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Capra 5 at PennState, May 31, 2002

Gravitational Self-force Problem in the mode decomposition approach

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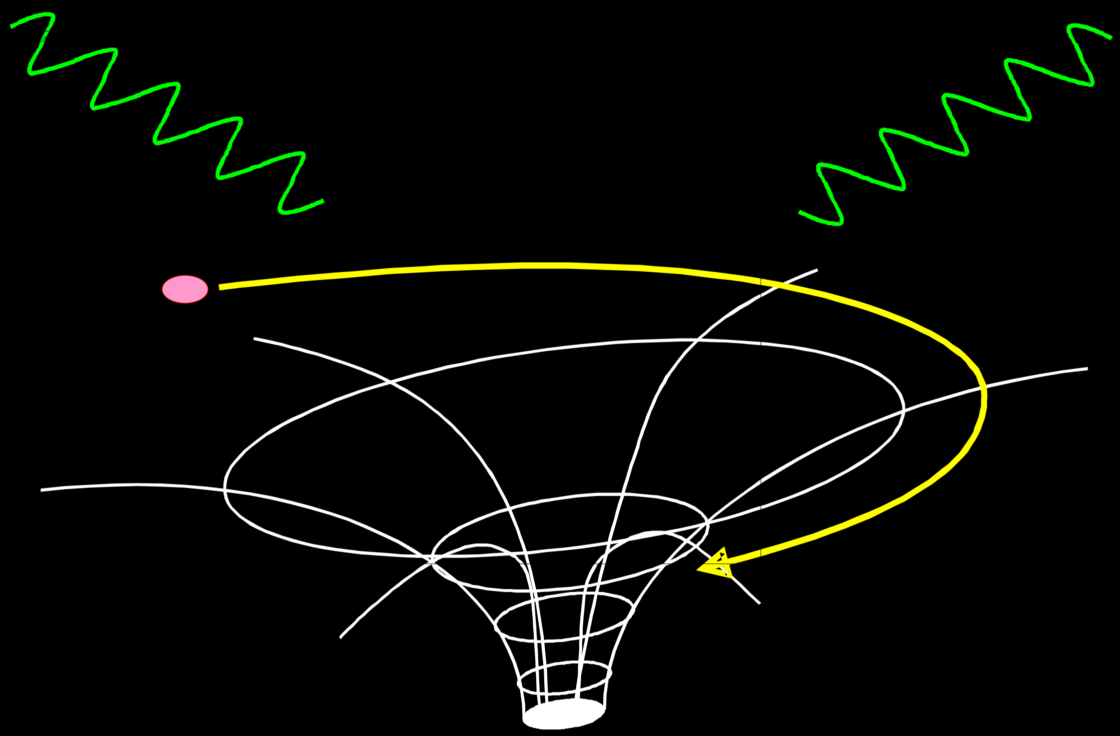
Gravitational Self-force Problem in the mode decomposition approach

