Gravitational self force for circular orbits of a non-rotating black hole

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- **A decomposition of a Green's function for the perturbed Einstein equations**
- **A particular normal coordinate system**
- **"Source" fields**
- **Gauge invariant quantities for a circular orbit of the Schwarzschild geometry**
- **Regularization parameters**
- **Scalar field numerical results for a circular orbit**
- **Gravitational field self force numerical results**

Decomposition

Assume a vacuum solution of the Einstein equations for the background geometry, g^{o}_{ab}. A test particle of small mass μ moves along a geodesic Γ of g^{o}_{ab}.

- Solve the perturbed Einstein equations to find the retarded metric perturbation h^{ret}_{ab} caused by μ.

- Decompose:
$$h^{\text{ret}}_{ab} = h^{\text{S}}_{ab} + h^{\text{R}}_{ab} \leftrightarrow h^{\text{dir}}_{ab} + h^{\text{tail}}_{ab}$$

- h^{S}_{ab} is a solution of the inhomogeneous perturbed Einstein equations and is singular at the particle. h^{R}_{ab} is a homogeneous solution and is smooth.

- The self force arises from $h^{\text{R}}_{ab} = h^{\text{ret}}_{ab} - h^{\text{S}}_{ab}$, as the particle moves along a geodesic of $g^{\text{o}}_{ab} + h^{\text{R}}_{ab}$.

$$h^{\mathsf{R}}_{ab} = h^{\mathsf{ret}}_{ab} - h^{\mathsf{S}}_{ab}$$

- h^{R}_{ab} is the **R**egularized **R**adiation **R**eaction **R**emainder.

- h^{S}_{ab} is the **S**ingular **S**ource **S**ubtrahend.

Normal coordinates are locally inertial

$$g_{ab} = \eta_{ab} + \frac{1}{2} x^i x^j g^0_{ab,ij} + \frac{1}{6} x^i x^j x^k g^0_{ab,ijk} + O(r^4),\ r \to 0,$$

$i, j, \ldots$ are spatial indices. x, y and z are locally Cartesian and map to r, θ, ϕ
$\mathcal{R} =$ length scale of the background geometry.

Thorne, Hartle and Zhang choice of normal coordinates is defined in a neighborhood about a geodesic in a vacuum spacetime.

$$g_{ab} = \eta_{ab} + {}_2H_{ab} + {}_3H_{ab} + O(r^4/\mathcal{R}^4), \quad r \to 0,$$

$${}_2H_{ab} dx^a dx^b = -\mathcal{E}_{ij} x^i x^j (dt^2 + \delta_{kl} dx^k dx^l) + \frac{4}{3} \epsilon_{kpq} \mathcal{B}^q{}_i x^p x^i dt\, dx^k,$$

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$${}_3H_{ab}dx^a dx^b = -\frac{1}{3}\mathcal{E}_{ijk}x^i x^j x^k(dt^2 + \delta_{lm}dx^l dx^m) + \frac{2}{3}\epsilon_{kpq}\mathcal{B}^q{}_{ij}x^p x^i x^j dt\, dx^m,$$

$\mathcal{E}$ and $\mathcal{B}$ are spatial, symmetric, tracefree and related to the Riemann tensor and its derivatives evaluated on the geodesic; in particular, $\mathcal{E}_{ij} = R_{titj}$ and $\mathcal{B}_{ij} = \epsilon_i{}^{pq}R_{pqjt}/2$.

Tidal distortions of a Schwarzschild black hole

Put a small black hole, $\mu \ll \mathcal{R}$, on a geodesic of the background geometry. The Schwarzschild metric is

$$g_{ab}^{\text{schw}} = -(1-2\mu/r)dt^2 + dr^2/(1-2\mu/r) + r^2 d\Omega^2$$

Quadrupole tidal distortion of the black hole by the background geometry distorts the metric by ${}_2h_{ab}$, with ${}_2h_{ab} \to {}_2H_{ab}$ for $\mu \ll r \ll \mathcal{R}$

$$\begin{aligned} {}_2h_{ab}dx^a dx^b &= -\mathcal{E}_{ij}x^i x^j \left[(1-2\mu/r)^2 dt^2 + dr^2 + (r^2 - 2\mu^2)(d\theta^2 + \sin^2\theta d\phi^2)\right] \\ &\quad + \frac{4}{3}\epsilon_{kpq}\mathcal{B}^q{}_i x^p x^i (1-2\mu/r)dt\, dx^k. \end{aligned}$$

This is an elementary result from the Regge-Wheeler perturbation analysis of a black hole.

