Orbital evolution of a test particle around a black hole: comparison of the local force and the conservation laws approaches

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Outline:

- * Motivation: Possibility that higher-order corrections to the waveforms (beyond what is available from conservation laws) are important
- * How to find the waveforms to higher order using the local self force
  - direct integration
  - perturbation expansion
- * The higher-order corrections to $\frac{dN_{cyc}}{d(\ln f)}$ and the waveforms
- * Implications for LISA

CAPRA 5 5/31/2002.

Motivation

Consider a pointlike particle of mass μ in equatorial and circular orbit around a Schwarzschild black hole of mass M, and assume $\varepsilon \equiv \frac{\mu}{M} \ll 1$.
The orbit is damped due to radiation reaction (RR).

* The "RR without RR forces" (RRWORRF) approach yields:

- Kepler's law to $O(1)$ [no correction at $O(\varepsilon)$, part of the $O(\varepsilon^2)$ correction]
- Corrections to the radial velocity to $O(\varepsilon)$
- Corrections to the orbit and waveforms to $O(\varepsilon^{-1})$
- $dN_{cyc}/d(\ln f)$ to $O(\varepsilon^{-1})$
- Violation of conservation laws to $O(\varepsilon^2)$, e.g., μ is unconserved to $O(\varepsilon^2)$

* Some are interested in the local self force in order to find $\dot{Q}$ (which cannot be obtained using balance arguments), and then use $\dot{E}, \dot{L}_z, \dot{Q}$ to evolve the orbit and find the waveforms.
That program will give the waveforms to $O(\varepsilon^{-1})$.

* In general, there is no way to obtain higher-order corrections:

@ ∞, fields associated with conservative self force effects decay faster than those associated with dissipation. When we integrate over a sphere at large r and take $r \to \infty$, we in practice discard all conservative effects.

- * Are corrections to the waveforms at $O(1)$ important?
  - The cross correlation of the data stream to a theoretical template plummets if the two slip by 1 radian (assuming a single template)

$$\frac{dN_{cyc}}{d(\ln f)} \sim O(\underbrace{\varepsilon^{-1})\left[1+O(\varepsilon)\right.}_{\displaystyle\downarrow \atop \text{can be about 1 radian @ the ISCO}}\left.\right]$$

  - Similar higher-order corrections may be of interest for extremely eccentric orbits (cumulative precession of periastron)

  - What are the corrections to

* $\frac{dN_{cyc}}{d(\ln f)}$

* (envelope)$^{\cdot}\,\frac{r}{\omega}$

* The waveform ?

We consider a simple toy model, in which the particle is endowed with scalar charge q.
For this case $\varepsilon = q^2/(\mu M)$.

How to find the orbit using the local self force?

1) Direct integration

Newton's law $\frac{Du^\alpha}{d\tau} = \mu^{-1} f^{SF}_\beta g^{\alpha\beta}$

$u^\alpha = u^\alpha_0 + \mu^{-1}\int f^\alpha_{eff.}\, d\tau$, where $f^\alpha_{eff.} = -\Gamma^\alpha_{\rho\sigma} u^\rho u^\sigma + f^{SF}_\beta g^{\alpha\beta}$

Problems with this approach:

* Need to parallel transport the integrand, and to choose a preferred path

$$\int_{\tau_0}^{\tau} f^\alpha_{eff}\, d\tau' = \int_{\phi_0}^{\phi} \left[f^\alpha_{eff}(\phi') + \Gamma^\alpha_{\beta\phi} f^\beta_{eff}(\phi')(\phi' - \phi_0) \right] \frac{d\phi'}{\omega u^t}$$

  It is most natural to choose the parallel transport path to be a circular arc. Then this integral becomes "easy", as $f^\alpha_{eff} = f^\alpha_{eff}(r)$, $\Gamma = \Gamma(r)$.

* To find the orbit $x^\alpha(\tau)$ one needs to integrate again. A short cut: use $u^\alpha u_\alpha = -1$ with the new u^α, and solve for r. [If written as $g^{\alpha\beta} u_\alpha u_\beta = -1$, this is a cubic equation.]

* This method is not very accurate, because in the radial component of the EOM, the self force f^{SF}_r is just a tiny correction to the "Newtonian force" $-\Gamma^r_{\alpha\beta} u^\alpha u^\beta$.