1: Introduction



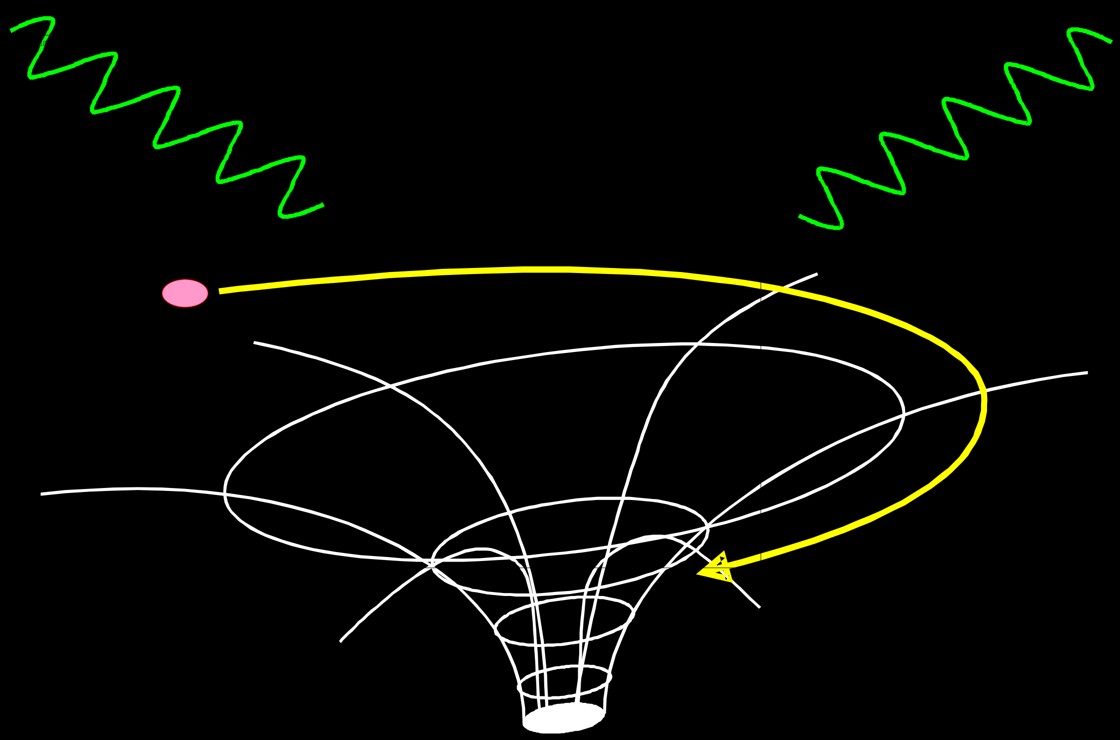
LISA primary target

A compact object (~1M) is inspiralling into a super-massive blackhole (~10^6M).

*extreme mass ratio
*eccentric orbit
*relativistic motion

We need to know what gravitational waves are expected to be detected.

Gravitational Self-force Problem in the mode decomposition approach



We use a perturbation analysis.

A supermassive blackhole is taken to be a background (Kerr blackhole), and we deal with a spiralling star as a source of the perturbation.

We consider the evolution of the orbit under the radiation reaction.

$$\frac{D}{d\tau} V_{\alpha} = ?$$

Gravitational Self-force Problem in the mode decomposition approach

2: Subtraction Procedure

$$F_\alpha(\tau) = \lim_{x \to z(\tau)} \left(F_\alpha[\mathbf{h}^{full}](x) - F_\alpha[\mathbf{h}^{S}](x) \right)$$

$$F_\alpha[\mathbf{h}] = -\frac{\mu}{2}\left(h_{\alpha\beta;\gamma} + h_{\gamma\alpha;\beta} - h_{\beta\gamma;\alpha} \right) V^\beta V^\gamma$$

$h^{full}{}_{\alpha\beta}$: Full Metric perturbation

$h^{dir}{}_{\alpha\beta}$: Direct Part of perturbation

$$h^{S}{}_{\alpha\beta}(x) = \mu \left[\frac{\bar{g}_{\alpha\mu}(x, z_{ret}) \bar{g}_{\beta\nu}(x, z_{ret}) V^{\mu}_{ret} V^{\nu}_{ret}}{\sigma_{;\lambda}(x, z_{ret}) V^{\lambda}_{ret}} \right] + \mu \int_{\tau_{ret}}^{\tau_{adv}} d\tau v_{\alpha\beta\mu\nu}(x, z_{(\tau)}) V^{\mu}{}_{(\tau\tau} V^{\nu}{}_{(\tau\tau}$$

*One can take the limit only after the subtraction.

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We replace the divergence by the sum of infinite harmonic series where each term is finite, and subtract the direct part term-by-term.

$$h_{\alpha\beta}^{full}(x) = \sum_{pLM} h_{pLM}^{(full)}(t,r) Y_{\alpha\beta}^{pLM}(\theta,\phi),$$

$$h_{\alpha\beta}^{S}(x) = \sum_{pLM} h_{pLM}^{(S)}(t,r) Y_{\alpha\beta}^{pLM}(\theta,\phi)$$

$$F_\alpha(\tau) = F_\alpha\left[\sum_{pLM}\left(h_{pLM}^{(full)} - h_{pLM}^{(S)}\right)\mathbf{Y}^{pLM}\right](z)$$

Gravitational Self-force Problem in the mode decomposition approach

We determine the S part by using a local analysis.

