Mode-sum approach: a year's progress

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with

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Adiabatic orbital evolution via self-force

Adiabatic inspiral ($t_{\substack{radiation\\reaction}} << t_{dynamic}$) described as a sequence of geodesics:

$$\mathbf{z}(\tau) = \mathbf{z}_{geod}[\tau; E(\tau), L_z(\tau), Q(\tau)].$$

Local rate of change of E, L_z, Q given by

Need to obtain this!

$$\boxed{\dot{C} = \mu^{-1} \frac{\partial C}{\partial u_\alpha} F_\alpha}$$

(E.g., Ori 96)

w/ $C(u_\alpha, x^\alpha) \equiv \{E(u_\alpha, x^\alpha), L_z(u_\alpha, x^\alpha), Q(u_\alpha, x^\alpha)\}$

Talk Plan

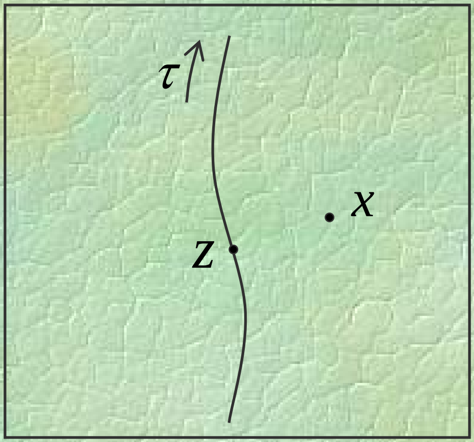
- Review of the mode sum approach
- Derivation of the RP: “direct force” approach
- Derivation of the RP: *l*-mode Green’s function approach
- Analytic approximation
- Gauge problem - and resolution strategy
- Implementation for radial trajectories
- Implementation for circular orbits (in progress)

Mode-sum prescription

$$F_{self} = \lim_{x\to z} F_{tail}(x)$$

$$= \lim_{x\to z}\left[F_{full}(x) - F_{dir}(x)\right]$$

$$= \sum_{l=0}^{\infty} \lim_{x\to z}\left[F^{l}_{full}(x) - F^{l}_{dir}(x)\right]$$

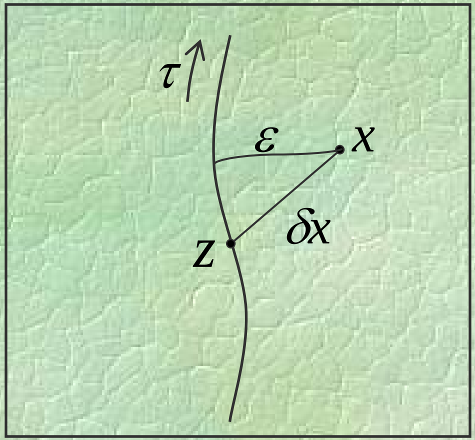


$$= \sum_{l=0}^{\infty}\left[F^{l}_{full}(x\to z) - (AL + B + C/L)\right] - \underbrace{\sum_{l=0}^{\infty}\left[F^{l}_{dir}(x\to z) - \left(AL + B + C/L\right)\right]}_{D}$$

$l+1/2$

$$\boxed{F_{self} = \sum_{l}\left[F^{l}_{full}(x\to z) - AL - B - C/L\right] - D}$$

Derivation of the RP : Scalar field
(based on the analytic form of the direct force)



$$F_{\alpha}^{dir}(x) = q\,\Phi^{dir}_{,\alpha}, \qquad \Phi^{dir}(x) = q\left(\frac{1+O(\delta x^2)}{\varepsilon}\right)$$

[Mino, Nakano, & Sasaki, 2001]

