Reconstruction of metric perturbations in Kerr spacetime

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The goal:

Constructing the MP from the Teukolsky variables ψ_0, ψ_4 .

Some useful notation:

$$\begin{aligned}\Delta &\equiv r^2 - 2Mr + a^2 \,, \\ \Sigma &\equiv r^2 + a^2 \cos^2\theta \,, \\ \rho &\equiv -1/(r - ia\cos\theta) \,,\end{aligned}$$

and also:

$$\psi_{+2} \equiv \psi_0 \;\;, \quad \psi_{-2} \equiv \rho^{-4}\psi_4 . \qquad (s=\pm 2)$$

Teukolsky formalism

The Master equation (formal):

$$W_{\pm 2}\left[\psi_{\pm 2}\right] = 4\pi\Sigma T_{\pm 2}\ ,$$

Definition of the operator $W_{\pm 2}$:

$$\begin{aligned}
W_s \equiv & -\Delta^{-s}\partial_r[\Delta^{s+1}\partial_r] + [(r^2+a^2)^2/\Delta - a^2\sin^2\theta]\,\partial_{tt} \\
& + 4aMr\Delta^{-1}\partial_{\varphi t} + [a^2/\Delta - \sin^{-2}\theta]\partial_{\varphi\varphi} \\
& -\sin^{-1}\theta\,\partial_\theta[\sin\theta\,\partial_\theta] - 2s\,[a(r-M)/\Delta + i\,\cos\theta\sin^{-2}\theta]\partial_\varphi \\
& -2s\,[M(r^2-a^2)/\Delta - r - ia\cos\theta]\partial_t + (s^2\cot^2\theta - s)\,,
\end{aligned}$$

Separation into radial and angular parts:

$$\psi_{\pm 2} = \sum_{lm\omega} R^{lm\omega}_{\pm 2}(r) S^{lm\omega}_{\pm 2}(\theta) e^{i(m\varphi - \omega t)}\,,$$

along with

$$4\pi\Sigma T_{\pm 2} = \sum_{lm\omega} T^{lm\omega}_{\pm 2}(r) S^{lm\omega}_{\pm 2}(\theta) e^{i(m\varphi - \omega t)}\,.$$

The radial Teukolsky equation:

$$P^{lm\omega}_{\pm 2}\left[R^{lm\omega}_{\pm 2}(r)\right] = T^{lm\omega}_{\pm 2}(r)\ ,$$

where

$$P^{lm\omega}_{s} \equiv \Delta^{-s}\partial_r[\Delta^{s+1}\partial_r] + V^{lm\omega}_{s}(r)\ .$$

Reconstruction formalism

(Chrzanowski 1975, Wald 1978)

Ingoing radiation gauge: $h_{\alpha\beta} l^\beta = h_\beta{}^\beta = 0$

In vacuum, there exists a potential Ψ, and differential operator Π such that

$$h_{\alpha\beta} = \Pi\,[\Psi]\,.$$

Ψ must satisfy **two** equations:

(i) Vacuum Teukolsky equation for $s = -2$:

$$W_{-2}\,[\Psi] = 0\,, \tag{1}$$

(ii) The differential equation (Wald, Lousto&Whiting)

$$D^4[\bar{\Psi}] = \psi_0\ , \tag{2}$$

where D is the differential operator

$$D = l^\mu \partial_\mu = \frac{r^2 + a^2}{\Delta}\partial_t + \partial_r + (a/\Delta)\partial_\varphi\,,$$

l^μ is the outgoing Kinnersly's tetrad vector.

Our goal: constructing Ψ from ψ_0.

Two

~~Three~~ remarks:

- there are two equations for one unknown! (mutually-consistent in **vacuum** only).
- Ψ satisfies the same field equation as ψ_{-2}, but yet it is **not** ψ_{-2} .

We shall construct Ψ for each mode $lm\omega$ of ψ_0.

