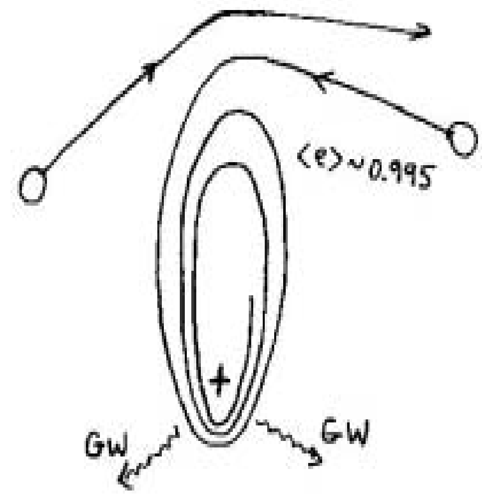
# First cut at data-analysis problem for LISA inspiral sources

Leor Barack & Curt Cutler

# Capture of a CO by a SMBH: Mechanism, & features to have in mind

- SMBHs abundant (table); Some evidence that rapidly rotating.
- Capture process begins when CO kicked into "loss cone" through multibody scattering
- Orbit still highly eccentric (e=0.4-0.6) when enters LISA band.
  (Hils & Bender 1995; Sigurdson & Rees 1997)
- $10^5$-$10^6$ orbits during last year, inside LISA band.
  (Finn & Thorne 2000)
- Gradual circularization, but still substantial eccentricity at LSO (Cutler, Kennefick, & Poisson 1994)



# CB inspiral: a foundation for LISA's science requirement

- LIST (Dec2001): Floor of LISA's noise curve to be determined by requirement of detecting CB inspirals.
- Based on research done till winter 2003-2004 (toward finalizing LISA's design) LIST will decide between noise *requirement* and noise *goal* ($\times$4 lower).
- Need to know: Detection rate *vs*. LISA's noise floor.
- LIST identified this task as "especially urgent".

# 3 elements in determining detection rate:

"astrophysics"

**Event rate**

Hils & Bender (1995);
Sigurdson & Rees (1997);
Freitag (2003):

- MSeq: ~ 5/Myr/gxy
- WD: ~ 0.5/Myr/gxy
- NS,BH: ~0.05/Myr/gxy

"source modeling"/GR

**S/N for a given source**

Finn & Thorne (2000):

$$S/N \sim 10\left(m/M_{sun}\right)\left(1Gpc/r\right)$$

(For a 1-yr mission;
based on study of circular
equatorial orbits)

- 1-10 events/yr out to 1 Gpc with S/N t 20
- 0.5-5 MainSeq/yr at Sgr A* (!), with S/N t 10

data analysis

**Ease of data analysis
(How much S/N need for
confident detection?)**

?

**Detection rate**

# “Progressive” plan for study of LISA inspiral data analysis problem

AK → NK → TB

## Analytic Kludge

Barack & Cutler

- Newtonian orbits, improved by including radiation reaction, periastron precession, Lense-Thirring effect –- all based on PN expressions
- Quadrupole emission

## Numerical Kludge

Gair, Hughes, Kennefick & T. Creighton // Babak & Satyaprakash

- Precise geodesics, with RR a la Hughes
- Quadrupole wave emission (improved using Press’ formula)

## Teukolsky-Based

Hughes, Kennefick, Glampedakis

- Waveforms and RR based on Teukolsky formalism

▪ Groups communicate through regular telecons

(reports at http://manuel.tapir.caltech.edu/listwg1/)

# Searching by “matched filtering”

- “Optimal” technique (minimizes false-alarm ¥ true signal dismissals) when precise waveforms are known – as with inspirals
- Need to build up a discrete set of templates, to cover parameter space (PS) in an “efficient” manner
- Basic challenge: Large PS, long integration time fi need a huge number of templates:

$$N_{temp} \sim (\#\,\text{wave cycles})^{\#par} \sim (\Delta t \cdot \nu)^{\#par} \sim (3\cdot 10^{7} \cdot 3\cdot 10^{-3})^{10} \sim 10^{50}$$

- fi Can’t search coherently over an entire year of data!
- Solution: Apply “Stacked” search, as in NS search for LIGO (Brady & Creighton)
- Basic information for designing stacked search: $\boxed{N_{temp}(\Delta t)}$