2) Perturbation expansion

Recall that we are interested in $\omega(t), \dot{r}(t)$ to $O(\varepsilon^2)$.

One geometrical constraint:

1. $u^\alpha u_\alpha = -1$ use to eliminate u^t

Three equations of motion:

2. $\frac{Du^t}{d\tau} = \mu^{-1} f_t^{SF} g^{tt}$ use to eliminate $\dot{u}^t$

3. $\frac{Du^r}{d\tau} = \mu^{-1} f_r^{SF} g^{rr}$

4. $\frac{Du^\phi}{d\tau} = \mu^{-1} f_\phi^{SF} g^{\phi\phi}$

These are the two equations for the two unknowns

Define : $\omega^2 = \frac{M}{r^3} + \sigma$

↳ deviation from Kepler's law

An overdot denotes derivative w.r.t. t

A prime denotes derivative w.r.t. r

The equations to solve:

$$\ddot{r} - \frac{3M}{r^2}\frac{\dot{r}}{1-\frac{2M}{r}} - r\left(1-2\frac{M}{r}\right)\sigma = \frac{1}{\mu u^{t2}}\left[\left(1-\frac{2M}{r}\right)f_r^{SF} + \frac{\dot{r}}{1-\frac{2M}{r}} f_t^{SF}\right]$$

$$\dot{\sigma} - \frac{3M}{r^4}\dot{r} + 2\frac{\frac{M}{r^3}+\sigma}{1-2\frac{M}{r}}\left[\frac{2\dot{r}}{r}\left(1-\frac{3M}{r}\right) - \frac{f_t^{SF}}{\mu u^{t2}}\right] = 2\frac{\frac{M}{r^3}+\sigma}{\mu r^3 u^{t2}} f_\phi^{SF}$$

These are not too bad: If desired, they can be solved numerically: Start @ some r and specify lowest order solution [$O(\varepsilon)$], and integrate (say) inward. This will give a solution good to $O(\varepsilon^2)$ [difference of $O(\varepsilon^3)$].

* A better approach: solve by iterations, or order by order.

* An even better approach: Notice that this is not a true evolution problem. We can re-express everything as functions of r, and get more elegant equations.

$$\sigma = \sigma(r) = \sigma_{(1)}(r) + \sigma_{(2)}(r) \qquad \dot{\sigma} = \sigma' V$$

$$\dot{r} = V(r) = V_{(1)}(r) + V_{(2)}(r) \qquad \ddot{r} = V' V$$

$$f_\alpha^{SF} = f_\alpha^{(1)} + f_\alpha^{(2)}$$

This is a formal expansion, as $f_\alpha^{(2)}$ is as yet unknown

The new equations:

$$VV' - 3\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}V^2 - r\left(1-\frac{2M}{r}\right)\sigma - \frac{1}{\mu u^{t2}}\left[\left(1-\frac{2M}{r}\right)f_r^{SF} + \frac{V}{1-\frac{2M}{r}} f_t^{SF}\right] = 0$$

$$\sigma' V - 3\frac{M}{r^4}V + 2\frac{M/r^3+\sigma}{1-\frac{2M}{r}}\left[\frac{2}{r}V\left(1-\frac{3M}{r}\right) - \frac{f_t^{SF}}{\mu u^{t2}}\right] - 2\frac{M/r^3+\sigma}{\mu r^2 u^{t2}} f_\phi^{SF} = 0$$

with $$u^{t2} = \frac{1}{1-3\frac{M}{r} - r^2\sigma - \frac{V^2}{1-\frac{2M}{r}}}$$

Solving order by order (in ε) we find:

$$V = 2\frac{r}{\mu M}\,\frac{r-3M}{r-6M}\left[\sqrt{\frac{M}{r^3}}(r-2M)f_\phi^{(1)} + M f_t^{(1)}\right]$$
$$+\frac{r(r-3M)}{\mu^3M^2(r-6M)^2}\Big\{2\sqrt{\frac{M}{r^3}}\,r^2(r-2M)^2(r-3M)\,f_\phi^{(1)}f_r^{(1)'} + 4Mr^2(r-3M)f_t^{(1)}f_r^{(1)}$$
$$+\sqrt{\frac{M}{r^3}}\,r(5r-6M)(r-2M)(r-3M)\,f_\phi^{(1)}f_r^{(1)} + 2Mr^2(r-2M)(r-3M)\,f_t^{(1)}f_r^{(1)'}$$
$$+2\sqrt{\frac{M}{r^3}}\,\mu M(r-2M)(r-6M)\,\underline{f_\phi^{(2)}} + 2\mu M^2(r-6M)\,\underline{f_t^{(2)}}\Big\} + O(\varepsilon^3)$$