Vacuum perturbations

Using the Teukolsky-Starobinsky relations, we construct a formal solution (to both equations):

$$\bar{\Psi} = p^{-1}\Delta^2 (D^\dagger)^4 [\Delta^2 \psi_0] \,, \quad \text{(For each } lm\omega\text{)}$$

where p is the real constant

$$p = \lambda^2(\lambda+2)^2 - 8\omega^2\lambda[\alpha^2(5\lambda+6) - 12a^2] + 144\omega^4\alpha^4 + 144\omega^2 M^2 \,,$$

where $\alpha^2 \equiv a^2 - am/\omega$.

$D^\dagger$ is the operator adjoint to D:

$$D^\dagger = -\frac{r^2+a^2}{\Delta}\partial_t + \partial_r - (a/\Delta)\partial_\varphi \,.$$

This is a solution for each mode $lm\omega$ of Ψ.

Radial part of $\bar{\Psi}$

Decompose $\bar{\Psi}$ as

$$\bar{\Psi} = \hat{R}^{lm\omega}_{-2} S^{lm\omega}_{+2}(\theta) e^{i(m\varphi - \omega t)} .$$

Then the above two equations read

(i) the radial Teukolsky equation

$W_{-2}[\Psi]=0$: $$P^{lm\omega}_{-2}[\hat{R}^{lm\omega}_{-2}(r)] = 0 , \tag{1}$$

(ii) the 4th-order equation

$D^4[\bar{\Psi}]=\psi_0$: $$(D_{m\omega})^4[\hat{R}^{lm\omega}_{-2}] = R^{lm\omega}_{+2}(r) , \tag{2}$$

where $D_{m\omega}$ is the radial part of D:

$$D_{m\omega} = \partial_r + iK/\Delta \ , \ D^{\dagger}_{m\omega} = \partial_r - iK/\Delta$$

where

$$K \equiv am - (r^2 + a^2)\omega .$$

The above solution

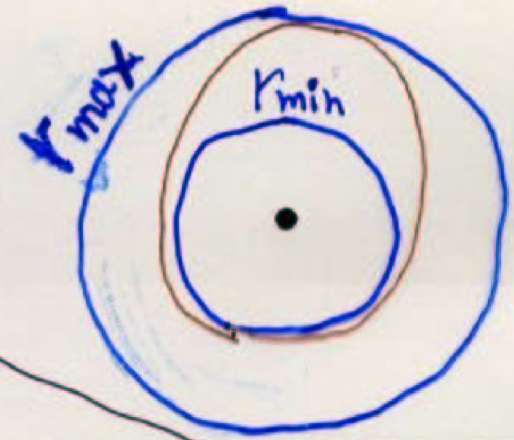
$$\bar{\Psi} = p^{-1}\Delta^2 (D^{\dagger})^4 [\Delta^2 \psi_0] ,$$

reads, for the radial function:

$$\hat{R}^{lm\omega}_{-2} = p^{-1}\Delta^2 (D^{\dagger}_{m\omega})^4 \left[\Delta^2 R^{lm\omega}_{+2}(r)\right] .$$

$$P[\hat{R}_{-2}^{lm\omega}]=0$$

$$(D_{m\omega})^4[\hat{R}_{-2}^{lm\omega}]=R_{+2}^{lm\omega}$$



Perturbations with sources

Assume: Matter confined to $r_{\min} \leq r \leq r_{\max}$.

Construction in two stages:

$W_{-2}[\Psi]=0$
$D^4\bar{\Psi}=\psi_0$

- First construct the above vacuum solution in the range $r > r_{\max}$.
- Then, extend it into $r < r_{\max}$.

At $r < r_{\max}$, the radial function $\hat{R}_{-2}^{lm\omega}$ cannot satisfy both equations.

Therefore, we extend it as a solution of the equation $D^4[\bar{\Psi}] = \psi_0$.

