# Retarded normal coordinates

Work in progress (1/4 completed);
in collaboration with Claude Barrabès

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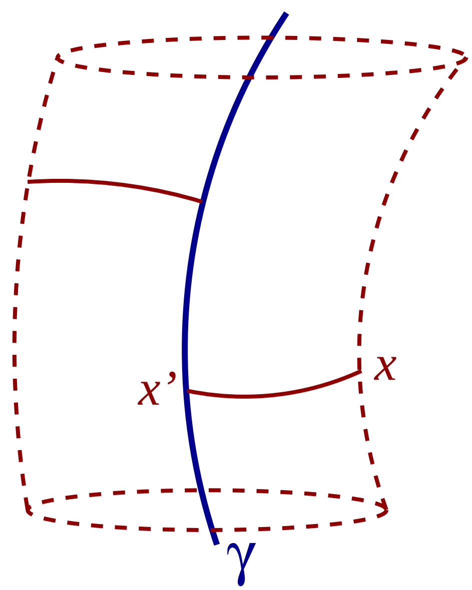
# Motivation

In his classic 1938 paper, Dirac calculated the self-force acting on an electrically charged particle by invoking energy-momentum conservation across a world tube that surrounds the particle's world line.

The world tube is constructed by emitting **spacelike** geodesics in the directions orthogonal to the world line; the tube is at a fixed spacelike distance away from the world line.



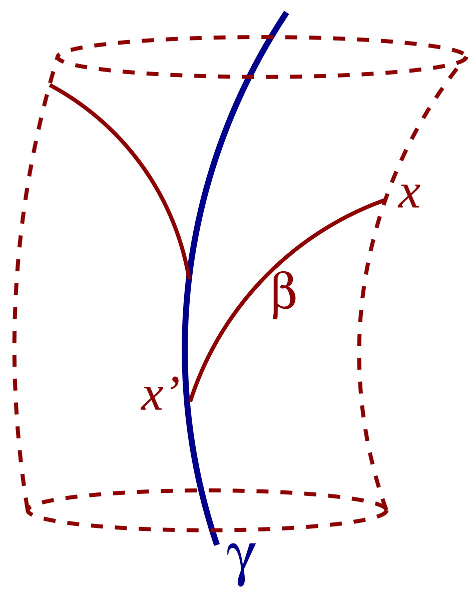
By relating field quantities at $x$ to the state of the particle at $x'$ (with $x$ and $x'$ linked by a spacelike geodesic), Dirac brought unnecessary complications to the computations.

Dirac did it the hard way.

As a consequence, DeWitt and Brehme did it the hard way.

As a consequence, Mino, Sasaki, and Tanaka did it the hard way.

Calculations in flat spacetime are much simplified if the world tube is constructed with **null** geodesics instead [Teitelboim, Villarroel, van Weert (1980); E.P. gr-qc/9912045].




The field quantities at $x$ are much more naturally related to the state of the particle at $x'$ if $x$ and $x'$ are linked by a null geodesic.

Calculations in curved spacetime will also benefit from the use of world tubes constructed from null geodesics.

To implement this idea, it is useful to construct a coordinate system based on null geodesics emanating from the world line.

These coordinates — **retarded normal coordinates** — will be defined in a (normal) neighbourhood of the world line.

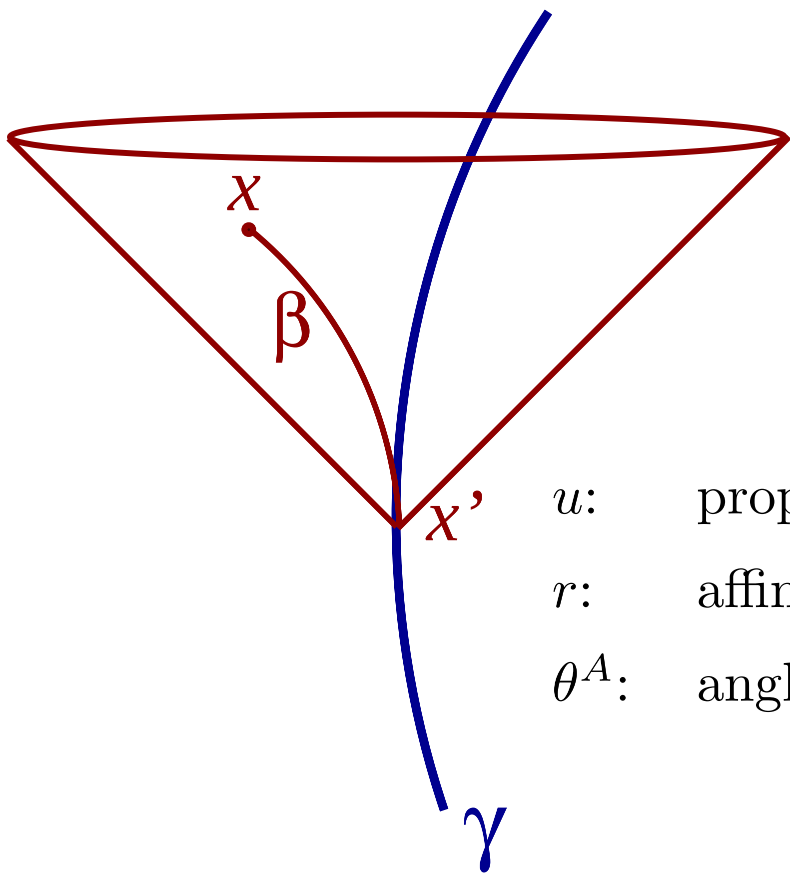
This idea, a variation on the theme of Fermi normal coordinates, is spelled out in Synge's 1964 book.

