Towards Integrating the Inhomogeneous Teukolsky Equation in the Time Domain

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Motivation

- There is a need to understand the type of signals that will be observed by LISA;
- This entails calculating the gravitational waveforms emitted by a small compact object orbiting a super-massive Kerr black hole;
- Our aim is to develop a numerical code that, upon input of the particle's motion, returns the waveform associated with this motion.
- WE ARE NOT TRYING TO RECONSTRUCT THE METRIC PERTURBATIONS...

Motivation: Teukolsky's equation

The Teukolsky equation describes the evolution of curvature perturbations in the background of a Kerr black hole due to a perturbation of integer spin s (Teukolsky, 1972,1973). Schematically,

$$\mathcal{D}\Psi(x^{\mu}) = S(x^{\mu}),$$

where

1. Ψ for $s = -2$ is associated with the gravitational-wave content of the spacetime in the far zone;

2. $\mathcal{D} \equiv \mathcal{D}(\partial/\partial x^{\alpha}, x^{\alpha}, a)$ is a second order differential operator that describes the motion of waves in the spacetime;

3. $S(x^{\mu})$ is the known source of the perturbation. The source term is singular at the particle location, since the particle is represented by Dirac δ-functions.

Motivation: Teukolsky's equation (Cont'd)

Approach 2:

- Use $\Psi(t,r,\theta,\varphi) = \Phi(t,r,\theta)_m e^{im\varphi}$

- Need to integrate in 2+1 dimensions with singular source term.

- In 2+1 D, the field diverges as $\ln(R)$ as one approaches the particles location ($R \to 0$). This cannot be handled very well with finite difference methods $\to$ use smeared δ-functions in the source term.

- There has been quite a lot of work on integrating the vacuum Teukolsky equation, either in Boyer-Lindquist coordinates (Krivan *et al.*, 1997), or in horizon penetrating coordinates (Campanelli *et al.*, 2001).

A first step: Schwarzschild background

Because of spherical symmetry, the Schwarzschild case is much simpler than the Kerr case. The biggest advantage is that we can use $\Psi = \Phi_l(t,r)Y_{lm}(\theta,\varphi)$:

1. We can do 1+1 D integration: the field is only discontinuous at the particle's position. Finite difference can handle this type of behaviour, which allows us to study the effects of a smeared source term on Φ. (Lousto and Price)

2. We can integrate in 2+1 D with $\Psi = \Phi(t,r,\theta)e^{im\varphi}$ and compare with the results from 1+1 D.

Scalar charge in Schwarzschild

We start by studying the radiation emitted by a scalar charge orbiting a Schwarzschild black hole.

The scalar field Ψ obeys:

$$g^{\alpha\beta}\Psi_{;\alpha\beta} = -4\pi q \int d\tau \frac{\delta^4(x^\mu - Z^\mu(\tau))}{\sqrt{-g}},$$

where

- q is the scalar charge (which we set to 1 from now on);
- τ is proper time along the particle's path (assumed to be a geodesic);
- $Z^\mu(\tau)$ are the known coordinates of the scalar charge;
- We use $\Psi = \frac{\Phi_l(t,r)}{r} Y_{lm}(\theta, \varphi)$.

Scalar charge in Schwarzschild

We solve the previous equation using standard finite difference methods. We will present results in 1+1 D for the following three cases:

1. Standard Schwarzschild coordinates, exact source term (Lousto and Price,1997): LP

2. Standard Schwarzschild coordinates, smeared source term (Lax-Wendroff): LW

3. Horizon Penetrating coordinates, smeared source term (Lax-Wendroff): HPC.

Wave equation in standard coordinates

The line element is:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2,$$

where f(r)=1-2M/r.

The wave equation takes the form:

$$\ddot{\Phi} - \Phi'' + V_l(r)\Phi = S(r,t),$$

where $V_l(r) = f(r)(l(l+1) + 2M/r)/r^2$, a dot denotes a time derivative, a prime a derivative with respect to r^*, the usual tortoise coordinate, and