# Our “Analytic Kludge” model waveforms

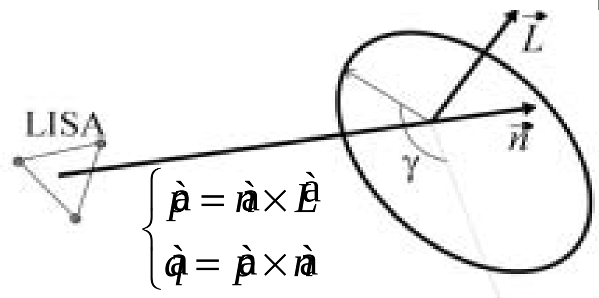
MOTIVATION: Get quick, order-of-magnitude answers:

- $N_{\mathrm{temp}}(\Delta t)$
- Required S/N for a yr-long mission
- LISA resolution for physical parameters of inspiral sources
- Evaluate problem of self-confusion
- Design hierarchical/stacked search strategy and gauge its efficiency

✓ Model simple enough to allow incorporation of full (14d) par. space

# AK waveforms

- Quadrupole radiation from Keplerian orbits
  (Peters & Mathews 1963)




$$\begin{cases} \hat{p} = \hat{n} \times \hat{L} \\ \hat{q} = \hat{p} \times \hat{n} \end{cases}$$

$$h_{\alpha\beta}(t) = \frac{1}{D} \sum_{n=1}^{\infty} \left( A_n^{+}(t) H_{\alpha\beta}^{+} + A_n^{\times}(t) H_{\alpha\beta}^{\times} \right)$$

$$\begin{cases} A_n^{\times}(t) = (\hat{L} \cdot \hat{n}) \left\{ 2\ddot{I}_n^{xy} \cos(2\gamma) + (\ddot{I}_n^{xx} - \ddot{I}_n^{yy}) \sin(2\gamma) \right\} \\ A_n^{+}(t) = -[1 + (\hat{L} \cdot \hat{n})^2] \left\{ (\ddot{I}_n^{xx} - \ddot{I}_n^{yy}) \cos(2\gamma)/2 + \ddot{I}_n^{xy} \sin(2\gamma) \right\} + [1 - (\hat{L} \cdot \hat{n})^2] (\ddot{I}_n^{xx} + \ddot{I}_n^{yy})/2 \end{cases}$$

where, E.g.,

$$\ddot{I}_n^{xy} = -\mu (2\pi \nu M)^{2/3} n (1 - e^2)^{1/2} \sin[n(2\pi \nu t + \varphi_0)] \times \left[ J_{n-2}(ne) - 2J_n(ne) + J_{n+2}(ne) \right]$$

- Mode distribution of power emitted:

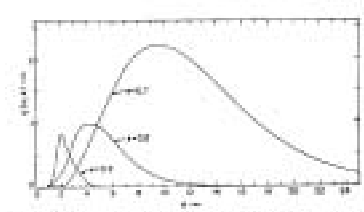


FIG. 3. g(n,e), the relative power radiated into the nth harmonic for e=0.2, 0.5, and 0.7.

# Features added “by hand”

- ✓ Inspiral: $\frac{d\nu}{dt} = \frac{96}{10\pi}\left(\mu/M^3\right)\left(2\pi\nu M\right)^{11/3}\left[\frac{1+(73/24)e^2+(37/96)e^4}{(1-e^2)^{7/2}}\right] + ... + \propto \vec{L}\cdot\vec{S} + ...$ 3.5 PN

  [Junker & Schäfer; Ryan]

- ✓ Circularization: $\frac{de}{dt} = -\frac{1}{15}\left(\mu/M\right)\left(2\pi\nu M\right)^{8/3} e\left(304+121e^2\right)\left(1-e^2\right)^{-5/2} + ... + \propto \vec{L}\cdot\vec{S} + ...$ 3.5 PN

  [Junker & Schäfer; Brumberg]

- ✓ Periastron precession: $\frac{d\gamma}{dt} = 6\pi\nu\left(2\pi\nu M\right)^{2/3}\left(1-e^2\right)^{-1} + ... + \propto \vec{L}\cdot\vec{S} + \propto \vec{L}\times\vec{S} + ...$ 2 PN