Calculation procedure:

1) evaluate the direct part by the local coordinate expansion method
2) apply the mode decomposition (mode-sum) formula

$$F_\alpha[\mathbf{h}^S] = \sum_{n_1 n_2 n_3 n_4 n_5} F_\alpha^{n_1 n_2 n_3 n_4 n_5} \frac{(t-t_0)^{n_1}(r-r_0)^{n_2}(\theta-\theta_0)^{n_3}(\phi-\phi_0)^{n_4}}{\xi^{2n_5+1}} + O(y)$$

$$\xi = \sqrt{g_{\alpha\beta} y^\alpha y^\beta + \tfrac{1}{2}(g_{\alpha\beta} V^\alpha y^\beta)^2}$$

Reference: L.Barack, Y.Mino, H.Nakano, A.Ori, M.Sasaki, gr-qc/0111001
Y.Mino, H.Nakano, M.Sasaki, gr-qc/0111074
L.Barack, A.Ori, gr-qc/0204093

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3: Gauge Problem

- Finite Algebraically Local Gauge Condition -

$$F_\alpha(\tau) = \lim_{x \to z(\tau)} \left(F_\alpha[\mathbf{h}^{full(harm)}](x) - F_\alpha[\mathbf{h}^{S(harm)}](x) \right)$$

Force is a gauge dependent variable. We must adopt a gauge condition in the regularization calculation in a consistent manner. If we ignore this condition, we may have a residual divergence.
The regularization scheme is defined in the harmonic gauge, therefore, we need a formula to calculate the metric perturbation in the harmonic gauge condition.

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$$F_\alpha(\tau) = F_\alpha[\mathbf{h}^{R.(harm)}](z),$$
$$h_{\alpha\beta}^{R(harm)}(x) = h_{\alpha\beta}^{full(harm)}(x) - h_{\alpha\beta}^{S(harm)}(x) \qquad (x \neq z(\tau))$$

The self-force is interpreted as a force by the R-part metric perturbation. At this stage, we can add a finite gauge transformation. (The gauge transformation must be defined in a whole geometry.)

$$F_\alpha(\tau) = F_\alpha[\mathbf{h}^{R(harm)} + \nabla\boldsymbol{\xi}](z)$$
$$= F_\alpha[\mathbf{h}^{R(harm)}](z) + \ddot{\xi}_\alpha(z)$$

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I found a nice choice of the finite gauge!

$$\boldsymbol{\xi} = \boldsymbol{\xi}^{*\to RW}[\mathbf{h}^{R(harm)}]$$

We use the algebraic gauge transformation given in the Zerilli's paper to R-part of the metric perturbation in the harmonic gauge.

$$\begin{aligned}
&F_\alpha(\tau)\\
&= F_\alpha\left[\mathbf{h}^{R(harm)} + \nabla\boldsymbol{\xi}^{*\to RW}[\mathbf{h}^{R(harm)}]\right](z)\\
&= \lim_{x\to z} F_\alpha\left[\mathbf{h}^{full(harm)} - \mathbf{h}^{S(harm)} + \nabla\boldsymbol{\xi}^{*\to RW}[\mathbf{h}^{full(harm)} - \mathbf{h}^{S(harm)}]\right](x)\\
&= \lim_{x\to z} F_\alpha\left[\mathbf{h}^{full(harm)} + \nabla\boldsymbol{\xi}^{*\to RW}[\mathbf{h}^{full(harm)}] - \mathbf{h}^{S(harm)} - \nabla\boldsymbol{\xi}^{*\to RW}[\mathbf{h}^{S(harm)}]\right](x)\\
&= \lim_{x\to z} F_\alpha\left[\mathbf{h}^{full(RW)} - \mathbf{h}^{S(RW)}\right](x)
\end{aligned}$$

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Comparison with the master variable- regularization

$$F_\alpha(\tau) = F_\alpha\left[\sum_{lm} \mathbf{h}_{lm}^{(RW)}\left[R_{lm}^{full} - R_{lm}^{S}\right]\right](z)$$

For this scheme, we need to evaluate non-vanishing part of the S-part in the Regge-Wheeler gauge.

$$\sum_{lm} \mathbf{h}_{lm}^{(RW)}\left[R_{lm}^{S}\right] = \cdots + O(y)$$

?