Octupole tidal distortion of the black hole is

$$
\begin{aligned}
{}_3h_{ab}dx^a dx^b &= -\frac{1}{3}\mathcal{E}_{ijk}x^i x^j x^k \big[(1-2\mu/r)^2(1-\mu/r)dt^2 + (1-\mu/r)dr^2 \\
&\quad +(r^2 - 2\mu r + 4\mu^3/5r)(d\theta^2 + \sin^2\theta d\phi^2)\big] \\
&\quad +\frac{2}{3}\epsilon_{kpq}\mathcal{B}^q{}_{ij}x^p x^i x^j (1-2\mu/r)(1-4\mu/3r)dt\, dx^k.
\end{aligned}
$$

NOTES:

- The singular part of h^{S}_{ab} is the part of the Schwarzschild geometry which is linear in μ.

- Additions to h^{S}_{ab} are the parts of ${}_2h_{ab}$ which are linear in $\mu/r \times \mathcal{E}_{ij}x^ix^j$ and $\mathcal{B}_{ij}x^ix^j$ and similar terms from ${}_3h_{ab}$.

- Or, h^{S}_{ab} is the tidally distorted Coulomb field of the black hole.

- ${}_2h^{\mu}_{ab}$ is the quadrupole tidal distortion of any object's Coulomb field, as long as the object has no appreciable higher multipole moments.

$$h^{\mathsf{S}}_{ab} \equiv {}_0h^{\mu}_{ab} + {}_2h^{\mu}_{ab} + {}_3h^{\mu}_{ab},$$

$$h^S_{ab} \equiv {}_0h^\mu_{ab} + {}_2h^\mu_{ab} + {}_3h^\mu_{ab},$$

superscript μ means "linear in μ"

$$\begin{aligned}
{}_0h^\mu_{ab}dx^a dx^b &= 2\mu/r(dt^2 + dr^2) \\
{}_2h^\mu_{ab}dx^a dx^b &= \frac{4\mu}{r}\mathcal{E}_{ij}x^i x^j dt^2 - \frac{8\mu}{3r}\epsilon_{kpq}\mathcal{B}^q{}_i x^p x^i dt\, dx^k \\
{}_3h^\mu_{ab}dx^a dx^b &= \frac{\mu}{3r}\mathcal{E}_{ijk}x^i x^j x^k\left[5dt^2 + dr^2 + 2r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \\
&\quad -\frac{20\mu}{9r}\epsilon_{kpq}\mathcal{B}^q{}_{ij}x^p x^i x^j dt\, dx^k,
\end{aligned}$$

The coordinates are the locally inertial THZ coordinates which follow the test particle. The gauge transformation to the harmonic gauge is straightforward.

Asymptotic matching

Where $\mu \ll r \ll \mathcal{R}$, the geometry of a small mass moving along a geodesic should be well described either by the background metric perturbed by a small mass, or by the Schwarzschild metric perturbed by weak tidal distortions. Thus the metric perturbation of the background geometry, in this region, should be approximately the part of the perturbed Schwarzschild geometry which is linear in μ.

Self force on a circular orbit of the Schwarzschild geometry.

- Consider a small mass μ moving along a particular geodesic of the Schwarzschild geometry.
- Use tensor spherical harmonics to solve for the ℓ, m retarded metric perturbation $h^{\mathsf{ret}}_{\ell m\, ab}$ caused by the particle.
- Find the THZ coordinates as functions of the Schwarzschild coordinates for the geodesic.
- Express the source field h^{S}_{ab} in terms of the Schwarzschild coordinates and decompose it in terms of tensor harmonics, $h^{\mathsf{S}}_{\ell m\, ab}$.
- The tensor harmonic decomposition of the reaction field is $h^{\mathsf{R}}_{\ell m\, ab} = h^{\mathsf{ret}}_{\ell m\, ab} - h^{\mathsf{S}}_{\ell m\, ab}$

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- The self force acting on the particle results in the worldline being modified to be a geodesic of the perturbed geometry

$$g^{\mathsf{o}}_{ab} + h^{\mathsf{R}}_{ab} = g^{\mathsf{o}}_{ab} + \sum_{\ell m} h^{\mathsf{R}}_{\ell m\, ab}$$