  [Junker & Schäfer; Brumberg; Ryan]

- ✓ Spin-orbit precession: $\frac{d\grave{E}}{dt} = \frac{4\pi^2\nu^2}{\mu}\left(1-e^2\right)^{-3/2}\vec{S}\times\grave{E}(t)$ 2 PN

  [Barker & O’Connell]

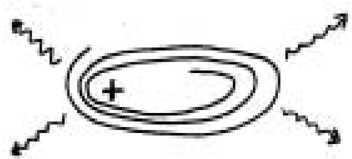
- ✓ Doppler modulation due to LISA motion: $\Phi(t) \rightarrow \Phi(t) + \delta\Phi_D(t)$

  [Cutler 1998]

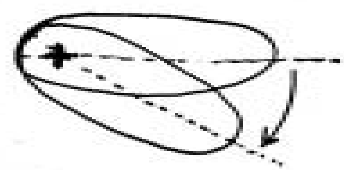
# Features of kludged model - summary

- Our model features (qualitatively) most characteristics of Kerr orbits, like

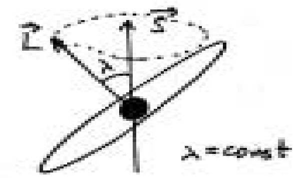
inspiral



Periastron precession

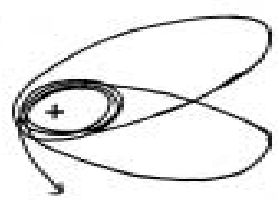


Spin-Orbit coupling

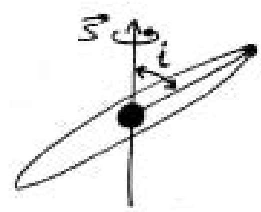


ï Features missing:

"Zoom-Whirl" effect



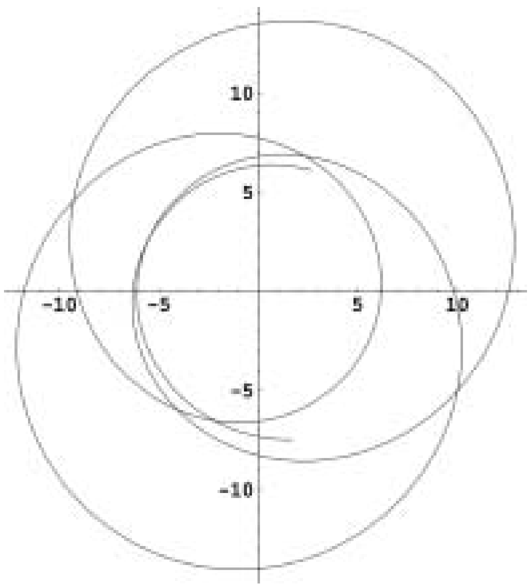
Evolution of inclination angle



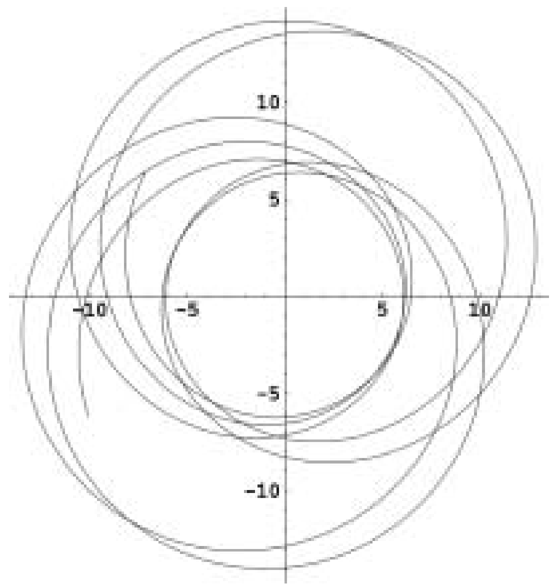
(but these may be less important for our problem)