In the present scheme,

$$F_\alpha(\tau) = F_\alpha\left[\sum_{lm}\left[\mathbf{h}_{lm}^{full(harm)} - \mathbf{h}_{lm}^{S(harm)}\right]\right](z) + \ddot{\xi}_\alpha$$

we only need the non-vanishing part of the S-part in the harmonic gauge.

$$\sum_{lm} \mathbf{h}_{lm}^{S(harm)} = \cdots + O(y)$$

Gravitational Self-force Problem in the mode decomposition approach

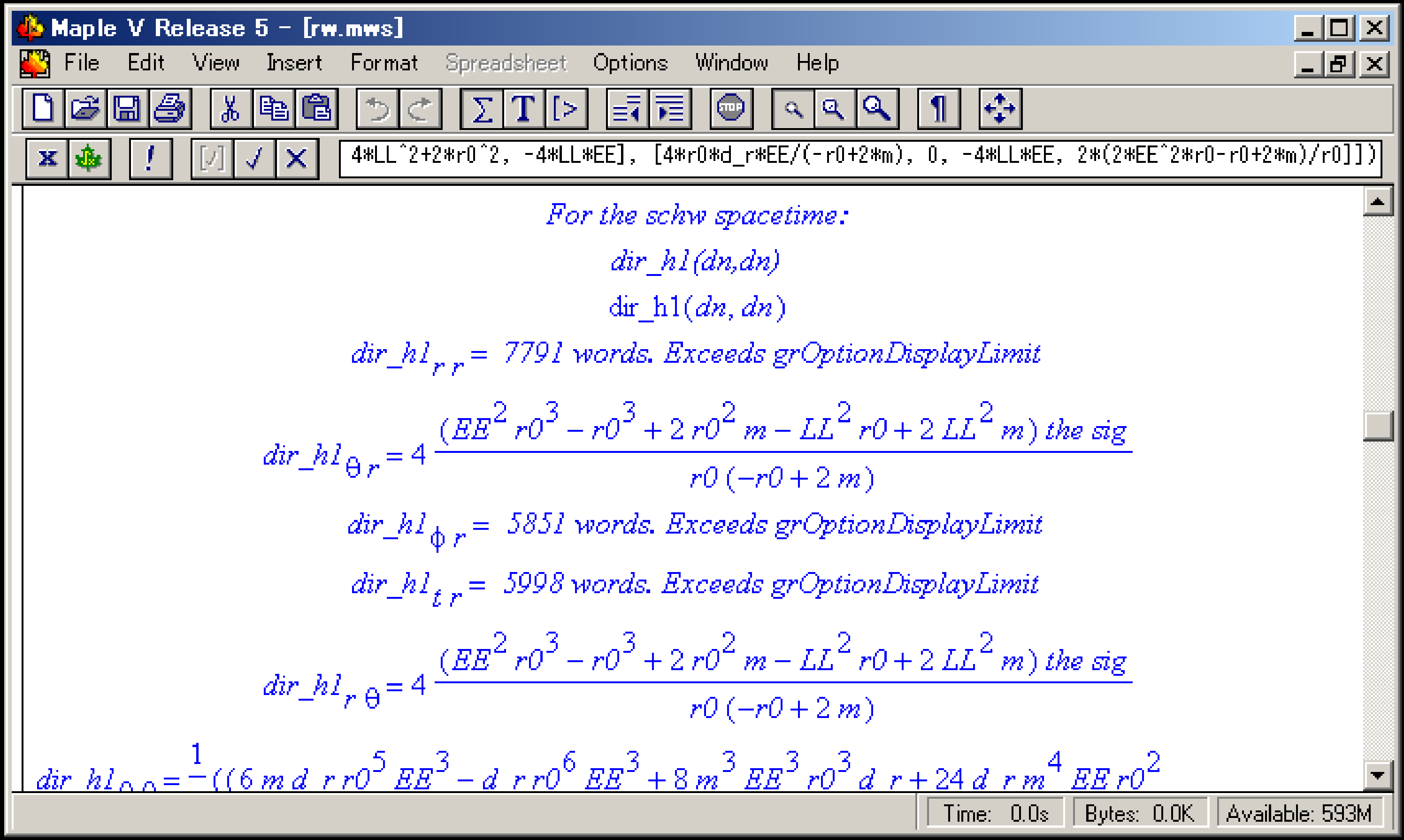
$$\mathbf{h}^{S(harm)} + \nabla\xi^{*\to RW}[\mathbf{h}^{S(harm)}]$$