Geodesics of the perturbed Schwarzschild geometry

The Geodesic equation is

$$\frac{du_a}{ds} = \frac{1}{2} u^b u^c \frac{\partial}{\partial x^a} (g^{\circ}_{bc} + h^{\mathsf{R}}_{bc})_\Gamma$$

$\partial/\partial t$ and $\partial/\partial\phi$ are not Killing vectors of $g^{\circ}_{ab} + h^{\mathsf{R}}_{ab}$, but define

$$u_t = -E, \quad \text{and} \quad u_\phi = J$$

then

$$\frac{dE}{ds} = -\frac{1}{2} u^b u^c \frac{\partial h^{\mathsf{R}}_{ab}}{\partial t}$$

$$\frac{dJ}{ds} = \frac{1}{2} u^b u^c \frac{\partial h^{\mathsf{R}}_{ab}}{\partial \phi}$$

For circular orbits

$$(E + u^a h^{\mathsf{R}}_{at})^2 = \frac{(R-2M)^2}{R(R-3M)}(1 + u^a u^b h^{\mathsf{R}}_{ab} - \frac{1}{2} R u^a u^b \partial h^{\mathsf{R}}_{ab}/\partial r)$$

and

$$(J - u^a h^{\mathsf{R}}_{a\phi})^2 = \frac{R^2 M}{R-3M}(1 + u^a u^b h^{\mathsf{R}}_{ab}) - \frac{R^3(R-2M)}{2(R-3M)} u^a u^b \partial h^{\mathsf{R}}_{ab}/\partial r.$$

The angular velocity Ω of a circular orbit as measured at infinity is

$$\Omega^2 = (d\phi/dt)^2 = (u^\phi/u^t)^2 = M/R^3 - \frac{R-3M}{2R^2} u^a u^b \partial h^{\mathsf{R}}_{ab}/\partial r.$$

The expressions given above for

$$\Omega \qquad dE/ds \qquad dJ/ds \qquad \text{and} \qquad E - \Omega J$$

are all gauge invariant when evaluated at the particle—this is a technical result based on gauge transformations in the Regge-Wheeler formalism. Expressions for u^t and u^ϕ are also gauge invariant, the radius of the orbit R is not.

Scalar field and circular orbits of the Schwarzschild geometry.

The self force from a scalar field on a particle of scalar charge q is

$$\mathcal{F}_a = q\nabla_a\psi^{\mathsf{R}}$$

where

$$\psi^{\mathsf{R}} = \psi^{\mathsf{ret}} - \psi^{\mathsf{S}}$$

With multipole expansions

$$\psi^{\mathsf{R}}_{\ell m}(r,t) = \psi^{\mathsf{ret}}_{\ell m}(r,t) - \psi^{\mathsf{S}}_{\ell m}(r,t),$$

the self force is found by evaluating

$$\begin{aligned} \mathcal{F}^{\mathsf{self}}_a &= \nabla_a \sum_{\ell m} \psi^{\mathsf{R}}_{\ell m} Y_{\ell m} \\ &= \nabla_a \sum_{\ell m} (\psi^{\mathsf{ret}}_{\ell m} - \psi^{\mathsf{S}}_{\ell m}) Y_{\ell m} \end{aligned}$$

at the source point z.

After summing over m

$$\mathcal{F}_a^{\textsf{self}} = \sum_\ell \left(\mathcal{F}_{\ell a}^{\textsf{ret}} - \mathcal{F}_{\ell a}^{\textsf{S}}\right)$$

The difference in multipole moments must be taken before the summation over ℓ.
The regularization parameters are derived from the multipole components of $\nabla_a \psi^S$ evaluated at the source point.

$$\begin{aligned}\lim_{x\to z} \mathcal{F}_{\ell r}^{\textsf{S}} &= \left(\ell + \frac{1}{2}\right) A_r + B_r - \frac{2\sqrt{2}D_r}{(2\ell-1)(2\ell+3)} \\ &\quad + \frac{E_r^1 B_{3/2}}{(2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5)} + O(\ell^{-6}).\end{aligned}$$

$$A_r = -\mathrm{sgn}(\Delta)\frac{[r_\circ(r_\circ - 3M)]^{1/2}}{{r_\circ}^2(r_\circ - 2M)}$$

$$B_r = -\left[\frac{r_\circ - 3M}{{r_\circ}^4(r_\circ - 2M)}\right]^{1/2}\left[F_{1/2} - \frac{(r_\circ - 3M)F_{3/2}}{r_\circ - 2M}\right],$$