$$\sigma = -\frac{r-3M}{\mu r^2}f_r^{(1)} + \frac{r-3M}{\mu^2Mr^2\sqrt{\frac{M}{r^3}}(r-2M)^2(r-6M)^3}\Big\{-\left(\frac{M}{r}\right)^{\frac{3}{2}}(r-2M)^2(r-6M)^3\left(\mu\underline{f_r^{(2)}} + r f_r^{(1)2}\right)$$
$$+2(r^3-19Mr^2+66M^2r-72M^3)\left[2M(r-2M)f_t^{(1)}f_\phi^{(1)} + \sqrt{\frac{M}{r^3}}(r-2M)^2f_\phi^{(1)2} + \sqrt{\frac{M}{r^3}}\,\mu r^3 f_t^{(1)2}\right]$$
$$+(r-2M)(r-3M)(r-6M)\Big[4Mr(r-2M)\left(f_\phi^{(1)}f_t^{(1)'} + f_\phi^{(1)'}f_t^{(1)}\right)$$
$$+4\left(\frac{M}{r}\right)^{\frac{1}{2}}\left(\mu r^3 f_t^{(1)}f_t^{(1)'} + (r-2M)^2 f_\phi^{(1)}f_\phi^{(1)'}\right)\Big]\Big\} + O(\varepsilon^3)$$

A few points to notice:

- $V_{(1)}$ is fully described by dissipation
- $\sigma_{(1)}$ is fully described by conservative effects
- The unknown $f_\alpha^{(2)}$ become negligible approaching the ISCO
- Except for those: $V_{(2)}$ vanishes if conservative effects are ignored, but $\sigma_{(2)}$ is fully described by dissipation.
- To $O(\varepsilon^2)$, not only $f_\alpha^{(1)}$ is needed, but also the gradients $f_\alpha^{(1)'}$.
- In $V_{(2)}$, the unknown terms are unimportant also at $r>6M$, because their coefficients are much smaller than those of the other terms.

The higher-order corrections to the waveforms

The orbit can be found by

$$t(r) = \int_{r_{start}}^{r} \frac{dt}{dr}\, dr \qquad\qquad \phi(t) = \int_{r_{start}}^{r} \frac{d\phi}{dr}\, dr$$

$$\frac{dt}{dr} = \frac{1}{V(r)} = \frac{1}{V_{(1)}} - \frac{V^{(2)}}{V_{(1)}^{2}} + O(\varepsilon)$$

$$\frac{d\phi}{dr} = \frac{\omega(\ )}{V(r)} = \sqrt{\frac{M}{r^3}}\left[\frac{1}{V_{(1)}} + \left(\frac{1}{2}\frac{\sigma_{(1)} r^3}{M V_{(1)}} - \frac{V^{(2)}}{V_{(1)}^{2}}\right)\right] + O(\varepsilon)$$

These integrals are easy to do with 4th-order Runge-Kutta. After having $[r, t(r), \phi(r)]$, invert to get $[t, r(t), \phi(t)]$.

$$\frac{dN_{cyc}}{d(\ln f)} \equiv \frac{1}{(2)\pi}\frac{\omega^2}{\dot{\omega}}$$

$$= -\frac{2}{(2)3\pi}\left(\frac{M}{r}\right)^{\frac{1}{2}}\frac{1}{V_{(1)}} - \frac{2}{(2)3\pi}\left(\frac{M}{r}\right)^{\frac{1}{2}}\left[\frac{3}{2}\frac{\sigma^{(1)} r^3}{M V_{(1)}} - \frac{V^{(2)}}{V_{(1)}^{2}} + \frac{1}{3}\frac{\sigma^{(1)\prime} r^4}{M V_{(1)}}\right] + O(\varepsilon)$$

$$\frac{\dot{r}}{r\omega} = -\frac{V_{(1)}}{\left(\frac{M}{r}\right)^{1/2}} + \frac{1}{\sqrt{\frac{M}{r^3}}}\left[\frac{1}{2}\frac{V^{(1)}\sigma^{(1)} r^2}{M} - \frac{V^{(2)}}{r}\right] + O(\varepsilon^3)$$

(2): additional factor in the scalar field case, where $\varepsilon = \gamma^2/(\gamma+1)$.