Introducing $\varepsilon = \varepsilon_0 + O(\delta x^2), \quad P_{\alpha}^{(n)} \propto \delta x^n,$

one obtaines (Barack, Mino, Nakano, Ori, & Sasaki, 2002)

$$F_{\alpha}^{dir} = \varepsilon_0^{-3} P_{\alpha}^{(1)} + \varepsilon_0^{-5} P_{\alpha}^{(4)} + \varepsilon_0^{-7} P_{\alpha}^{(7)} + O(\delta x)$$

Now expand in Y_{lm}, sum over m, and take x ✸ z:

$$F_{\alpha l}^{dir}(x \to z) = A_{\alpha} L + B_{\alpha}$$

Scalar RP in Schwarzschild: summary

(Coordinates chosen such that the orbit is equatorial)

$$A_{\pm r} = \mp \frac{q^2}{r^2} \frac{\mathcal{E}}{fV}, \qquad A_{\pm t} = \pm \frac{q^2}{r^2} \frac{\dot{r}}{V}, \qquad A_{\varphi} = 0$$

$$B_r = \frac{q^2}{r^2} \frac{(\dot{r}^2 - 2\mathcal{E}^2)\hat{K}(w) + (\dot{r}^2 + \mathcal{E}^2)\hat{E}(w)}{\pi f (1 + \mathcal{L}^2 / r^2)^{3/2}}$$

$$B_t = \frac{q^2}{r^2} \frac{\mathcal{E}\dot{r}\left[\hat{K}(w) - 2\hat{E}(w)\right]}{\pi V^{3/2}}$$

$$B_{\varphi} = \frac{q^2}{r} \frac{\dot{r}\left[\hat{K}(w) - \hat{E}(w)\right]}{\pi (\mathcal{L}/r) V^{1/2}}$$

$$C_{\alpha} = D_{\alpha} = 0$$

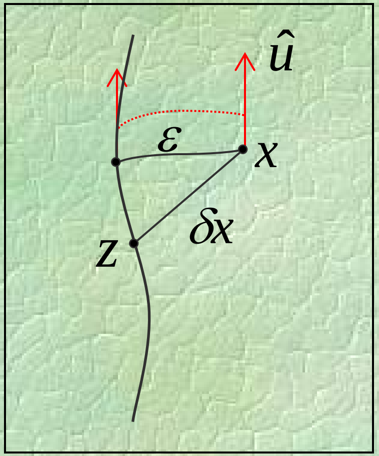
w/
$$\mathcal{E} = -u_t$$
$$\mathcal{L} = u_{\varphi}$$
$$\dot{r} = u^r$$
$$f = 1 - 2M/r$$
$$w = \mathcal{L}^2 / (\mathcal{L}^2 + r^2)$$
$$V = 1 + \mathcal{L}^2 / r^2$$
$\hat{E}, \hat{K}$ – elliptic intgs.

Derivation of the RP: gravitational case (based on the analytic form of the direct force)

$$F^{\alpha}_{self} = F^{\alpha}_{tail}(x \to z) = \left(\mu\, k^{\alpha\beta\gamma\delta}\, \bar{h}^{tail}_{\beta\gamma;\delta} \right)_{x \to z}$$

In harmonic gauge!

$$k^{\alpha\beta\gamma\delta} = \frac{1}{2} u^{\beta} u^{\gamma} g^{\alpha\delta} + \frac{1}{4} g^{\beta\gamma} g^{\alpha\delta} + \frac{1}{4} u^{\alpha} g^{\beta\gamma} u^{\delta} - g^{\alpha\beta} u^{\gamma} u^{\delta} - \frac{1}{2} u^{\alpha} u^{\beta} u^{\gamma} u^{\delta}$$



Similarly: $F^{\alpha}_{full,dir}(x) = \mu\, k^{\alpha\beta\gamma\delta}\, \bar{h}^{full,dir}_{\beta\gamma;\delta}$

$$\boxed{\bar{h}^{dir}_{\beta\gamma} = 4\mu\, \hat{u}_{\beta} \hat{u}_{\gamma} \frac{1 + O(\delta x^2)}{\varepsilon}}$$