# Sample orbits

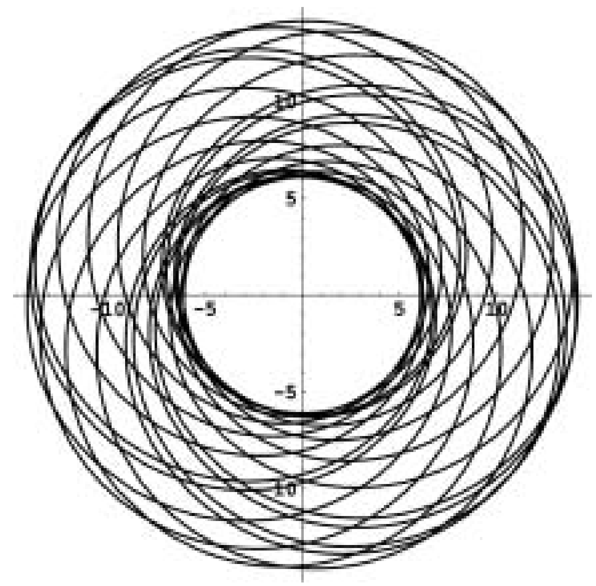
$\mu$=1$M_\odot$,  $M$= $10^6 M_\odot$,  $f$=$10^{-3}$ Hz,  $e$=0.4,  $\boldsymbol{S}$=0




1 hour



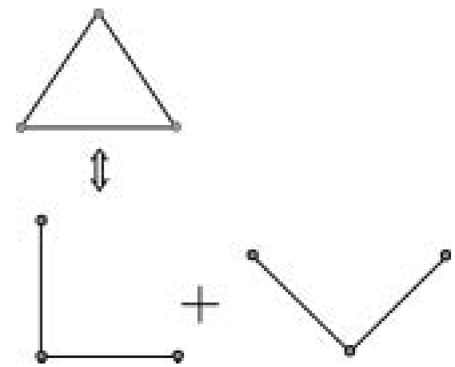

2 hour




6 hour

# LISA’s response (Cutler 1998)

- LISA ¤ two independent two-arm/90° interferometers with responses $h_I$, $h_{II}$,



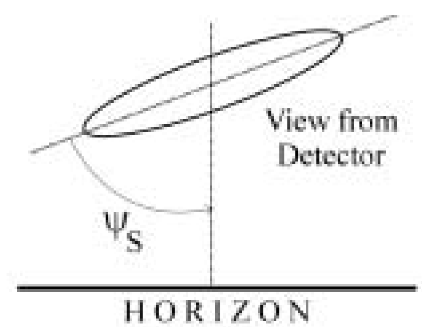
$$h_A(t) = \frac{1}{D}\frac{\sqrt{3}}{2}\sum_{n=1}^{\infty}\left[F_A^{+}(t)A_n^{+}(t) + F_A^{\times}(t)A_n^{\times}(t)\right]$$

$$\begin{cases} F_I^{+} = \frac{1}{2}(1+\cos^2\theta_S)\cos(2\phi_S)\cos(2\psi_S) - \cos\theta_S\sin(2\phi_S)\sin(2\psi_S) \\ F_I^{\times} = \frac{1}{2}(1+\cos^2\theta_S)\cos(2\phi_S)\sin(2\psi_S) + \cos\theta_S\sin(2\phi_S)\cos(2\psi_S) \end{cases}$$

$$F_{II} = F_I(\phi_S \to \phi_S - \pi/4)$$

$(\theta_S,\phi_S)$ – direction to source

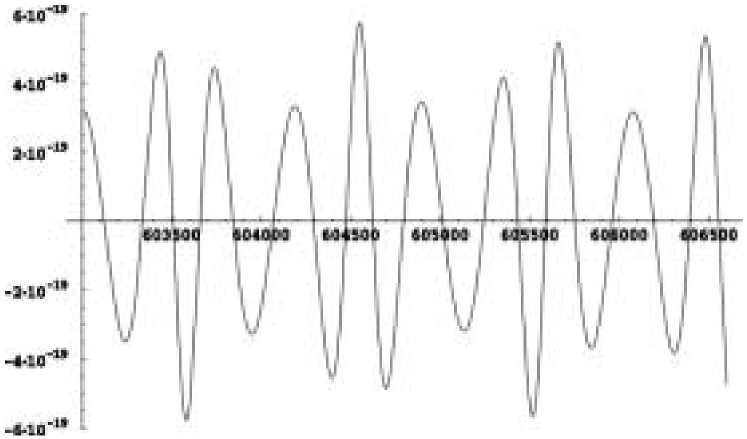
$\psi_S$ – “polarizarion angle”



