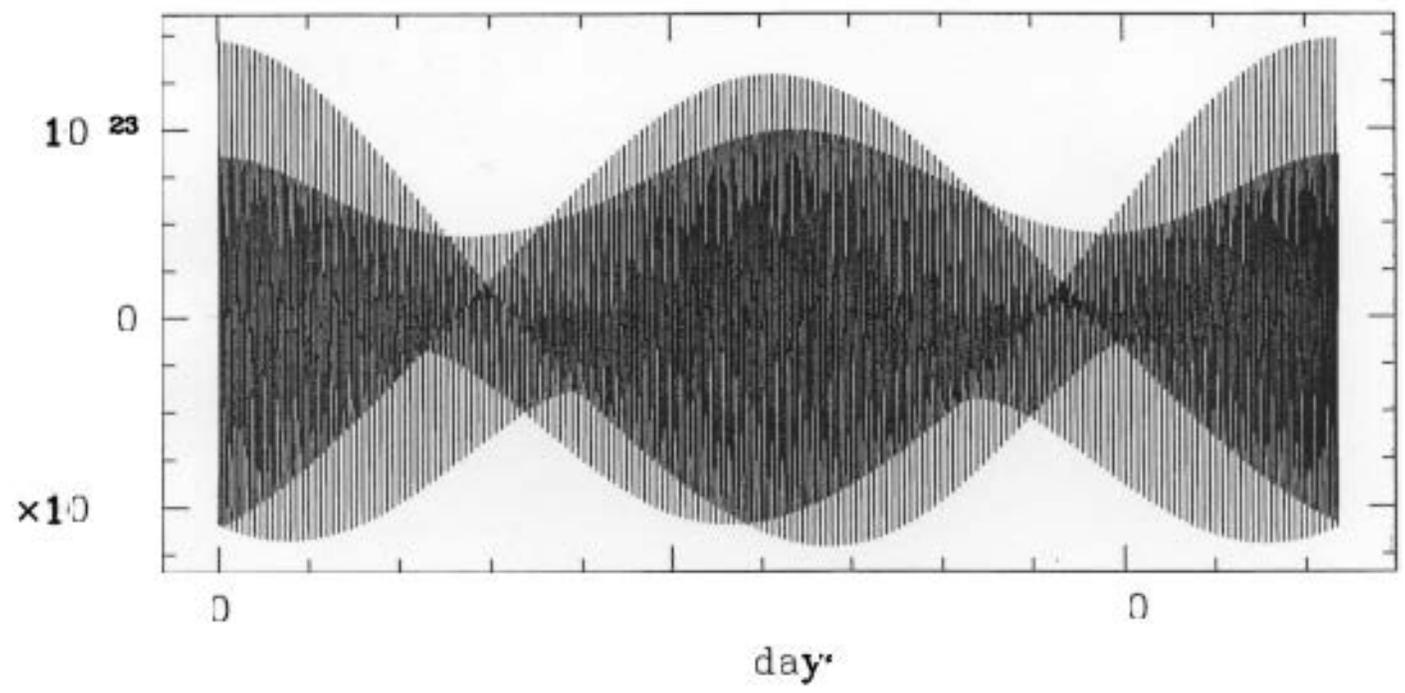
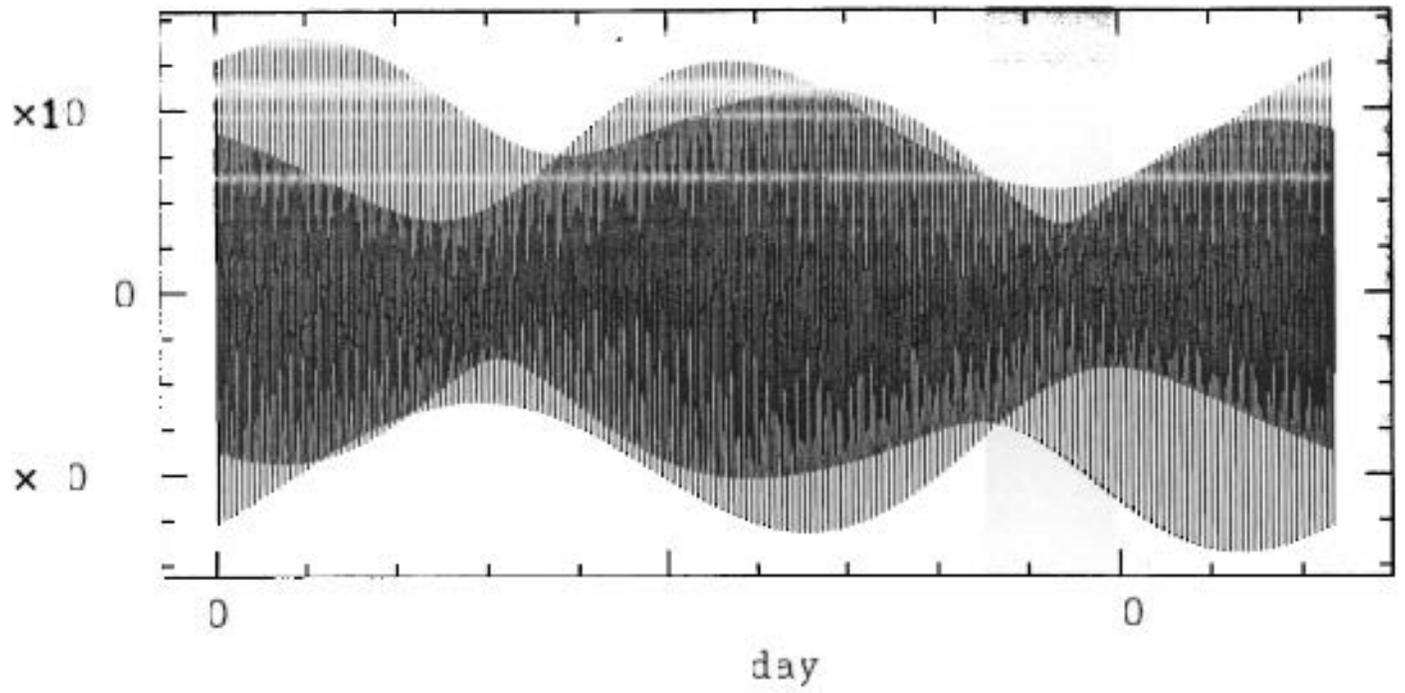
$$r = 7 \mu \quad a = 0.95 \mu$$

