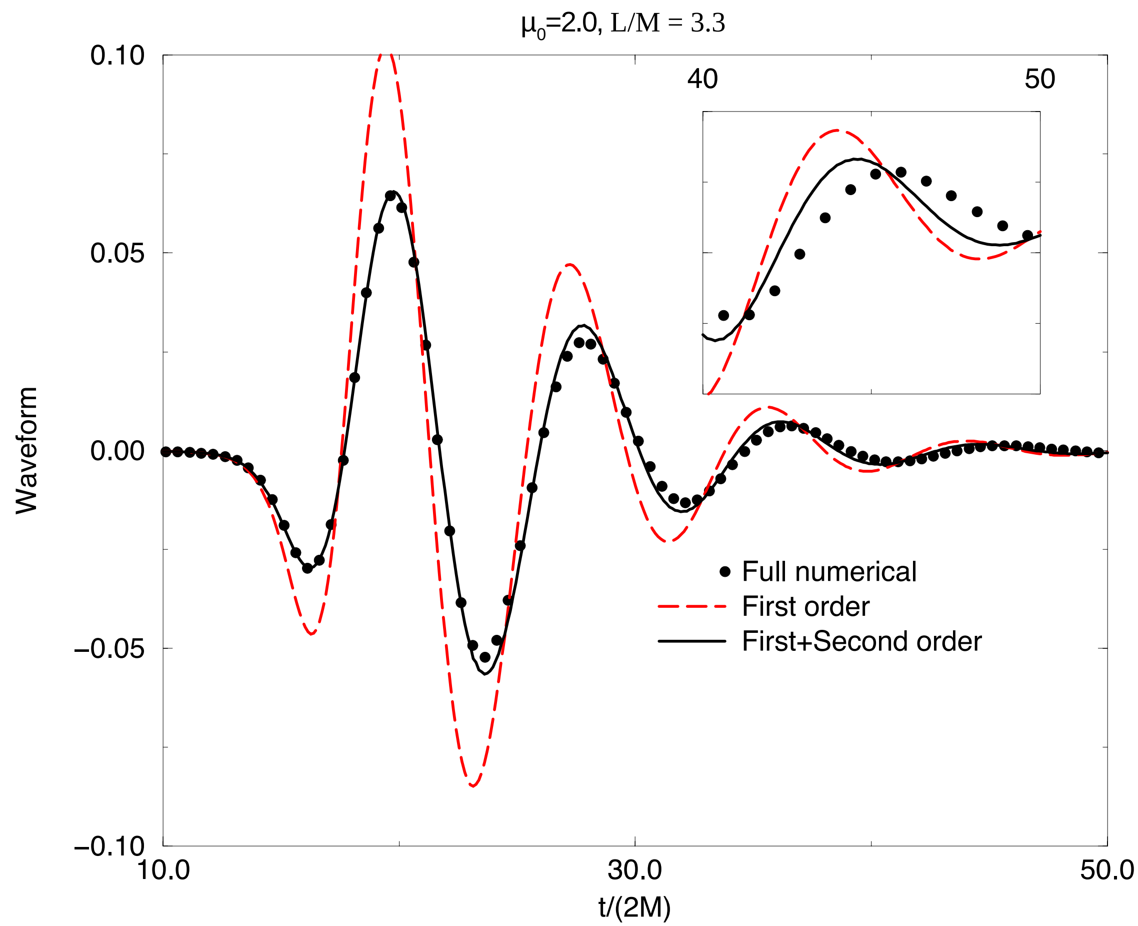
How far can perturbation theory go? Second Order Gravitational Perturbations of a Kerr Black Hole

Manuela Campanelli

The University of Texas at Brownsville

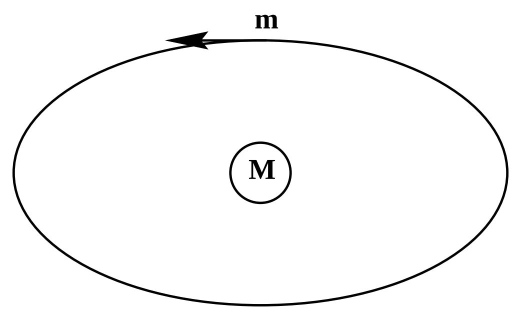
M. Campanelli and C.O. Lousto, PRD, **59**, 124022 (1999)