Notice that the orbit is fully described to $O(1)$ by $V_{(1)}$, $V_{(2)}$, and $\sigma_{(1)}$. $\sigma_{(2)}$ does not enter at that order. Recall that the unknown $f_{\alpha}^{(2)}$ contribute negligibly: we can happily ignore our ignorance of them.

$$\Delta \frac{dN_{cyc}}{d(\ln f)} = -\frac{2}{(2)3\pi}\left(\frac{M}{r}\right)^{\frac{1}{2}}\left[\frac{3}{2}\frac{\sigma^{(1)}}{V_{(1)}}\frac{r^3}{M} + \frac{1}{3}\frac{\sigma^{(1)\prime}}{V_{(1)}}\frac{r^4}{M} - \frac{V^{(2)}}{V_{(1)}^{2}}\right] + O(\varepsilon)$$

This is a useful quantity, as it measures how many cycles are required for the cross-correlation of the data stream with a template made to $O(\varepsilon^{-1})$ to plummet.

$$\frac{\Delta\left(\frac{V}{r\omega}\right)}{\left(\frac{V}{r\omega}\right)^{2}} = \left(\frac{M}{r}\right)^{\frac{1}{2}}\left[\frac{1}{2}\frac{\sigma^{(1)}}{V_{(1)}}\frac{r^3}{M} - \frac{V^{(2)}}{V_{(1)}^{2}}\right] + O(\varepsilon)$$

This quantity is the difference in the rates of change of the waves' envelopes.

Points to notice:

- These are dimensionless quantities to $O(1)$.
- Both quantities are independent of $\sigma_{(2)}$.
- They are made of terms which are linear in $\sigma^{(1)}, \sigma^{(1)\prime}, V^{(2)}$.
- These are fully determined (except for negligible terms) by the conservative effects.

The waveform: The waveform can be approximated by the "restricted waveform":

$$h = C\, v_{w}^{2}(t) \cos\left[\phi_{wave}(t)\right]$$

$$v_{w} = \left(\frac{d\phi}{dt}\right)^{\frac{1}{3}} \sim \frac{1}{r^{1/2}}$$

$$h \sim C\left[\frac{M}{r^3(t)} + \sigma(t)\right]^{\frac{1}{3}} \cos\left[\phi_{wave}(t)\right]$$