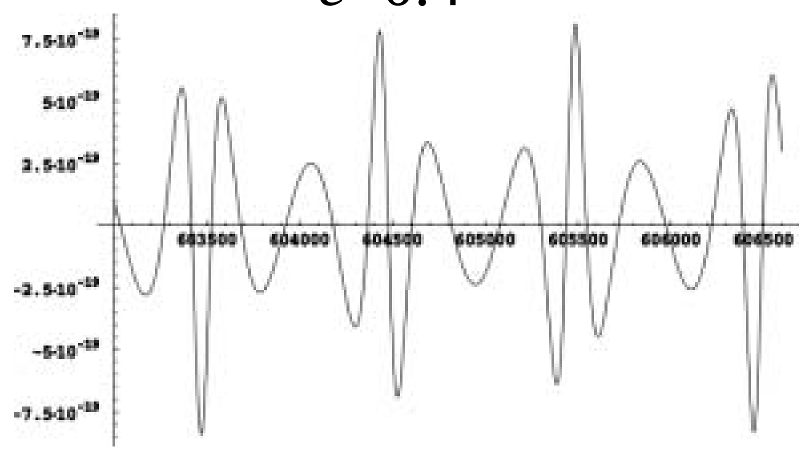
# Sample kludge waveforms (***S***=*0*)