$$\alpha \approx 60^\circ$$



$$r = 7M \quad \alpha = 0.05M$$

$$\gamma = 60^\circ$$

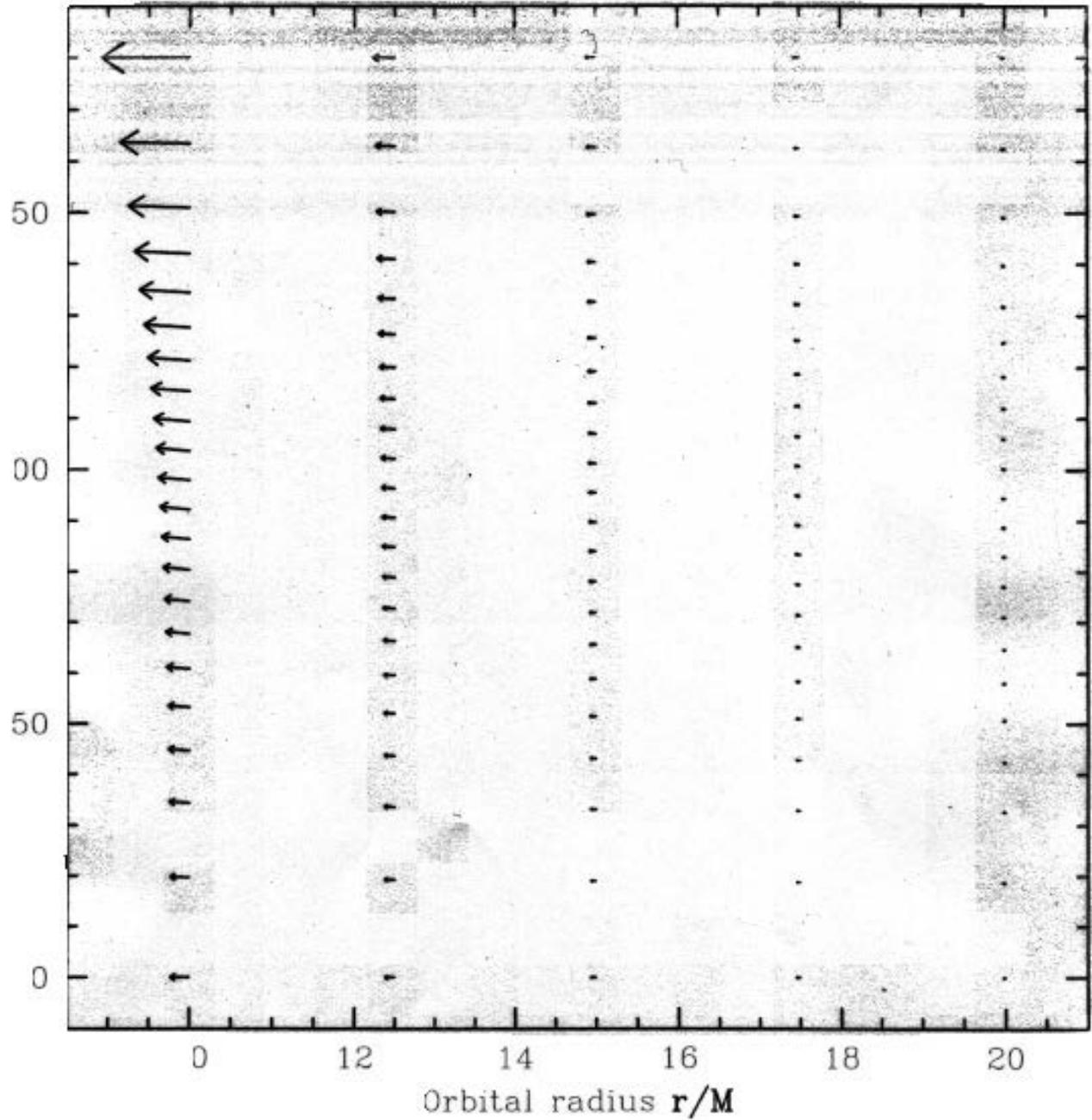


$$r = 7 \mu$$

$$a = 0.05 \mu$$

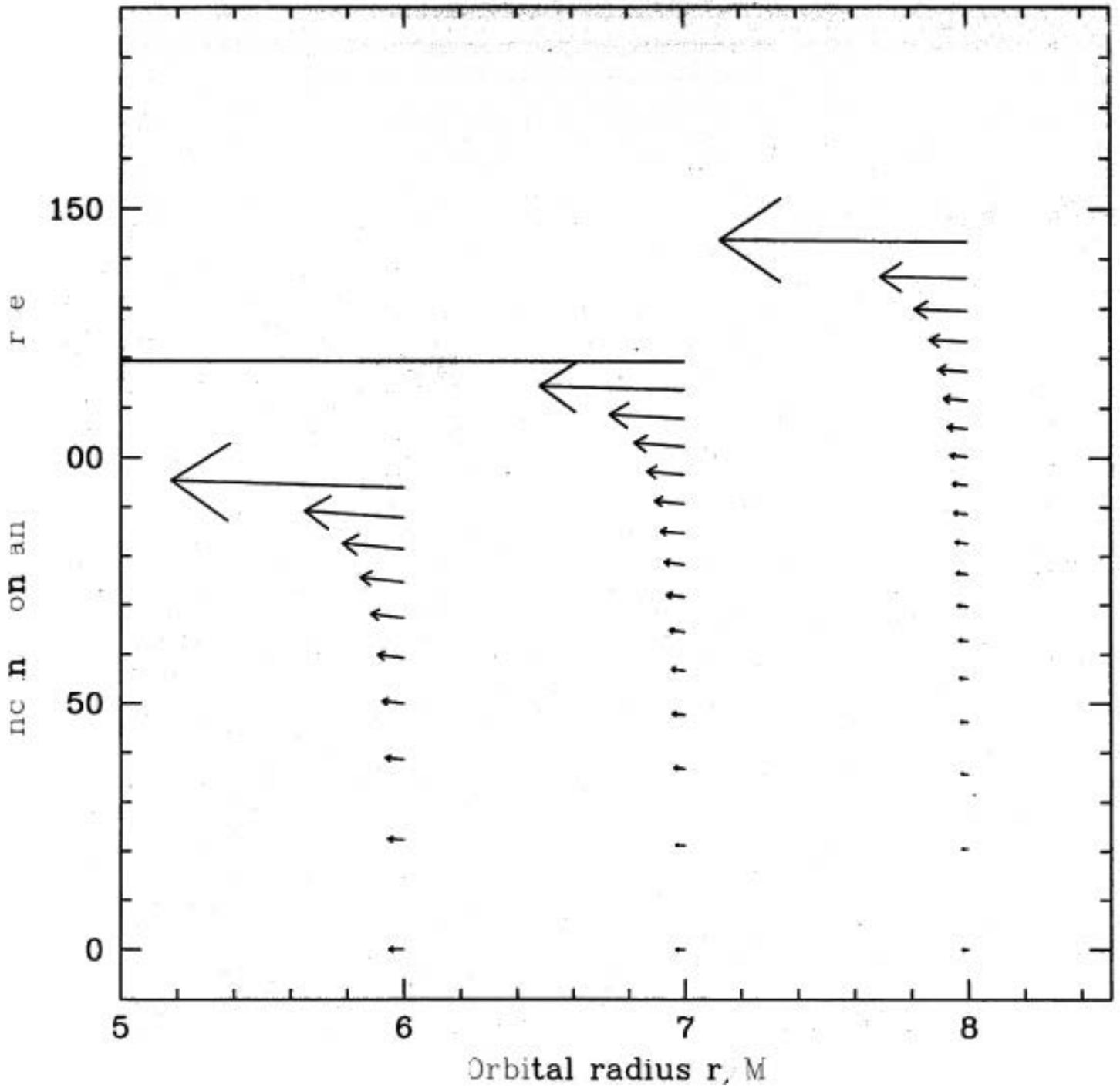
$$\lambda = 60^\circ$$

$a = 0.8M$



Very good agreement with Fintan Ryan's
post Newtonian expressions for τ as we
move to the weak field.

$$a = 0.8M$$



$r = 10M$