and

$$D_r = \left[\frac{2{r_\circ}^2(r_\circ - 2M)}{r_\circ - 3M}\right]^{1/2}\left[-\frac{M(r_\circ - 2M)F_{-1/2}}{2{r_\circ}^4(r_\circ - 3M)} - \frac{(r_\circ - M)(r_\circ - 4M)F_{1/2}}{8{r_\circ}^4(r_\circ - 2M)}\right.$$
$$\left. + \frac{(r_\circ - 3M)(5{r_\circ}^2 - 7r_\circ M - 14M^2)F_{3/2}}{16{r_\circ}^4(r_\circ - 2M)^2} - \frac{3(r_\circ - 3M)^2(r_\circ + M)F_{5/2}}{16{r_\circ}^4(r_\circ - 2M)^2}\right].$$

$B_{3/2}$ is a constant, $F_p = {}_2F_1[p, \frac{1}{2}; 1; M/(r_\circ - 2M)]$. The A_r and B_r terms agree with the recent results of Barack, Mino, Nakano, Ori and Sasaki.

The D_r and E_r^1 terms sum to zero. Their inclusion speeds up convergence of the sum.

Derivation of the regularization parameters

$$\psi^{\mathsf{S}} = \frac{q}{\rho}\left(1 + \frac{\lambda_{ijk}X^iX^jX^k}{\rho^2} + \frac{\lambda_{ijkl}X^iX^jX^kX^l}{\rho^2} + \frac{O(X^5)}{\rho^2}\right)$$

ρ^2 is the (spatial geodetic distance)^2 through $O(X^2)$.
The $\lambda_{ij\ldots}$ are dependent only upon the orbit and determined by ψ^{S}.
The X^i are locally the THZ x, y and z.
We need a multipole decomposition of $\partial_r\psi^S$.
If the particle is on the z-axis, only the $m = 0$ components contribute, and the $m = 0$ part of a typical term above can be written as

$$\frac{{r_{\circ}}^p(1 - \cos\Theta)^{p/2}(r - r_{\circ})^q}{\rho^n},$$

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$$\frac{r_{\circ}{}^{p}(1-\cos\Theta)^{p/2}(r-r_{\circ})^{q}}{\rho^{n}},$$

In Schwarzschild coordinates.

$$r_{\circ}(1-\cos\Theta)^{1/2} \approx \text{distance near} \quad \Theta = 0$$

and is C^{∞} elsewhere.

We particularly require the expansion

$$(1-\cos\Theta)^{m+1/2} = \sum_{\ell=0}^{\infty} A_\ell^{m+1/2} P_\ell(\cos\Theta)$$

$$\begin{aligned} A_\ell^{m+1/2} \quad &= \quad B_{m+1/2}(2\ell+1)/\left[(2\ell-2m-1)(2\ell-2m+1)\ldots\right. \\ &\qquad \left.(2\ell+2m+1)(2\ell+2m+3)\right], \end{aligned}$$