μ0=2.0, L/M = 3.3
Waveform
t/(2M)
0.10
0.05
0.00
−0.05
−0.10
10.0
30.0
50.0
40
50
Full numerical
First order
First+Second order

Why do we need second order perturbation theory?

$$\boxed{g_{\mu\nu} = g^{(0)}_{\mu\nu} + \varepsilon h^{(1)}_{\mu\nu} + \varepsilon^2 h^{(2)}_{\mu\nu} + O(\varepsilon^3)} \qquad \varepsilon \sim m/M$$



- Waveform Calculation: consistent next step to Radiation Reaction corrections of the geodesic trajectory $\sim O(m^2)$.
- 2nd order perturbations provide error bars to linearized theory and increases the accuracy and range of validity of the results (not so small mass ratio).
- Direct comparison with full GR numerical simulations (in the time domain).

First order curvature perturbations

$$\psi_4 = C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta, \qquad \psi^{(1)} = \rho^{-4}\psi_4^{(1)} \quad \text{(outgoing radiation)}$$

Teukolsky equation : $\boxed{\widehat{\mathcal{W}}\psi^{(1)} = T_p^{(1)}}$,

$$\widehat{\mathcal{W}} = \left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\vartheta\right]\partial_{tt} + \frac{4Mar}{\Delta}\partial_{t\varphi} - 4\left[r + ia\cos\vartheta - \frac{M(r^2-a^2)}{\Delta}\right]\partial_t$$
$$- \Delta^2\partial_r\left(\Delta^{-1}\partial_r\right) - \frac{1}{\sin\vartheta}\partial_\vartheta\left(\sin\vartheta\,\partial_\vartheta\right) - \left[\frac{1}{\sin^2\vartheta} - \frac{a^2}{\Delta}\right]\partial_{\varphi\varphi}$$
$$+ 4\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\vartheta}{\sin^2\vartheta}\right]\partial_\varphi + \left(4\cot^2\vartheta + 2\right),$$

- Stable numerical code to integrate it in the time domain as a 2+1 PDE.
- Formulas to impose Cauchy data: $\psi^{(1)} = \psi^{(1)}(h_{ij}, K_{ij}), \dot{\psi}^{(1)} = \dot{\psi}^{(1)}(h_{ij}, K_{ij})$
- $\psi_4^{(1)}$ gauge and tetrad invariant!

Second order curvature perturbations

$$\boxed{\widehat{\mathcal{W}}\psi_4^{(2)} = S[\psi^{(1)},\partial_t\psi^{(1)}] + T_p^{(2)} = S_{eff}}$$

$$\begin{aligned}
S \;=\;& \left[\overline{d}_3^{(0)}\left(\delta+4\beta-\tau\right)^{(1)} - \overline{d}_4^{(0)}\left(D+4\epsilon-\rho\right)^{(1)}\right]\psi_4^{(1)} \\
& - \left[\overline{d}_3^{(0)}\left(\Delta+4\mu+2\gamma\right)^{(1)} - \overline{d}_4^{(0)}\left(\overline{\delta}+4\pi+2\alpha\right)^{(1)}\right]\psi_3^{(1)} \\
& +3\left[\overline{d}_3^{(0)}\nu^{(1)} - \overline{d}_4^{(0)}\lambda^{(1)}\right]\psi_2^{(1)} - 3\psi_2^{(0)}\left[\left(\overline{d}_3-3\pi\right)^{(1)}\nu^{(1)} - \left(\overline{d}_4-3\mu\right)^{(1)}\lambda^{(1)}\right],
\end{aligned}$$