Calculation procedure of the direct part:

1) the local coordinate expansion of the direct part in the harmonic gauge
2) apply the mode decomposition formula
3) calculate the gauge transformation to the Regge-Wheeler gauge
4) evaluate the direct part of the force in the Regge-Wheeler gauge

Gravitational Self-force Problem in the mode decomposition approach

1) Local coordinate expansion in the harmonic gauge



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2) Mode decomposition of the metric the harmonic gauge

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```
> har_H0_1:=r0/(r0-2*m)*grcomponent(dir_h1(dn,dn),[t,t])
>             -2*m/(r0-2*m)^2*rr*sig*grcomponent(dir_h0(dn,dn),[t,t]);
> # sig^(-3/2) omitted
```

$$har\_H0\_1 := r0\left(\frac{1}{2}((-2\,EE^5\,r0^7\,d\_r + 104\,m^3\,r0^4\,EE\,d\_r - 120\,m^3\,EE^3\,r0^4\,d\_r + 12\,m\,EE^5\,r0^6\,d\_r$$

$$- 24\,m^2\,EE^5\,r0^5\,d\_r + 64\,EE^3\,m^4\,r0^3\,d\_r + 16\,m^3\,r0^4\,d\_r\,EE^5 + 48\,d\_r\,m^5\,EE\,r0^2$$

$$- 112\,m^4\,r0^3\,EE\,d\_r - 48\,m^2\,r0^5\,EE\,d\_r + 11\,m\,r0^6\,EE\,d\_r - 26\,m\,r0^6\,EE^3\,d\_r + 84\,m^2\,r0^5\,EE^3\,d\_r$$

$$+ 3\,r0^7\,EE^3\,d\_r - r0^7\,EE\,d\_r)\,tt^3)\Big/\left(r0^5\,(-r0+2\,m)^3\right) + \left(\frac{1}{2}((-2\,r0^7 + 9\,r0^7\,EE^2 - r0^5\,LL^2\right.$$

$$+ 208\,r0^4\,m^3 - 96\,r0^5\,m^2 + 22\,r0^6\,m + 258\,r0^5\,m^2\,EE^2 - 48\,LL^2\,r0^3\,m^2 + 11\,LL^2\,r0^4\,m$$

$$- 79\,r0^6\,m\,EE^2 + 104\,LL^2\,r0^2\,m^3 + 5\,r0^5\,EE^2\,LL^2 + 132\,LL^2\,r0^3\,m^2\,EE^2 - 42\,LL^2\,r0^4\,m\,EE^2$$

$$- 13\,EE^4\,r0^7 + 48\,m^5\,LL^2 + 6\,EE^6\,r0^7 - 224\,r0^3\,m^4 + 96\,r0^2\,m^5 - 184\,m^3\,EE^2\,r0^2\,LL^2$$

$$+ 48\,m^3\,EE^4\,r0^2\,LL^2 - 72\,m^2\,EE^4\,r0^3\,LL^2 + 36\,m\,EE^4\,r0^4\,LL^2 + 96\,m^4\,EE^2\,r0\,LL^2$$

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4) Evaluation of the force in the Regge-Wheeler gauge

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```
> F_ele_t_1:=simplify(subs(d=0,F_ele_t));
```