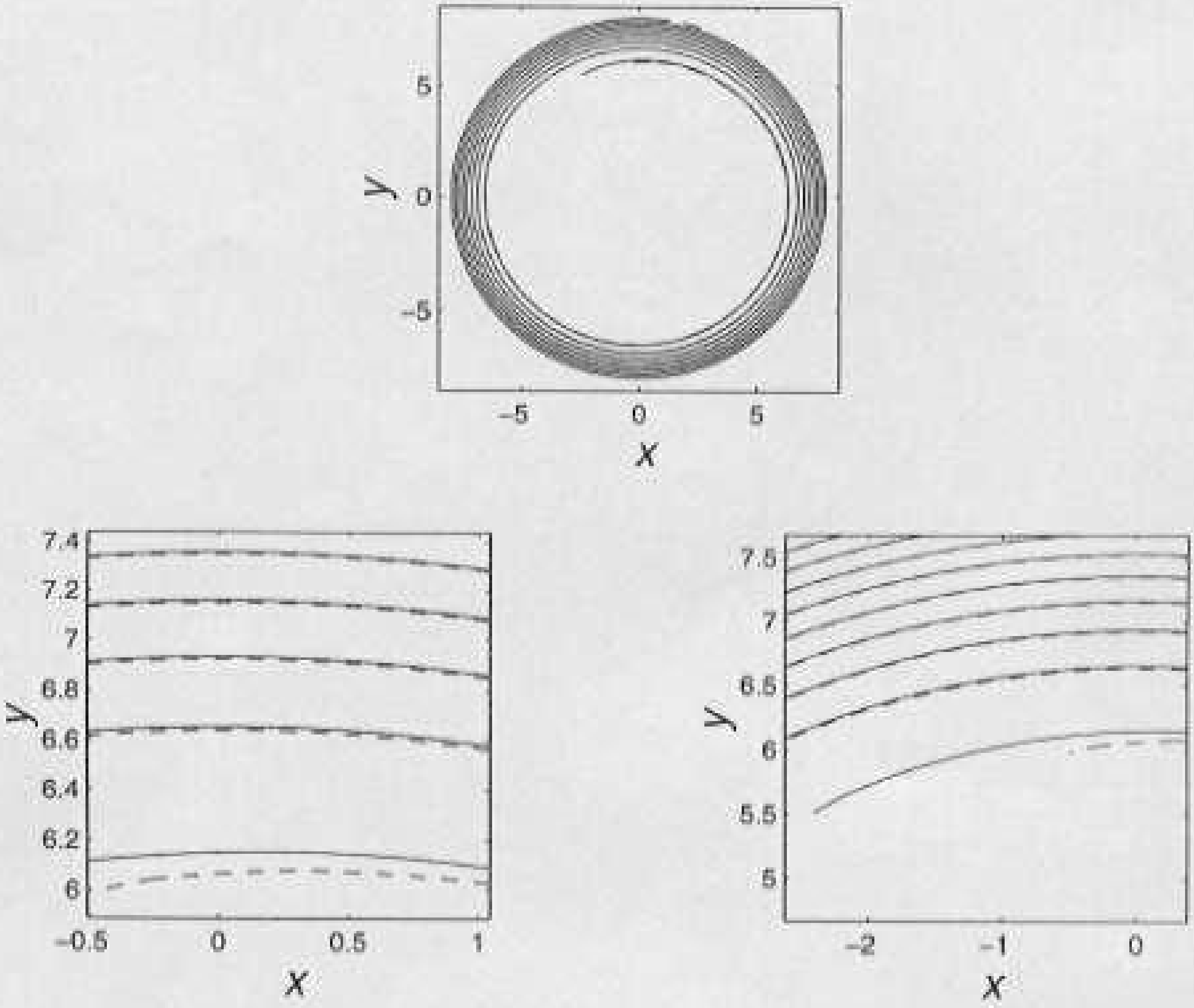

FIG. 1: The orbit of a scalar charge with $q^2/(\mu M) = 0.1$ in initially circular and equatorial orbit around a Schwarzschild black hole. The upper panel shows the last few orbits before the ISCO. The solid line is the orbit using our second-order results, and the dashed line is the orbit using the first-order approximation (RRWORRF). The lower panels display two enlargements of the same orbits: the one on the left emphasizes the difference in the r values, and the one on the right the difference in phase. Here, $x = (r/M)\cos\phi$ and $y = (r/M)\sin\phi$.