$\mu$=1M$_\odot$, $M$= 10$^6$M$_\odot$, $f$=10$^{-3}$ Hz, $\Delta t$=1 hour

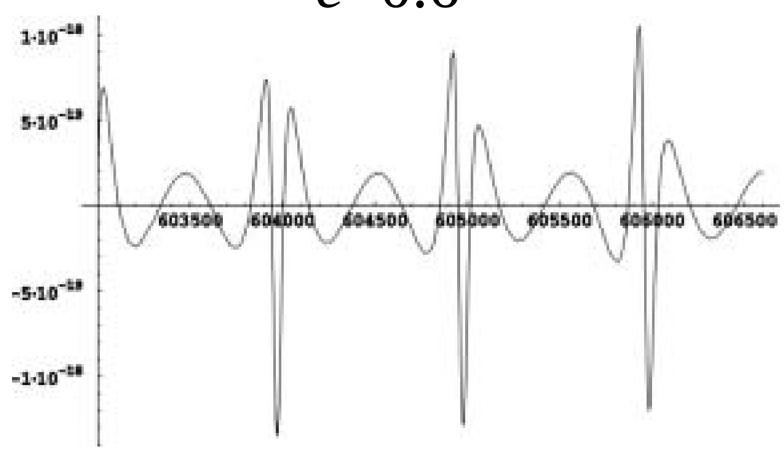
$e$=0.2




$e$=0.4

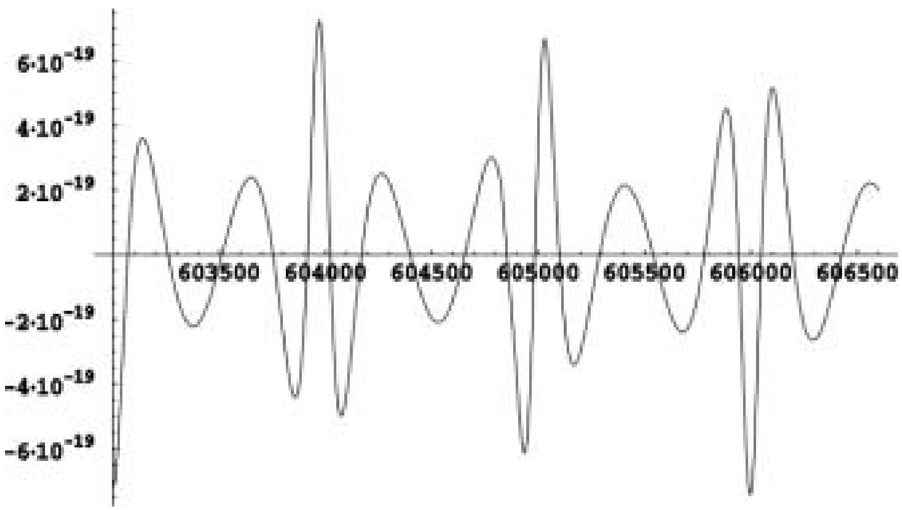



$e$=0.6

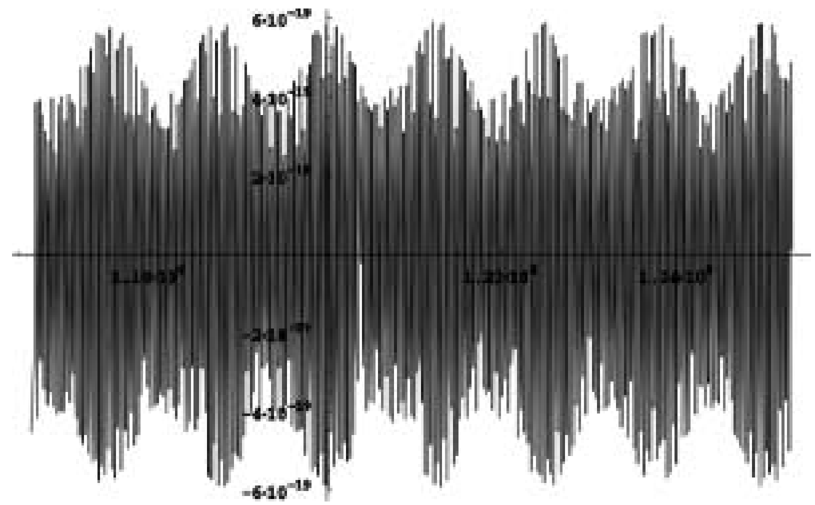


# Sample kludge waveforms ($S = M^2$)

$\mu = 1M_{\odot}$, $M = 10^6 M_{\odot}$, $f = 10^{-3}$ Hz, $e = 0.4$



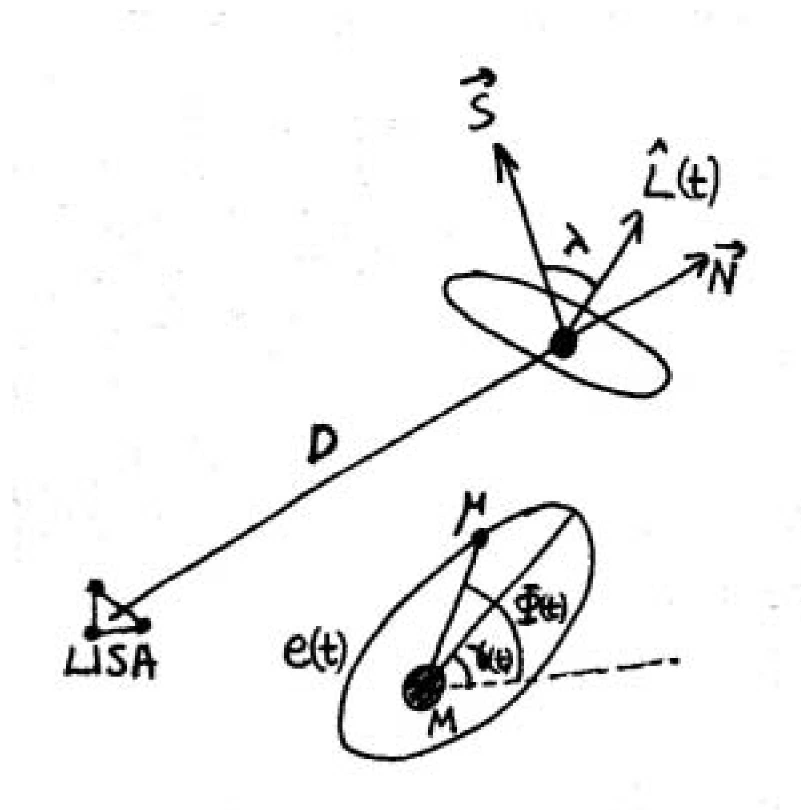

1 hour




1 day

# Parameter Space for inspiral problem

| | |
|---|---|
| $M, \mu$ | 2 |
| $\vec{N} = (D, \theta_S, \varphi_S)$ | 3 |
| $\vec{S} = (S, \theta_K, \varphi_K)$ | 3 |
| $t_0 \equiv t(\nu_0),$ | 1 |
| $e(t_0), \Phi(t_0), \gamma(t_0),$ | 3 |
| $\grave{L}_0 = [\lambda, \alpha(t_0)]$ | 2 |
| | 14 |