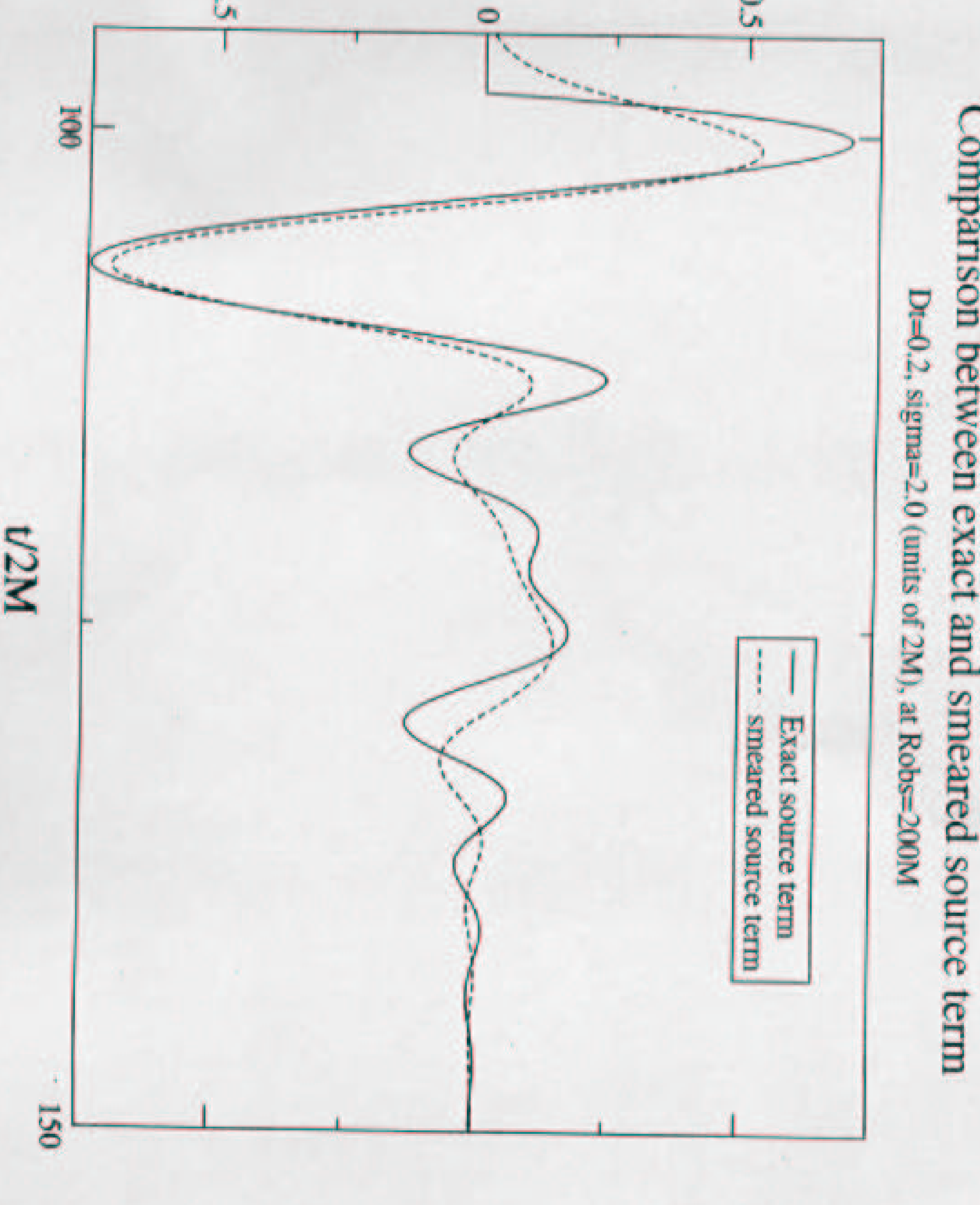
$$S(r,t) = \frac{4\pi}{u^t r} Y^*_{lm}(t)\delta(r^* - R^*_p(t)).$$