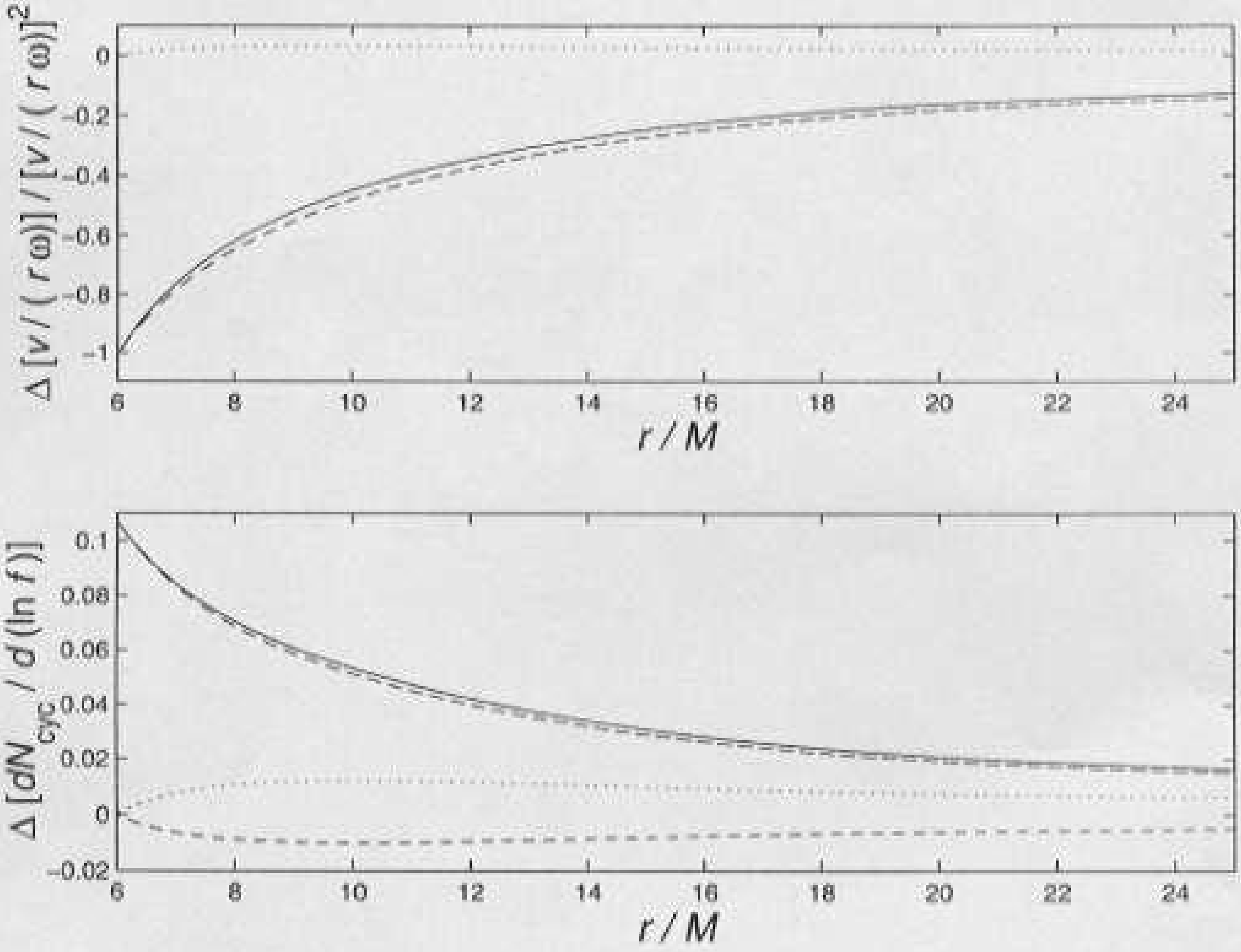

FIG. 2: The magnitude of the higher-order corrections to the orbit. Upper panel: $\Delta\left(\frac{v}{r\Omega}\right)/\left(\frac{v}{r\Omega}\right)^2$ as a function of r. The solid line is the full effect, the dotted line is the contribution of the term in Eq. (8) proportional to $\sigma_{(1)}$, and the dashed line is the contribution of the term proportional to $V_{(2)}$. Lower panel: $\Delta\, dN_{\rm cyc}/\,d(\ln f)$ as a function of r. The solid line is the full effect. The dotted line is the contribution of the term in Eq. (7) proportional to $\sigma'_{(1)}$, the dash-dotted line the contribution of the term proportional to $\sigma_{(1)}$, and the dashed line the contribution of the term proportional to $V_{(2)}$.