[Mino, Sasaki, & Tanaka, 1997]

Again, we obtain $F^{dir}_{\alpha}(x) = \varepsilon_0^{-3} P^{(1)}_{\alpha} + \varepsilon_0^{-5} P^{(4)}_{\alpha} + \varepsilon_0^{-7} P^{(7)}_{\alpha} + O(\delta x)$

Note: the $P^{(n)}$'s, hence the RP values, depend on our choice of the off-worldline extension of $k^{\alpha\beta\gamma\delta}$

Considerations in choosing the *k*-extension

➔ In our (non-elegant, yet convenient) scheme, the *l*-mode contributions are defined through decomposing each vectorial α-component in scalar Y^{lm}:

$$F^{\alpha}_{full} = \mu \sum_{l,m,i} \underline{k^{\alpha\beta\gamma\delta}(x)\left(\bar{h}^{(i)lm}(r,t)\, Y^{(i)lm}_{\beta\gamma}(\theta,\varphi)\right)_{;\delta}} = \mu \sum_{lm,l'm'} F^{\alpha lml'm'}_{full} Y^{l'm'}(\theta,\varphi) \equiv \mu \sum_{l'} F^{\alpha l'}_{full}$$

Wish to design $k(\theta,\varphi)$ such that decomposing this in Y^{lm} turns out simple!

➔ Derivation of the gravitational RP better be simple

➔ Numerical computation of the full modes better be simple

➔ Calculation of the “gauge difference” (see below) better be simple

Note: Final result (F^{self}) does not depend on the *k*-extension; one just needs to be sure to apply the same extension to both the RP and the full modes.

RP values in Schwarzschild : Grav. case

	k-extension	RP values	especially convenient for…
I	"fixed contravariant components" (in Schwarzcshild coordinates)	$R_\alpha^{grav} = (\delta_\alpha^\beta + u_\alpha u^\beta) R_\beta^{sca}$	Numerical calculation of the full modes
II	"revised **I**" (some *k* components multiplied by certain functions of θ)	Same as **I**	Calculation of the scalar-harmonic *l*-modes (assures finite mode coupling)
III	$g^{\alpha\beta}(x)$: "same as"; $u^\alpha(x)$: parallelly-propagated along a normal geodesic	$R_\alpha^{grav} = R_\alpha^{scalar}$	Calculating the "gauge difference"
IV	"revised **III"** (under study)	Same as **III**	Calculating the full modes & gauge difference, still with a finite mode-coupling

Deriving the RP via local analysis of the (*l*-mode) Green's function

- This method relies only on MST/QW's original tail formula,

$$\Rightarrow F_{dir}(\tau) = \lim_{\varepsilon\to 0}\left(\overbrace{\int_{-\infty}^{\tau^+}(\nabla G)d\tau'}^{\text{full}} - \overbrace{\int_{-\infty}^{\tau-\varepsilon}(\nabla G)d\tau'}^{\text{tail}} \right) = \lim_{\varepsilon\to 0}\int_{\tau-\varepsilon}^{\tau^+}(\nabla G)d\tau',$$

with the Green's function equation, $\Box G + RG = \delta^4$ ...

- G^l expanded as $G^l = G_0 + G_1 L^{-1} + G_2 L^{-2} + \cdots$, where $G_n = G_n\left[(L\delta x); z\right]$

- Analytic solutions for the G_n's (in terms of Bessel functions)

- G_0, G_1, G_2 suffice for obtaining *A*, *B*, *C*, & also *D*

Green's function method - cont.

- Method applied so far for radial & circular orbits in Schwarzschild, for both scalar [LB & Ori, 2000] and gravitational [LB 2001,2] cases
- Agreement in all RP values for all cases examined. Especially important is the independent verification of D=0, which cannot be easily checked numerically.
- Green's function method involes tedious calculations (though by now mostely automated) and difficult to apply to general orbits.
- It's main advantage: quite easily extendible to higher orders in the 1/L expansion! (Direct force method requires knowledge of higher-order Hadamard terms.)