Note: This is an **ordinary** differential equation. Since $D \equiv l^\mu \partial_\mu = \frac{d}{d\gamma}(\xi)$, it reads

$$\frac{d^4\bar{\Psi}}{d\gamma^4}(\xi) = \psi_0 \ ,$$

where γ is the affine parameter.

Since this equation is imposed everywhere,

$$(D_{m\omega})^4[\hat{R}_{-2}^{lm\omega}] = R_{+2}^{lm\omega}(r)$$

is satisfied at any r.

The Solution

[redacted]

$$\bar{\Psi} = \hat{R}_{-2}^{lm\omega} S_{+2}^{lm\omega}(\theta) e^{i(m\varphi - \omega t)} .$$

$$\hat{R}_{-2}^{lm\omega}(r) = \int_{r_{\min}}^{r_{\max}} T_{+2}^{lm\omega}(r') H(r,r') dr' ,$$

$$H(r,r') = H^{(out)}(r,r')\theta(r-r') + H^{(down)}(r,r')\theta(r'-r) ,$$

$$H^{(out)}(r,r') = a^{(out)}(r') R_{-2}^{(out)}(r) ,$$

and

$$\begin{aligned} H^{(down)}(r,r') &= a^{(down)}(r') R_{-2}^{(down)}(r) \\ &\quad + e^{-i(mu - \omega r_*)} \sum_{i=0}^{3} B_i(r') r^i . \end{aligned}$$

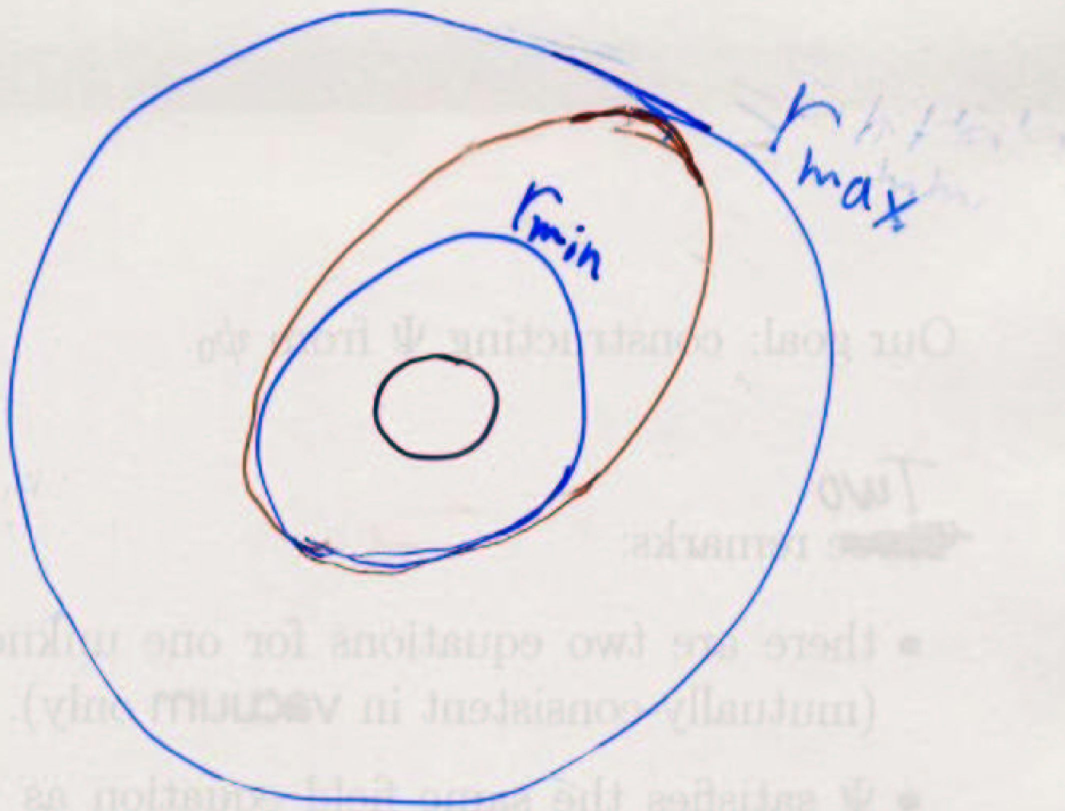
[we have set $a^{(a)} \equiv A^{(a)} C^{(a)}$].