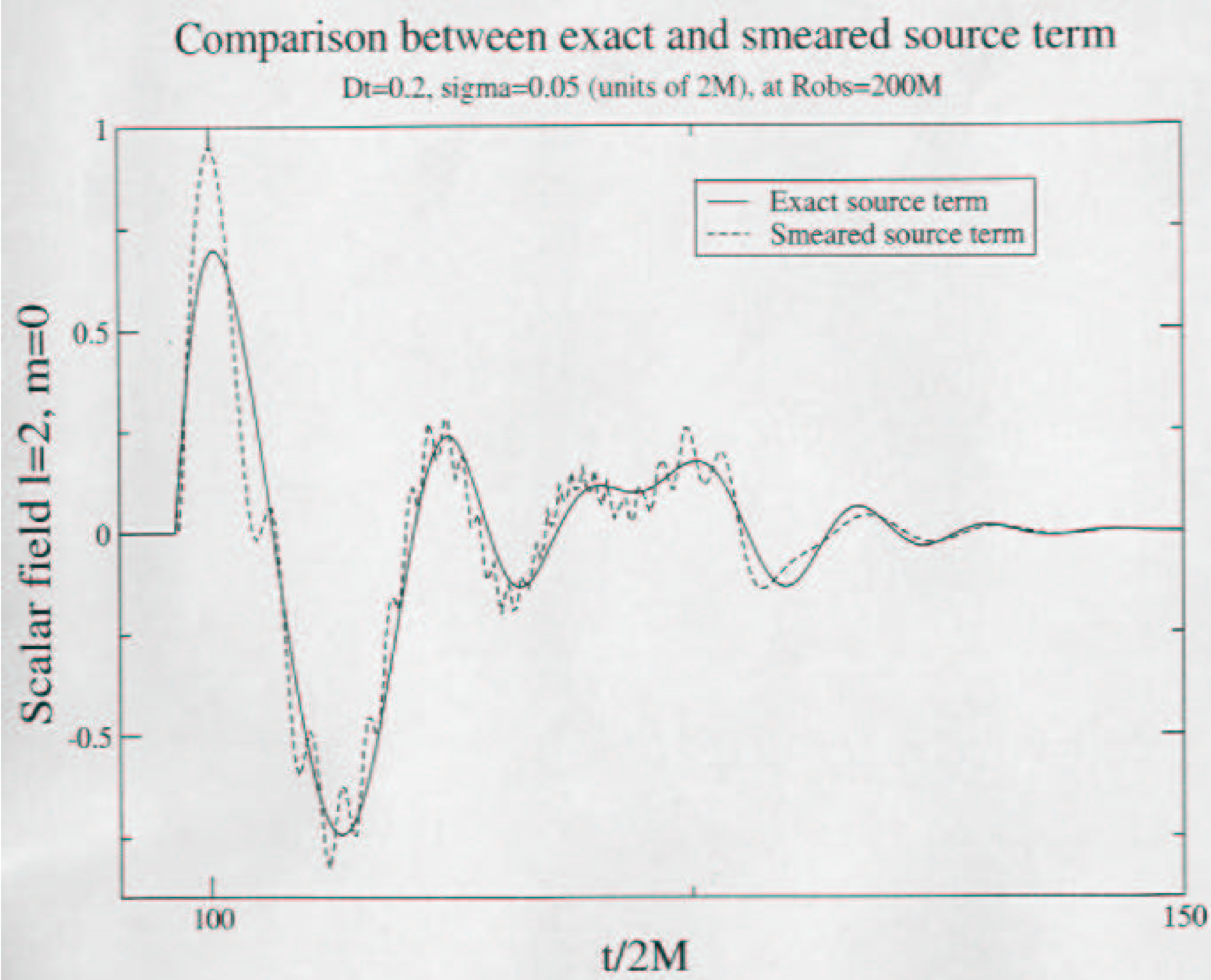
For a smeared particle:

$$\delta(x) = \frac{e^{-\frac{1}{2}(x/\sigma)^2}}{\sqrt{2\pi\sigma^2}},$$

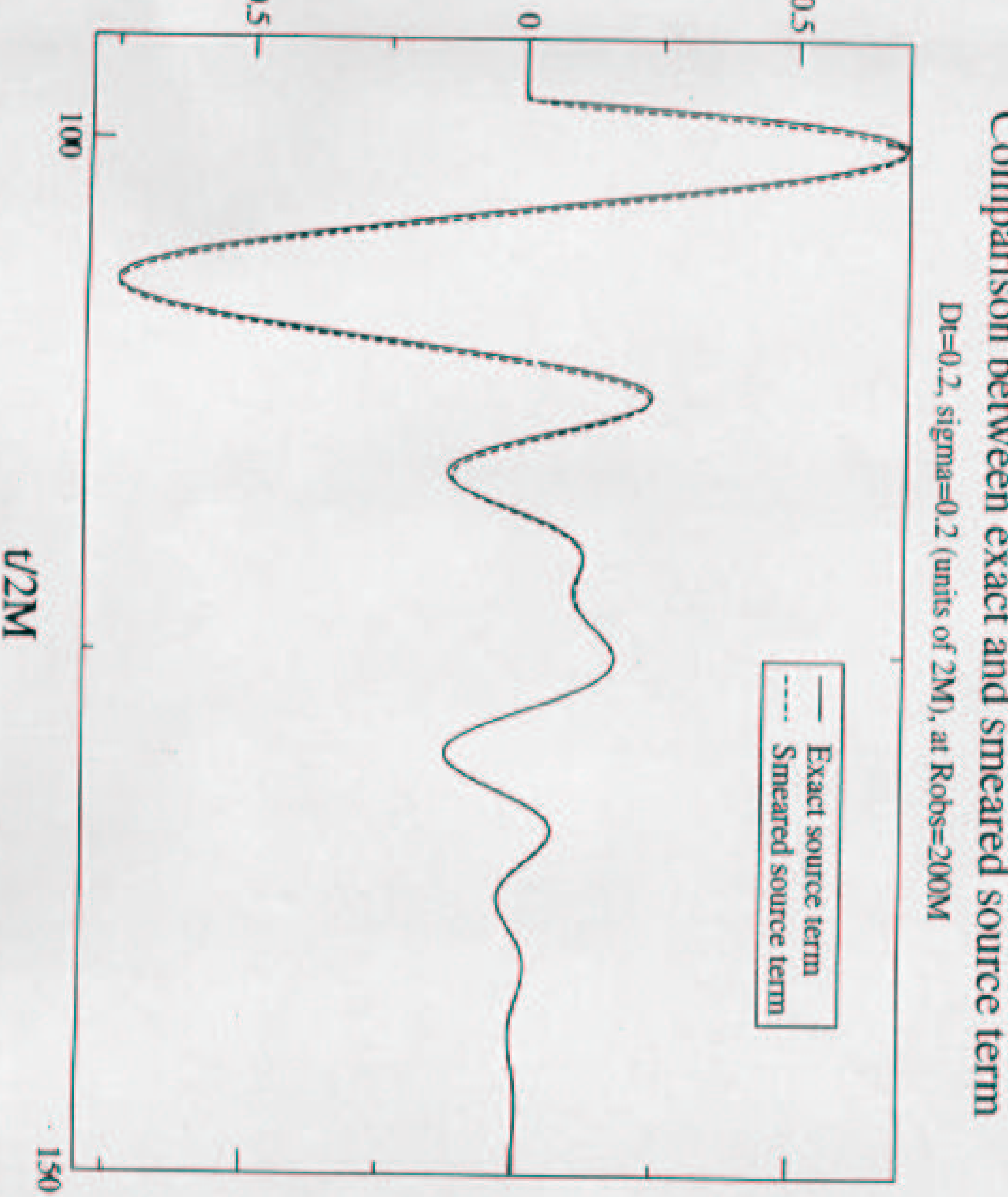
where σ is the particle's width, and $x = r^* - R^*_p(t)$.

Scalar field l=2, m=0




Comparison between exact and smeared source term
Dt=0.2, sigma=0.05 (units of 2M), at Robs=200M
Exact source term
Smeared source term
Scalar field l=2, m=0
t/2M
1
0.5
0
-0.5
100
150

Scalar field l=2, m=0



Wave equation in HP coordinates

The line element in Painlevé-Güllstrand (Martel and Poisson, 2001) coordinates is