Gauge problem

- Self force calculations require $h_{\alpha\beta}$ (or it's tensor-harmonic modes $h^{(i)lm}Y^{(i)lm}_{\alpha\beta}$) in the harmonic gauge.
- BH perturbations are not simple in that gauge, even in Schwarzschild: separable with respect to l,m but various elements of the tensor-harmonic basis remain coupled. Unclear how this set of coupled equations will behave in numerical integrations.
- Two possible strategies:
  - Either tackle the perturbation equations in the harmonic gauge ; or
  - Formalize and calculate the self force in a different gauge: e.g., Regge-Wheeler gauge (for Schwarzschild), or the radiation gauge (for Schawrzschild or Kerr)

Gauge transformation of the self force

[LB & Ori, 2001]

- $\delta h^{H\to R}_{\alpha\beta} = \xi_{\alpha;\beta} + \xi_{\beta;\alpha}$

Radiation or RW gauges

- Then, given $\xi^{H\to R}_{\alpha}$:

$$\boxed{(\delta F^{\alpha}_{self\,,\,full})^{H\to R} = -\mu[(g^{\alpha\lambda} + u^{\alpha}u^{\lambda})\ddot{\xi}_{\lambda} + R^{\alpha}{}_{\mu\lambda\nu}u^{\mu}\xi^{\lambda}u^{\nu}]}$$

- If δF attains a well defined limit $x\to z$, then $(RP)^R=(RP)^H$; however -

- δF not necessarily well defined at z. Examples:

  - $\delta F^{H\to Radiation}_{self}$ is direction-dependent even for a static particle in flat space
  - $\delta F^{H\to RW}_{self}$ is direction-dependent for all orbits in Schwarzschild, except strictly radial ones

Dealing with the gauge problem

➔ For radial trajectories in Schwarzschild - no gauge problem! One implements the mode-sum scheme with the full modes F^l_{full} calculated in the RW gauge, and with the same RP as in the H gauge.

➔ If δF is discontinuous (direction-dependent) but finite as $x \to z$: average over spatial directions?

➔ If δF admits at least one direction from which $x \to z$ is finite: define a "directional" force?

In both cases, a useful sense of the resulting quantities must be made by prescribing the construction of desired gauge- invariant quantities out of them. [Ori 02]

☆ A general strategy: Calculating the force in an "intermediate" gauge, obtained from the R-gauge by modifying merely its "direct" part (such that it resembles the one of the H-gauge). This is how it's done:

Self force in the "intermediate" gauge

- Solve $h_{\alpha\beta}^{(\hat{H})} = h_{\alpha\beta}^{(R)} + \xi_{\alpha;\beta} + \xi_{\beta;\alpha}$ for $\xi_{\alpha}^{R\to\hat{H}}$,

  where $h_{\alpha\beta}^{(\hat{H})}$ is the direct part of $h_{\alpha\beta}^{(H)}$

- Construct $\delta F^{R\to\hat{H}}(\xi^{R\to\hat{H}})$

- Now:

$$F_{self}^{H} = F_{full}^{H} - F_{dir}^{H}$$

$$+$$

$$\delta F_{self}^{H\to\hat{H}} = \delta F_{full}^{H\to\hat{H}} = \delta F_{full}^{H\to R} + \delta F_{full}^{R\to\hat{H}}$$

$$\boxed{F_{self}^{\hat{H}} = F_{full}^{R} + \delta F_{full}^{R\to\hat{H}} - F_{dir}^{H}}$$

Mode-sum scheme in the $\hat{H}$ gauge

$$F^{(\hat{H})}_{self} = \sum_l \left(F^{l(R)}_{full} + \delta F^{l(R\to\hat{H})}_{full} - \underbrace{AL - B - C/L\Big) - D}_{\text{H-gauge RP}}$$

R-gauge modes (easy to obtain)

l-modes of δF (to be obtained analytically)

- Note: Must apply the same k-extension for F^l, δF^l, and the RP!