$$
\begin{aligned}
F\_ele\_t\_0 := \frac{1}{2}\Big(& -r0^9\, r^3 \left(\frac{\partial}{\partial t}\mathrm{rw\_H2}(r,t)\right) + 2\left(\frac{\partial}{\partial r}\mathrm{rw\_H1}(r,t)\right) r0^9\, r^3 + I\, EE^3\, M\, \mathrm{rw\_H2}(r,t)\, r0^7\, LL\, r^3 \\
& + r0^9\, EE^3 \left(\frac{\partial}{\partial r}\mathrm{rw\_H2}(r,t)\right) d\_r\, r^3 - 2\, I\, \mathrm{rw\_H1}(r,t)\, M\, d\_r\, LL\, r^3\, r0^7 \\
& + 16\, I\, \mathrm{rw\_H1}(r,t)\, M\, d\_r\, LL\, r^3\, r0^6\, m - 48\, I\, \mathrm{rw\_H1}(r,t)\, M\, d\_r\, LL\, r^3\, r0^5\, m^2 \\
& + 64\, I\, \mathrm{rw\_H1}(r,t)\, M\, d\_r\, LL\, r^3\, r0^4\, m^3 - 32\, I\, \mathrm{rw\_H1}(r,t)\, M\, d\_r\, LL\, r^3\, r0^3\, m^4 \\
& + 4\, I\, \mathrm{rw\_H1}(r,t)\, M\, d\_r\, LL\, r^2\, r0^7\, m - 32\, I\, \mathrm{rw\_H1}(r,t)\, M\, d\_r\, LL\, r^2\, r0^6\, m^2 \\
& + 96\, I\, \mathrm{rw\_H1}(r,t)\, M\, d\_r\, LL\, r^2\, r0^5\, m^3 - 128\, I\, \mathrm{rw\_H1}(r,t)\, M\, d\_r\, LL\, r^2\, r0^4\, m^4 \\
& + 64\, I\, \mathrm{rw\_H1}(r,t)\, M\, d\_r\, LL\, r^2\, r0^3\, m^5 + 2\, I\, EE^2\, \mathrm{rw\_H1}(r,t)\, M\, d\_r\, LL\, r^3\, r0^7 \\
& - 8\, I\, EE^2\, \mathrm{rw\_H1}(r,t)\, M\, d\_r\, LL\, r^3\, r0^6\, m + 8\, I\, EE^2\, \mathrm{rw\_H1}(r,t)\, M\, d\_r\, LL\, r^3\, r0^5\, m^2 \\
& - 8\, I\, EE^2\, \mathrm{rw\_H1}(r,t)\, M\, d\_r\, LL\, r^2\, r0^7\, m + 32\, I\, EE^2\, \mathrm{rw\_H1}(r,t)\, M\, d\_r\, LL\, r^2\, r0^6\, m^2 \\
& - 32\, I\, EE^2\, \mathrm{rw\_H1}(r,t)\, M\, d\_r\, LL\, r^2\, r0^5\, m^3 + 8\, I\, EE^2\, \mathrm{rw\_H1}(r,t)\, M\, d\_r\, LL\, r\, r0^7\, m^2
\end{aligned}
$$

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The direct part of the force in the Regge-Wheeler gauge