$$ds^2 = -f(r)dt^2 + 2\sqrt{\frac{2M}{r}}dtdr + dr^2 + r^2 d\Omega^2,$$

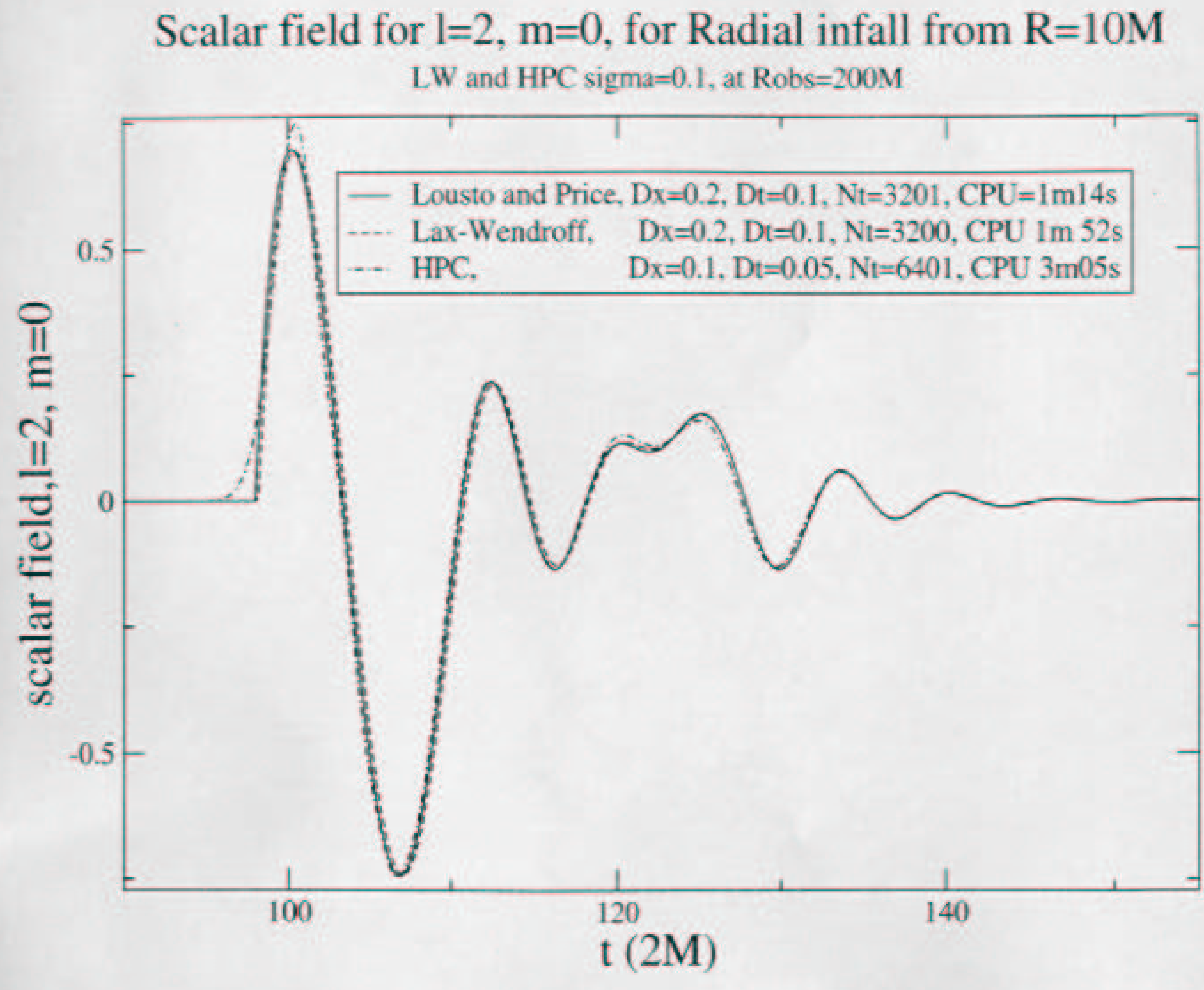
where, again, $f(r) = 1 - 2M/r$.

Wave equation:

$$\ddot{\Phi} - f(r)\Phi'' - 2\sqrt{2M/r}\dot{\Phi}' + \frac{1}{2r}\sqrt{2M/r}\dot{\Phi} - \frac{2M}{r^2}\Phi' + V_l(r)\Phi = S(r,t),$$

where $V_l(r) = (l(l+1) + 2M/r)/r^2$, a dot denotes a time derivative, a prime a derivative with respect to r, and