- **Preliminary Results for radiation gauge in Schwarzschild:**

δF^l has no contribution to A, B, or C !

Calculation of contribution to D (zero?) under way

Implementing the mode-sum method: radial trajectories (RW gauge)

- We worked entirely within the RW gauge:
  - *l*-mode MP derived (via twice differentiating) from Moncrief's scalar function ψ^l, satisfying $\psi^l_{,tt} - \psi^l_{,r^*r^*} + V(r)\psi^l = \text{Source}$
  - Then, *l*-mode full force derived through $F^{\alpha l}_{full} = \mu k^{\alpha\beta\gamma\delta} \bar{h}^l_{\beta\gamma;\delta}$
  - Mode-sum formula applied with H-gauge RP - see Lousto's talk.
- Parameters *A*, *B*, *C* for the RW modes obtained independently by applying the Green's function technique to Moncrief's equation. We found $(RP)^{RW} = (RP)^{H}$.

☆ Explicit demonstration of the RP's "gauge-invariance" property

Large l analytic approximation

$$F_{tail} = \sum_l \left(F^l_{full} - F^l_{dir} \right)$$

Conjecture: l-mode expansion of F_{tail} converges faster than any power of l.

- F^l_{full} and F^l_{dir} have the same $1/L$ expansion
- In particular:
  O(L^{-2}) term of F^l_{full} can be inferred from that of F^l_{dir}
- This, in turn, is obtained by extending the Green's function method one further order

(conjecturing here is not risky, since the result is tested numerically)

analytic approximation - cont.

- For radial trajectories in Schwarzschild (in RW gauge) we found (LB & Lousto, 02)

$$F^{rl}_{reg} \equiv F^{rl}_{bare} - A^r L - B^r - C^r / L$$
$$= -\frac{15}{16}\frac{\mu^2}{r^2}\mathsf{E}^2\left(\mathsf{E}^2 + 4M/r - 1\right)L^{-2} + O(L^{-4})$$

$$F^{reg}_{tl} = -\frac{15}{16}\mu^2\mathsf{E}\frac{d}{d\tau}\left(\dot{r}^2/r\right)L^{-2} + O(L^{-4})$$

In perfect agreement with numerical results! (see Lousto's talk)

- Integrated "energy loss" by *l*-mode at large *l* (for $r_0 = \infty$, $\square = 1$):

$$\int_{-\infty}^{\tau(EH)} \mu \dot{\mathsf{E}}^l \, d\tau = -\int_{-\infty}^{\tau(EH)} F^{reg}_{tl} \, d\tau = \frac{15}{32}(\mu/M)\mu L^{-2}$$

but gauge-dependent!

Implementing the mode-sum method: circular orbits ($\hat{H}$ gauge)

- Current status:

$$F_{self}^{(\hat{H})} = \sum_{l} \Big(\underbrace{F_{full}^{l(RW)}} + \underbrace{\delta F_{full}^{l(RW \to \hat{H})}} \underbrace{- AL - B - C/L \Big) - D} $$

Full modes in RW gauge:

Numerical calculation of the tensor-harmonic MP modes (via Moncrief). ✔

Still need to prescribe & carry out the construction of the "scalar-harmonic" force mode.

Gauge difference: being calculated (by solving for ξ, obtaining δF, and decomposing into modes.)

H-gauge RP ✔

What's next?

- Circular case will provide a first opportunity for comparing self-force and energy balance approaches.
- Generic orbits in Schwarzschild: already have the RP and numerical code; soon will have δF^l.
- Orbits in Kerr: key point - be able to obtain MP. Progress relies on this.