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$$
\begin{aligned}
F\_t\_ele1 := {} & \frac{1}{128} EE\, d\_r \sqrt{2} \Big(160\, LL^2\, \mathrm{F}\left(\tfrac{3}{2},\tfrac{3}{2},2\right) r0^{11} - 32\, \mathrm{F}\left(\tfrac{1}{2},\tfrac{1}{2},1\right) r0^{13} + 1152\, LL^4\, \mathrm{F}\left(\tfrac{3}{2},\tfrac{3}{2},2\right) r0^9 \\
& + 256\, \mathrm{F}\left(\tfrac{1}{2},\tfrac{1}{2},1\right) LL^{12}\, r0 - 512\, \mathrm{F}\left(\tfrac{1}{2},\tfrac{1}{2},1\right) LL^{12}\, m + 64\, \mathrm{F}\left(\tfrac{1}{2},\tfrac{1}{2},1\right) r0^{12}\, m + 2016\, LL^8\, \mathrm{F}\left(\tfrac{3}{2},\tfrac{3}{2},2\right) r0^5 \\
& + 2368\, LL^6\, \mathrm{F}\left(\tfrac{3}{2},\tfrac{3}{2},2\right) r0^7 + 128\, LL^{12}\, \mathrm{F}\left(\tfrac{3}{2},\tfrac{3}{2},2\right) r0 - 256\, LL^{12}\, \mathrm{F}\left(\tfrac{3}{2},\tfrac{3}{2},2\right) m - 64\, LL^2\, \mathrm{F}\left(\tfrac{1}{2},\tfrac{3}{2},2\right) r0^{11} \\
& - 240\, LL^8\, \mathrm{F}\left(\tfrac{5}{2},\tfrac{7}{2},4\right) r0^5 - 120\, LL^6\, \mathrm{F}\left(\tfrac{5}{2},\tfrac{7}{2},4\right) r0^7 - 1664\, LL^4\, \mathrm{F}\left(\tfrac{1}{2},\tfrac{3}{2},2\right) r0^9 - 5440\, LL^6\, \mathrm{F}\left(\tfrac{1}{2},\tfrac{3}{2},2\right) r0^7 \\
& - 7168\, LL^8\, \mathrm{F}\left(\tfrac{1}{2},\tfrac{3}{2},2\right) r0^5 - 4352\, LL^{10}\, \mathrm{F}\left(\tfrac{1}{2},\tfrac{3}{2},2\right) r0^3 - 1024\, LL^{12}\, \mathrm{F}\left(\tfrac{1}{2},\tfrac{3}{2},2\right) r0 \\
& + 2048\, LL^{12}\, \mathrm{F}\left(\tfrac{1}{2},\tfrac{3}{2},2\right) m - 288\, LL^8\, \mathrm{F}\left(\tfrac{3}{2},\tfrac{5}{2},3\right) r0^5 - 288\, LL^6\, \mathrm{F}\left(\tfrac{3}{2},\tfrac{5}{2},3\right) r0^7 + 2448\, LL^8\, \mathrm{F}\left(\tfrac{1}{2},\tfrac{5}{2},3\right) r0^5 \\
& + 1824\, LL^6\, \mathrm{F}\left(\tfrac{1}{2},\tfrac{5}{2},3\right) r0^7 + 528\, LL^4\, \mathrm{F}\left(\tfrac{1}{2},\tfrac{5}{2},3\right) r0^9 + 1536\, LL^{10}\, \mathrm{F}\left(\tfrac{1}{2},\tfrac{5}{2},3\right) r0^3 \\
& + 384\, LL^{12}\, \mathrm{F}\left(\tfrac{1}{2},\tfrac{5}{2},3\right) r0 - 768\, LL^{12}\, \mathrm{F}\left(\tfrac{1}{2},\tfrac{5}{2},3\right) m + 240\, LL^8\, \mathrm{F}\left(\tfrac{3}{2},\tfrac{7}{2},4\right) r0^5 + 240\, LL^6\, \mathrm{F}\left(\tfrac{3}{2},\tfrac{7}{2},4\right) r0^7
\end{aligned}
$$

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Gravitational Self-force Problem in the mode decomposition approach

4: Summary and Problems in Kerr

We consider a modified regularization for the self-force.
The gauge condition is defined to be Regge-Wheeler one
in the R-part of the metric perturbation.

$$
\begin{aligned}
&F_\alpha(\tau) \\
&= F_\alpha\left[\mathbf{h}^{R(harm)} + \nabla\boldsymbol{\xi}^{*\to RW}[\mathbf{h}^{R(harm)}]\right](z) \\
&= \lim_{x\to z} F_\alpha\left[\mathbf{h}^{full(RW)} - \mathbf{h}^{S(RW)}\right](x) \\
&= \sum_{lm\omega}\left(F_\alpha^{full(RW)}[R_{lm\omega}] - F_{\alpha,lm\omega}^{S(RW)}\right)
\end{aligned}
$$

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We do not have a *full* metric perturbation formalism in the 'Regge-Wheeler' gauge in a Kerr, but, it may be possible in a renormalized perturbative approach.

$$h_{\alpha\beta}^{full(RW,Kerr)} = h_{\alpha\beta}^{full(RW,Schw)} + a h_{\alpha\beta}^{full(RW,1)} + a^2 h_{\alpha\beta}^{full(RW,2)} + \cdots$$

$$\xi_{\alpha}^{(*\to RW,Kerr)} = \xi_{\alpha}^{(*\to RW,Schw)} + a \xi_{\alpha}^{(*\to RW,1)} + a^2 \xi_{\alpha}^{(*\to RW,2)} + \cdots$$

In the electromagnetic case, we have a systematic expansion with respect to the spin parameter (a), which has an infinite convergence radius. The gravitational case is in progress.