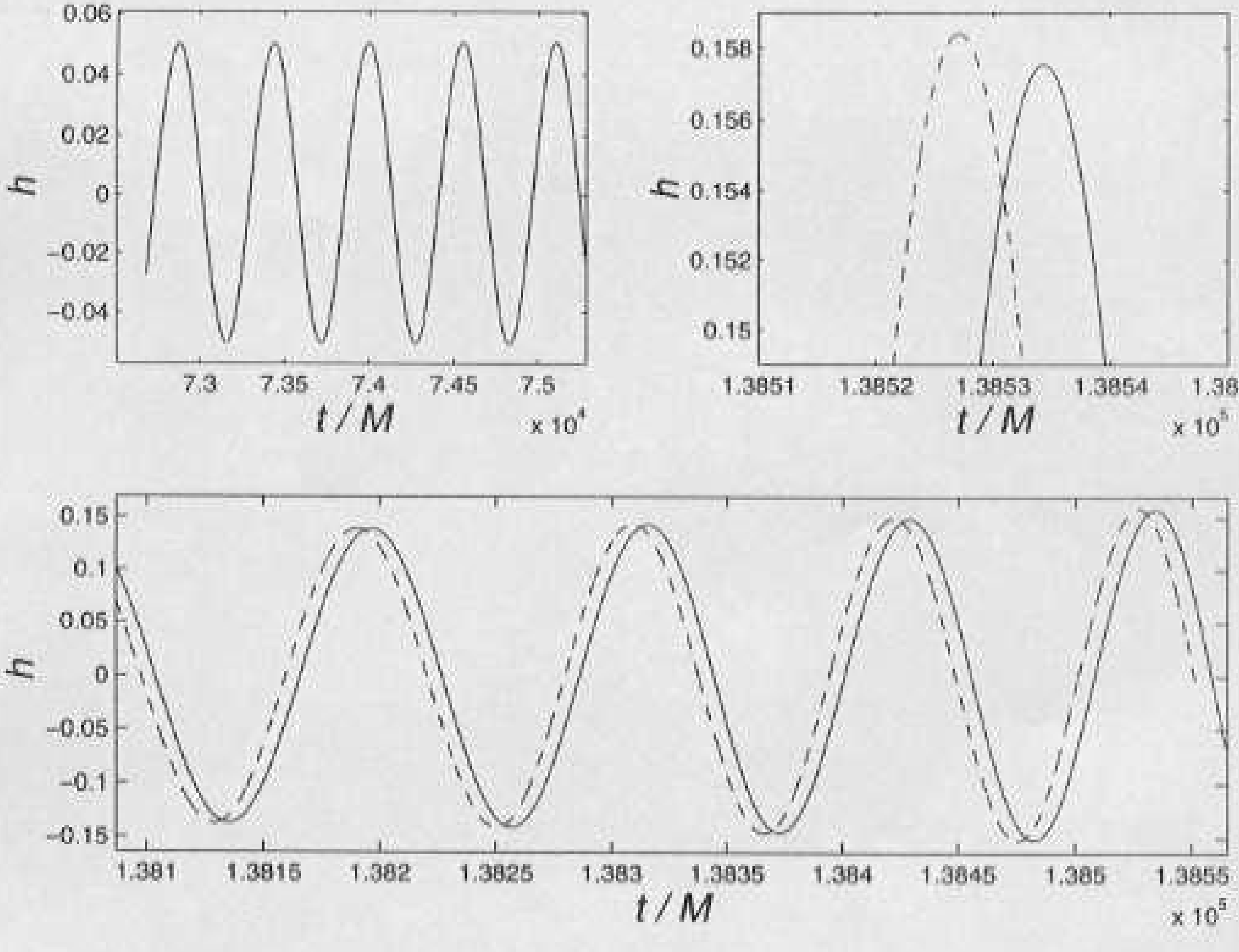

FIG. 3: The waveforms. In all panels the solid line describes the waveform to $O(1)$, and the dashed line describes the waveform to $O(\epsilon^{-1})$. The upper left panel describes the waves from an early part of the orbit (at $r \approx 20M$). The lower panel describes the last few cycles before the ISCO. The upper right panel is a magnification of the last maximum before the ISCO.

Implications for LISA

* The high-order correction to $\frac{dN_{cyc}}{d(\ln f)}$ is at $O(1)$, hence independent of ε.

* For all scalar charges and black holes (all ε), we find $\frac{dN_{cyc}}{d(\ln f)} \sim 0.11$ near the ISCO.

* In the gravitational case the $f_\alpha^{(1)}$ are as yet unbeknownst to us. All things being equal, we pick up another factor of 2 in $\frac{dN_{cyc}}{d(\ln f)}$. One may expect than even a stronger effect than in the scalar field toy-model case.

* An important issue is the possibility to use templates of $O(\varepsilon^{-1})$, to cross correlate against the data stream. One may hope, that templates to $O(\varepsilon^{-1})$ for slightly different parameters may be a good approximation, such that one does not have to prepare templates to $O(1)$. That is, one may hope that templates to $O(\varepsilon^{-1})$ for a certain value of ε may yield a template to $O(1)$ for another value of ε. It appears to be generally impossible with just one template, but multiple-template techniques will be just fine.

* For accurate astronomy, one would have to take $O(1)$ effects into consideration.