He didn't, however, push it to completion.

His goal was also slightly different: he was interested in a large neighbourhood of the world line in a weakly curved spacetime, while I'm interested in a small neighbourhood in an arbitrary spacetime.

# 2. Geometric construction

The retarded normal coordinates of $x$ are $(u, r, \theta^A)$.




$u$: proper time at $x'$

$r$: affine-parameter distance along $\beta$

$\theta^A$: angles that specify which null geodesic

# 3. Warning from flat spacetime

The transformation from Lorentzian coordinates $(t, x, y, z)$ to retarded coordinates $(u, r, \theta, \phi)$ in flat spacetime is

$$\begin{aligned} t &= u + r \\ x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

These are based on the geodesic $x = y = z = 0$.

The transformation brings the metric to the form

$$ds^2 = -du^2 - 2\,du dr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

The metric is **singular** on $\gamma$.

This means that tensor components are not defined on $\gamma$, and this property survives in curved spacetimes.

This difficulty is easily dealt with by introducing an **orthonormal tetrad**

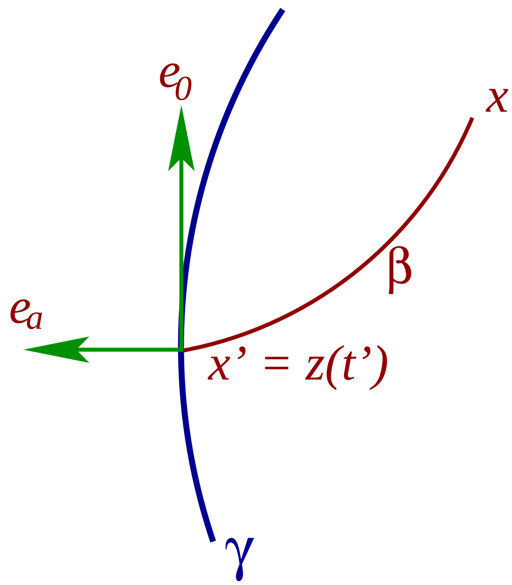
$$e_0 = \frac{\partial}{\partial t}, \quad e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

and working with **frame components** of tensors.

These will be well defined on and off $\gamma$.

Tetrads play a central role in the construction of the retarded null coordinates — they are the fundamental objects from which the metric is constructed.

# 4. Definition of RNC




Let $\gamma$ be an arbitrary world line $z^{\alpha'}(t')$ with tangent vector $u^{\alpha'} = dz^{\alpha'}/dt'$; $t'$ is proper time.

Let $(u^{\alpha'}, e_a^{\alpha'})$ be an orthonormal tetrad that is Fermi-Walker transported on $\gamma$.

Let $x$ be a point in the normal neighbourhood of $\gamma$.

Let $\beta$ be the unique null geodesic that connects $x$ to $\gamma$.

Let $x'$ be the point at which $\beta$ intersects $\gamma$.

Then the **quasi-cartesian** version of the retarded normal coordinates of the point $x$ are defined by

$$\boxed{u \equiv t' \equiv \text{proper time at } x'}$$

$$\boxed{\hat{x}^a \equiv -e_a^{\alpha'} \sigma_{\alpha'}(x', x)}$$

where $\sigma(x', x)$ is Synge's world function.

The statement that $x$ and $x'$ are linked by a null geodesic is

$$\boxed{\sigma(x', x) = 0}$$

To go from the quasi-cartesian coordinates $\hat{x}^a$ to the **quasi-spherical** coordinates $(r, \theta^A)$ we first define a radius

$$\boxed{r \equiv \sqrt{\delta_{ab}\hat{x}^a\hat{x}^b} = u^{\alpha'}\sigma_{\alpha'}}$$

This can be shown to be an **affine parameter** on all null geodesics $\beta$ that emanate from $x'$.

These geodesics are described by the relations $\hat{x}^a = r\,\Omega^a$, in which $\Omega^a$ is a constant unit vector: $\delta_{ab}\Omega^a\Omega^b = 1$.