$$S(r,t) = \frac{4\pi}{u^t r} Y^*_{lm}(t)\delta(r - R_p(t)).$$

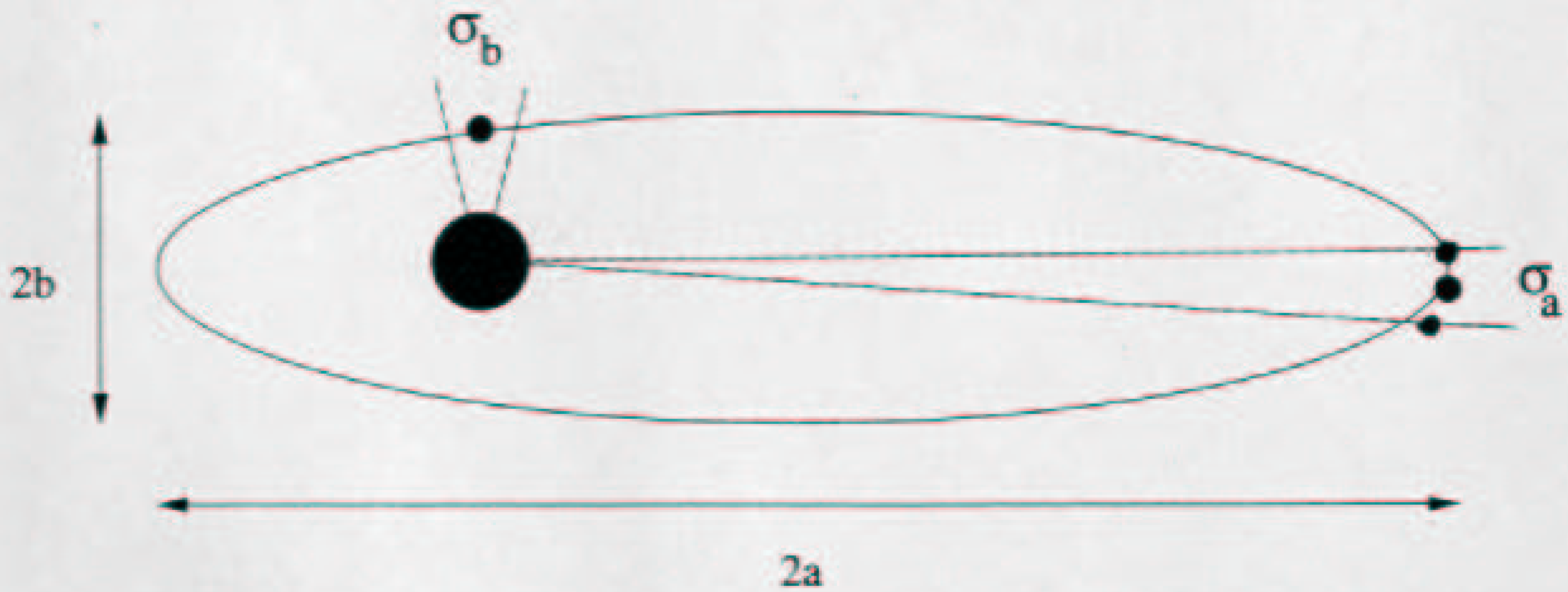


Scalar field for l=2, m=0, for Radial infall from R=10M
LW and HPC sigma=0.1, at Robs=200M
Lousto and Price, Dx=0.2, Dt=0.1, Nt=3201, CPU=1m14s
Lax-Wendroff, Dx=0.2, Dt=0.1, Nt=3200, CPU 1m 52s
HPC, Dx=0.1, Dt=0.05, Nt=6401, CPU 3m05s
scalar field,l=2, m=0
0.5
0
-0.5
100
120
140
t (2M)

Looking ahead

Consider an eccentric orbit in Newtonian gravity. The orbit is described by a and $b = \sqrt{1-e^2}a$ are the semi-major and semi-minor axis of the orbit.

It seems reasonable to ask that the size of the particle be much smaller than b, so that the particle is well resolved on the orbit, say $\sigma_a = b/50$.



$\sigma_a \sim$ distance between gridpoints

Spherical: $\left.\begin{array}{l}\sigma_a \sim a\,\delta\theta \\ \sigma_b \sim b\,\delta\theta\end{array}\right\}$ $\sigma_b = \sqrt{1-e^2}\,\sigma_a$

Looking ahead

If $a = 15$, and $e = 0.8$, than $b = 9$ and $\sigma_a = 0.18$, (in units of $2M$). As before we take $\Delta x = \sigma_a \approx 0.2$.

1. Spherical coordinates:
   - In spherical coordinates: $\Delta\theta \approx \Delta x/a \approx 0.012$
   - For a grid where $r \leq 250$, $0 \leq \theta \leq \pi$, this gives us 3.3e5 grid points;
   - At b, the particle now has a size given by $\sigma_b = \sqrt{1 - e^2}\sigma_a = 0.6\sigma_a$. The particle constantly changes size as it moves along its trajectory. . .
2. Cartesian coordinates:
   - We can take the $\Delta x = \Delta z = 0.2$.

- For $0 \leq \Delta x \leq 250$, and $-250 \leq z \leq 250$, we get 3.2e6 gridpoints.

- At b, we have $\sigma_b = \sigma_a$. The size of the particle is fixed.

Cartesian coordinates are more costly than spherical coordinates, but they have the advantage of fixing the size of the particle everywhere. The constant modulation in the size of the particle may introduce modulations in the amplitude and the phase of the waveforms.

Outlook

1. Perform 2+1 integration for scalar charge moving in Schwarzschild in both usual and horizon penetrating coordinates;

2. Compare the results obtained using spherical and Cartesian-like coordinates;

3. Move on to the 2+1 Kerr problem.