Six functions of r' are involved:

$$a^{(out)}(r'), \;\; a^{(down)}(r'), \;\; \text{and} \;\; B_i(r') \;\; (i = 0...3) .$$

They are constructed from the homogeneous $s = -2$ function: $R_{-2}^{(out)}(r')$ and $R_{-2}^{(down)}(r')$, and their first-order derivative.

Domain of validity



Ψ must satisfy <u>two</u> eqs.:

$$W_{-2}[\Psi] = 0$$

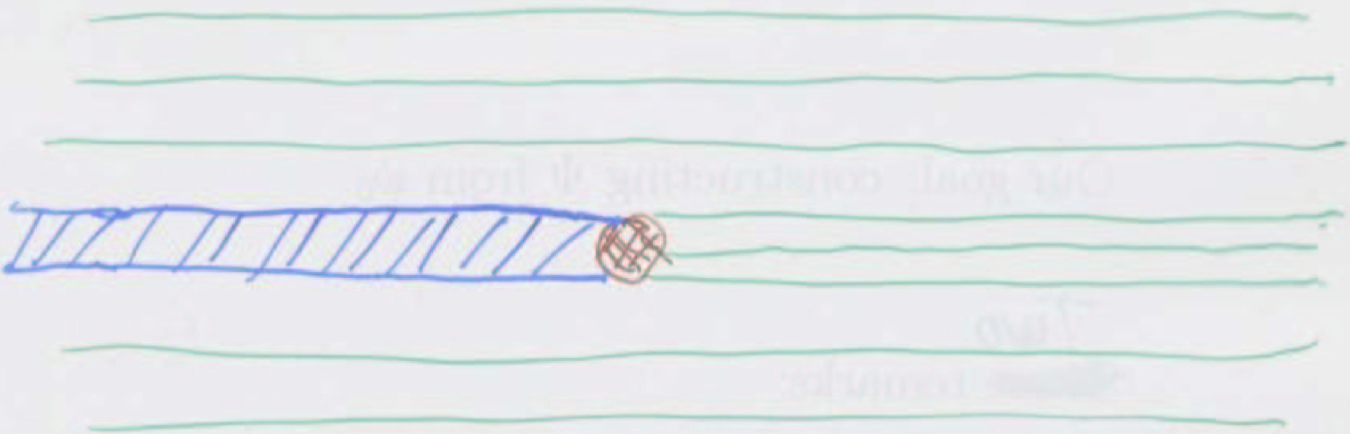
$$D^4[\bar{\Psi}] = \psi_0$$

second equation satisfied by construction.

this is an <u>ODE</u> along the null geodesics:

$$\frac{d^4 \bar{\Psi}}{d r^4}(\xi) = \psi_0$$

<u>Conclusion</u>: ψ is analytic everywhere-except in the "shadowed" region.



<u>Conclusion</u>:

ψ is valid (i.e., it satisfies <u>both</u> required eqs.) everywhere-except in the "shadowed" region.

In the point-like limit, the shadowed region becomes a <u>line singularity</u> (forming a 1+1 surface).

The same holds for $h_{\alpha\beta}$

Summary

- We have constructed the potential Ψ (and hence h) in the frequency domain
- valid everywhere - except in the "shadow" of the source
- In the point limit, Ψ becomes singular (a line singularity)
- The same holds for $h_{\alpha\beta}$ (in the radiation gauge)
- This singularity of radiation-gauge $h_{\alpha\beta}$ is inevitable.
- But it is merely a gauge singularity
- $h_{\alpha\beta}$ may be used for radiation-reaction.