where

$$B_{m+1/2} = (-1)^{m+1} 2^{m+3/2} \left[(2m+1)!!\right]^2.$$

$$\sum_{\ell=0}^{\infty} A_\ell^{m+1/2} = 0.$$

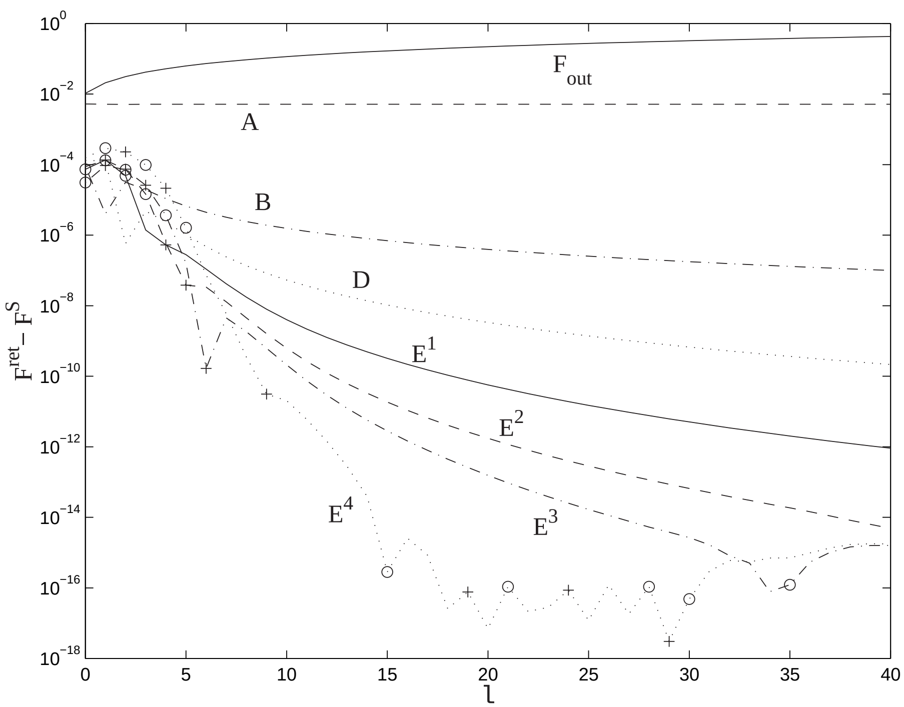

Figure 1: $\mathcal{F}^{\rm ret}_{\ell r}$ is displayed as a function of ℓ, along with the result of it being regularized by $A_r, B_r \ldots E^4_r$. In principle the self force could be determined by summing up the data points along curve B or any below it. A point where the data on a particular curve changes sign from being negative to positive is labeled with $+$, from positive to negative by $\circ$.

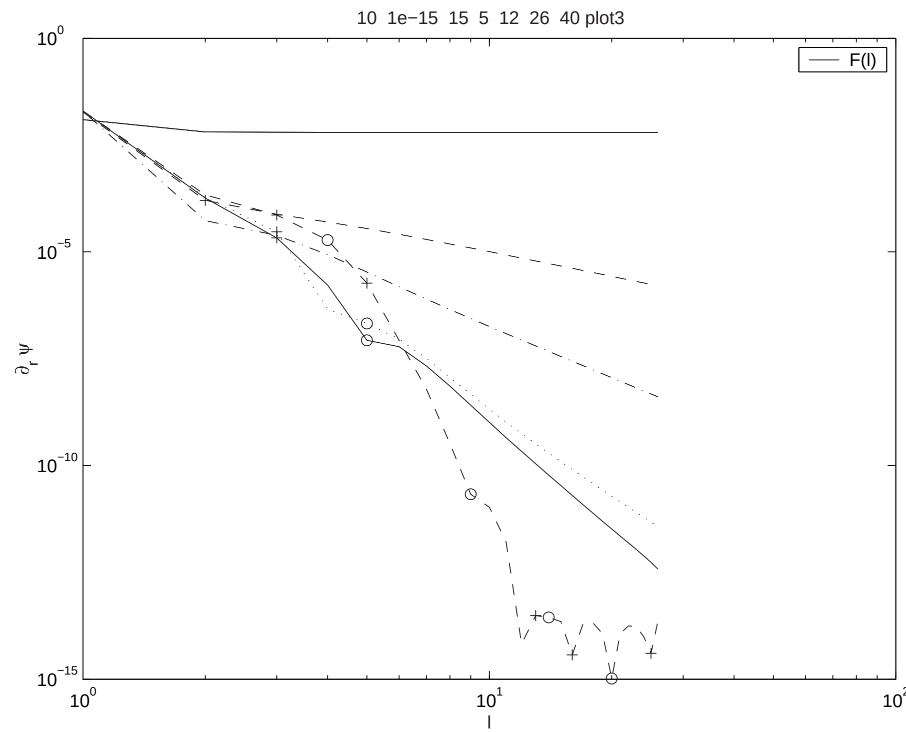

Figure 2: $\Delta\Omega^2$, the contribution to the change in orbital frequency, from the gravitational self force, for a test particle at $R_\text{o} = 10M$ is displayed as a function of ℓ, along with the result of it being regularized by $A_r, B_r \ldots E_r^4$; the labeling should be similar to that in the previous figure.

The last figure is based on preliminary, unvarnished results but reveals the ability to calculate self force effects without summing to very large values of ℓ. In principle $\Delta\Omega^2$ could be determined by summing up the data points along any curve except the top one. For this figure all of the regularization parameters have been determined by curve fitting (it would have been better to use the analytically known parameters before doing the fitting). Note that while $\Delta\Omega^2$ is independent of the choice of gauge the radius of the orbit $R_{\rm o}$ is not. Thus this figure alone conveys no interesting physical information — However a comparison of two gauge invariant quantities, such as those mentioned earlier, will be of interest.