The transformation to quasi-spherical coordinates is then

$$\boxed{\hat{x}^a(r, \theta^A) = r\Omega^a(\theta^A)}$$

where $\theta^A$ are two angles that parameterize the vector $\Omega^a$.

For example, $\Omega^a = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

# 5. Metric in RNC

The metric at $x$ is computed by first constructing $(e^\alpha_0, e^\alpha_a)$, an orthonormal tetrad obtained by parallel transport of $(u^{\alpha'}, e^{\alpha'}_a)$ on the null geodesic $\beta$.

The metric is expressed in terms of **frame components of the Riemann tensor** evaluated on $\gamma$.

For example, it involves

$$R_{a0b0}(u) \equiv R_{\alpha'\gamma'\beta'\delta'}(x')e^{\alpha'}_a u^{\gamma'} e^{\beta'}_b u^{\delta'}$$

The dependence of the metric on $u$ comes from these frame components.

The metric is expressed as an expansion in powers of $r$.

The dependence on the angles comes from the unit vector $\Omega^a(\theta^A)$.

We have

$$ds^2 = g_{uu}\, du^2 - 2\, dudr + 2g_{uA}\, dud\theta^A + g_{AB}\, d\theta^A d\theta^B$$

with

$$\begin{aligned}
g_{uu} &= -1 - r^2 R_{c0d0}\Omega^c\Omega^d + O(r^3) \\
g_{uA} &= \frac{2}{3} r^3 \big( R_{a0c0}\Omega^c + R_{acd0}\Omega^c\Omega^d \big)\Omega^a_A + O(r^4) \\
g_{AB} &= r^2\Omega_{AB} - \frac{1}{3} r^4 \big( R_{a0b0} + R_{a0bc}\Omega^c + R_{b0ac}\Omega^c \\
&\qquad + R_{acbd}\Omega^c\Omega^d \big)\Omega^a_A\Omega^b_B + O(r^5)
\end{aligned}$$

and

$$\Omega^a_A \equiv \frac{\partial\Omega^a}{\partial\theta^A}, \qquad \Omega_{AB} \equiv \delta_{ab}\Omega^a_A\Omega^b_B = \mathrm{diag}(1, \sin^2\theta)$$

These results, and those below, assume that the world line $\gamma$ is a **geodesic**, but there is no difficulty in generalizing to arbitrary world lines.

Because the metric is obtained from a tetrad, we have immediate access to the **parallel propagator** on $\beta$:

$$g^{\alpha}{}_{\alpha'}(x, x') = -e^{\alpha}_{0}(x) u_{\alpha'}(x') + e^{\alpha}_{a}(x) e^{a}_{\alpha'}(x')$$

The retarded normal coordinates permit an easy construction of world tubes of constant $r$.

These have a surface element given by

$$d\Sigma_{\alpha} = r_{,\alpha} \left[ 1 - \frac{1}{6} r^2 \big( R_{00} + 2R_{0a}\Omega^a + R_{ab}\Omega^a\Omega^b \big) + O(r^3) \right] r^2 du d\Omega$$

It involves the frame components of the Ricci tensor.

# 6. Electromagnetic field tensor

Straightforward computations based on the DeWitt-Brehme electromagnetic Green's functions yield the frame components of the retarded electromagnetic field tensor of a point electric charge:

$$
\begin{aligned}
F_{a0} &= \frac{e}{r^2}\Omega_a + \frac{e}{3}R_{c0d0}\Omega^c\Omega^d\Omega_a - \frac{e}{6}\big(5R_{a0c0}\Omega^c + R_{ac0d}\Omega^c\Omega^d\big) \\
&\quad + \frac{e}{6}\big(2R_{a0} - R_{ac}\Omega^c\big) + \frac{e}{12}\big(5R_{00} + R + R_{cd}\Omega^c\Omega^d\big)\Omega_a \\
&\quad + F_{a0}(\mathrm{tail}) + O(r) \\
F_{ab} &= \frac{e}{2}\big(R_{a0bc} - R_{b0ac}\big)\Omega^c + \frac{e}{2}\big(R_{a0c0}\Omega_b - R_{b0c0}\Omega_a\big)\Omega^c \\
&\quad - \frac{e}{2}\big(R_{a0}\Omega_b - R_{b0}\Omega_a\big) + F_{ab}(\mathrm{tail}) + O(r)
\end{aligned}
$$

These can be substituted into the electromagnetic stress-energy tensor for integration across a world tube of constant $r$.

# 7. What's left to do

- Generalize results to arbitrary world lines (easy).
- Complete the derivation of the DeWitt-Brehme equations of motion (straightforward but tedious).
- Implement the Quinn-Wald comparison axiom (first attempt failed, perhaps because of computational error; neighbourhood identification might be tricky).
- Consider scalar and gravitational self-forces (straightforward but tedious).
- See if the RNC simplify the computation of mode-sum regularization parameters (????).

In the end, no new result will be derived with this framework, but I believe that it is the **natural** framework for self-force calculations.