# Counting templates: geometric approach

(Cutler & Flanagan 1994, Owen 1996)

- Introduce inner product in Para. Space: $\langle a|b\rangle \equiv 2\int_0^\infty df \, \frac{\tilde{a}^*(f)\,\tilde{b}(f) + c.c.}{S(f)}$

ï $S(f)$ is spectral density of detector's noise $n(t)$
(assumed stationary & Gaussian): $E[\tilde{n}(f)\tilde{n}^*(f')] = \frac{1}{2}\delta(f - f')S(f)$

- Discretize PS with (normalized) templates $|u(t)\rangle$: $\langle u|u\rangle = 1$

  Can show $\langle u|n\rangle$ is a random variable with $\overline{\langle u\,|\,n\rangle} = 0, \quad rms\langle u\,|\,n\rangle = 1$

- For signal s(t) filtered by u(t) define $S/N \equiv \frac{\langle s\,|\,u\rangle}{rms\langle n\,|\,u\rangle} = \langle s\,|\,u\rangle$

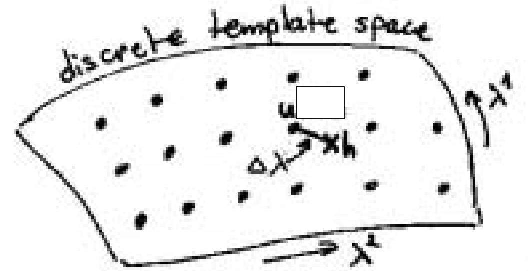
- Write detector's output as $s(t)=n(t)+\mathrm{A}h(t)$, where A is ($t$-indep.) amplitude and $h(t)$ shape of GW, with $\langle h|h\rangle = 1$. $\Rightarrow$ $S/N = \langle n + \mathrm{A}\,h\,|\,u\rangle = \mathrm{A}\underline{\langle h\,|\,u\rangle}$

$0 \le \langle h\,|\,u\rangle \le 1$ measures effectiveness of $|u(t)\rangle$ in searching for $h(t)$

# Geometric approach (cont.)

- Discretization *mismatch* [=Relative loss of S/N due to use of a discrete template family]:

$$\frac{\Delta(S/N)}{(S/N)_{opt}} = \frac{\mathrm{A}\langle u\,|\,u\rangle - \mathrm{A}\langle h\,|\,u\rangle}{\mathrm{A}\langle u\,|\,u\rangle} = 1 - \langle h|u\rangle \equiv \underline{\mathrm{M}\,(\Delta\lambda^i)}$$



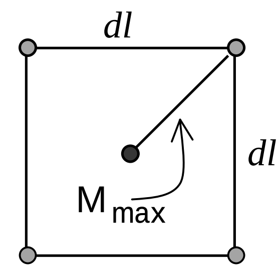

- Local approx. for mismatch: $\mathrm{M} \approx \Gamma_{ij}\Delta\lambda^i\Delta\lambda^j + O(\Delta\lambda^3)$

where the "metric" is $$\Gamma_{ij} = \frac{1}{2}\frac{\partial^2 \mathrm{M}}{\partial\lambda^i\partial\lambda^j}\bigg|_{\Delta\vec{\lambda}=0} = -\frac{1}{2}\left\langle \frac{\partial^2 \mathrm{h}}{\partial\lambda^i\partial\lambda^j}\bigg| h \right\rangle\bigg|_{\Delta\vec{\lambda}=0} = \frac{1}{2}\left\langle \frac{\partial h}{\partial\lambda^i}\bigg|\frac{\partial h}{\partial\lambda^j}\right\rangle\bigg|_{\Delta\vec{\lambda}=0}$$