$$\overline{d}_3 = (\bar{\delta}+3\alpha+\bar{\beta}+4\pi-\bar{\tau}), \quad \overline{d}_4 = (\sigma+4\mu+\overline{\mu}+3\gamma-\overline{\gamma}),$$

$$\begin{aligned}
T_p^{(2)} \;=\;& \sum_{p=1}^{2}\Big\{\overline{d}_4^{(0)}\left[\left(\overline{\delta}-2\overline{\tau}+2\alpha\right)^{(2-p)}T_{n\overline{m}}^{(p)} - \left(\Delta+2\gamma-2\overline{\gamma}+\overline{\mu}\right)^{(2-p)}T_{\overline{m}\overline{m}}^{(p)}\right] \\
& + \overline{d}_3^{(0)}\left[\left(\Delta+2\gamma+2\overline{\mu}\right)^{(2-p)}T_{n\overline{m}}^{(p)} - \left(\overline{\delta}-\overline{\tau}+2\overline{\beta}+2\alpha\right)^{(2-p)}T_{nn}^{(p)}\right]\Big\},
\end{aligned}$$

Problem: how do we compute the effective source term?

$$S_{eff} = S[\psi^{(1)}_{4...0}, \alpha^{(1)}....\sigma^{(1)}, D^{(1)}...\Delta^{(1)}] + T^{(2)},$$

- We choose a tetrad $(l_\mu, n_\mu, m_\mu, \overline{m}_\mu)^{(1)}$ to express S in terms metric perturbations $h^{(1)}_{\mu\nu}$.
- We need to choose a gauge to compute the metric perturbations. [Outgoing radiation gauge:]

$$\begin{aligned}(h^{(1)}_{\mu\nu})^{\;ORG} = \quad & 2Re\Big[\rho^{-4}\left\{-n_\mu n_\nu(\overline{\delta} - 3\alpha - \overline{\beta} + 5\pi)(\overline{\delta} - 4\alpha + \pi)\right. \\ & - \overline{m}_\mu \overline{m}_\nu(\Delta + 5\mu - 3\gamma + \overline{\gamma})(\Delta + \mu - 4\gamma) \\ & + n_{(\mu}\overline{m}_{\nu)}[(\overline{\delta} - 3\alpha + \overline{\beta} + 5\pi + \overline{\tau})(\Delta + \mu - 4\gamma) \\ & \left. + \,(\Delta + 5\mu - \overline{\mu}\right\}(\Psi_{ORG})\Big]\end{aligned}$$

- We need to compute the potential Ψ_{ORG} corresponding to our physical problem (in the time domain)!

- $\psi_4^{(2)}$ not invariant under 1st order coord. tranf.

$$\widetilde{\psi_4^{(2)}} \to \psi_4^{(2)} + \partial_\mu \psi_4^{(1)} \xi_{(1)}^{\mu}.$$

- $\psi_4^{(2)}$ not invariant under tetrad rotations:

$$\widetilde{\psi_4^{(2)}} \to \psi_4^{(2)} + 2[(A-1) - i\Lambda]\psi_4^{(1)} + 4\overline{a}\psi_3^{(1)} + 6\overline{a}^2\psi_2^{(0)}.$$

Restoring gauge and tetrad invariance to 2nd order:

$$\boxed{\widehat{\mathcal{W}}[\psi^{(2)} + Q] = S + \widehat{\mathcal{W}}[Q] = \widehat{\mathcal{W}}[\psi_I] = S_I}$$

Choice of the invariant

$$\psi_I^{(2)} = \psi^{(2)} + Q - Q^{AF}$$

Interpretation of the invariant waveform

$$(\psi_I^{(2)})_{AF} = (\psi_4^{(2)})_{AF}$$

Energy and momentum radiated computation with the usual expressions.

Summary

- Work in an AF gauge (for instance the ORG)
- The RW gauge used in 2nd order perturbations of Schwarzshild is *not* AF
- Need to work out the regularization of the source term for the particle
- Its numerical implementation is an intensive problem
- Work in the time domain!