ï $\sqrt{\det(\Gamma_{ij})}$ is proper "template density"

- Template spacing determined by prescribing $\mathrm{M}_{\max}$:

$$\mathrm{M}_{\max} = N(dl/2)^2 \quad \Rightarrow \quad dl = 2\sqrt{\mathrm{M}_{\max}/N}$$

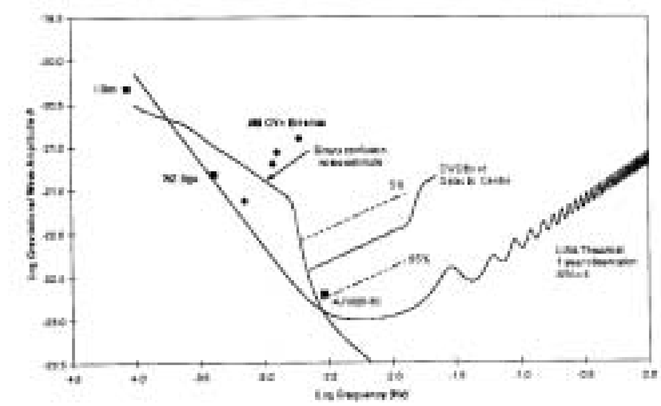



- Then, total # of templates is $$\boxed{\mathrm{N}_{temp} = \frac{\int \sqrt{\Gamma}\, d^N\lambda}{(2\sqrt{\mathrm{M}_{\max}/N})^N}}$$

# Implementation with AK waveforms

- Short integration time $\rightarrow$ work in time domain
- Employ LISA noise model (Hughes et al 2001):

$$S(f) = S_{inst} + S_{exgal}^{conf} + S_{gal}^{conf}$$
$$S_{inst} = 4.85\times10^{-57} f^{-4} + 8.38\times10^{-47} + 4.85\times10^{-43} f^{2}$$
$$S_{exgal}^{conf} = 1.1\times10^{-46} f^{-7/3}$$
$$S_{gal}^{conf} = \ldots$$



- No need to search over $D \rightarrow$ ignore this parameter
- $t_0$ easy to search $\rightarrow$ project $t_0$ (=$\lambda^0$) out (a la Owen 96) :

$$\gamma_{ij} = \Gamma_{ij} - \frac{\Gamma_{i0}\Gamma_{j0}}{\Gamma_{ij}}$$

- Calculate 12D matrix $\gamma_{ij}$

# Sample data set

(for one point in parameter space)

**$\gamma$ matrix for one integration day ($\vec{S} = 0$)**

$$\gamma_{ij} = \begin{array}{c} \begin{array}{ccccccccc} \ln\mu & \ln M & e_0 & \gamma_0 & \Phi_0 & \cos\bar\theta_s & \bar\phi_s & \cos\bar\theta_L & \bar\phi_L \end{array} \\ \begin{pmatrix} 1.87 & -378 & -534 & -0.245 & 0.0861 & -5.44 & 0.541 & -1.47 & -0.0411 \\ -378 & 934000 & 1.33\cdot10^{6} & 521 & -183 & 14.4 & 623 & 2360 & -294. \\ -534 & 1.33\cdot10^{6} & 1.89\cdot10^{6} & 742 & -261 & 13.1 & 889 & 3360 & -420. \\ -0.245 & 521 & 742 & 0.292 & -0.103 & 0.0074 & 0.349 & 1.32 & -0.165 \\ 0.0861 & -183 & -261 & -0.103 & 0.0362 & -0.00261 & -0.123 & -0.465 & 0.0582 \\ -5.44 & 14.4 & 13.1 & 0.0074 & -0.00261 & 30.3 & -4.46 & 2.06 & 1.01 \\ 0.541 & 623 & 889 & 0.349 & -0.123 & -4.46 & 1.08 & 1.28 & -0.347 \\ -1.47 & 2360 & 3360 & 1.32 & -0.465 & 2.06 & 1.28 & 6.12 & -0.68 \\ -0.0411 & -294 & -420 & -0.165 & 0.0582 & 1.01 & -0.347 & -0.68 & 0.128 \end{pmatrix} \end{array}$$

$\sqrt{\det(\gamma_{ij})} = 3.77 \cdot 10^{-11}$

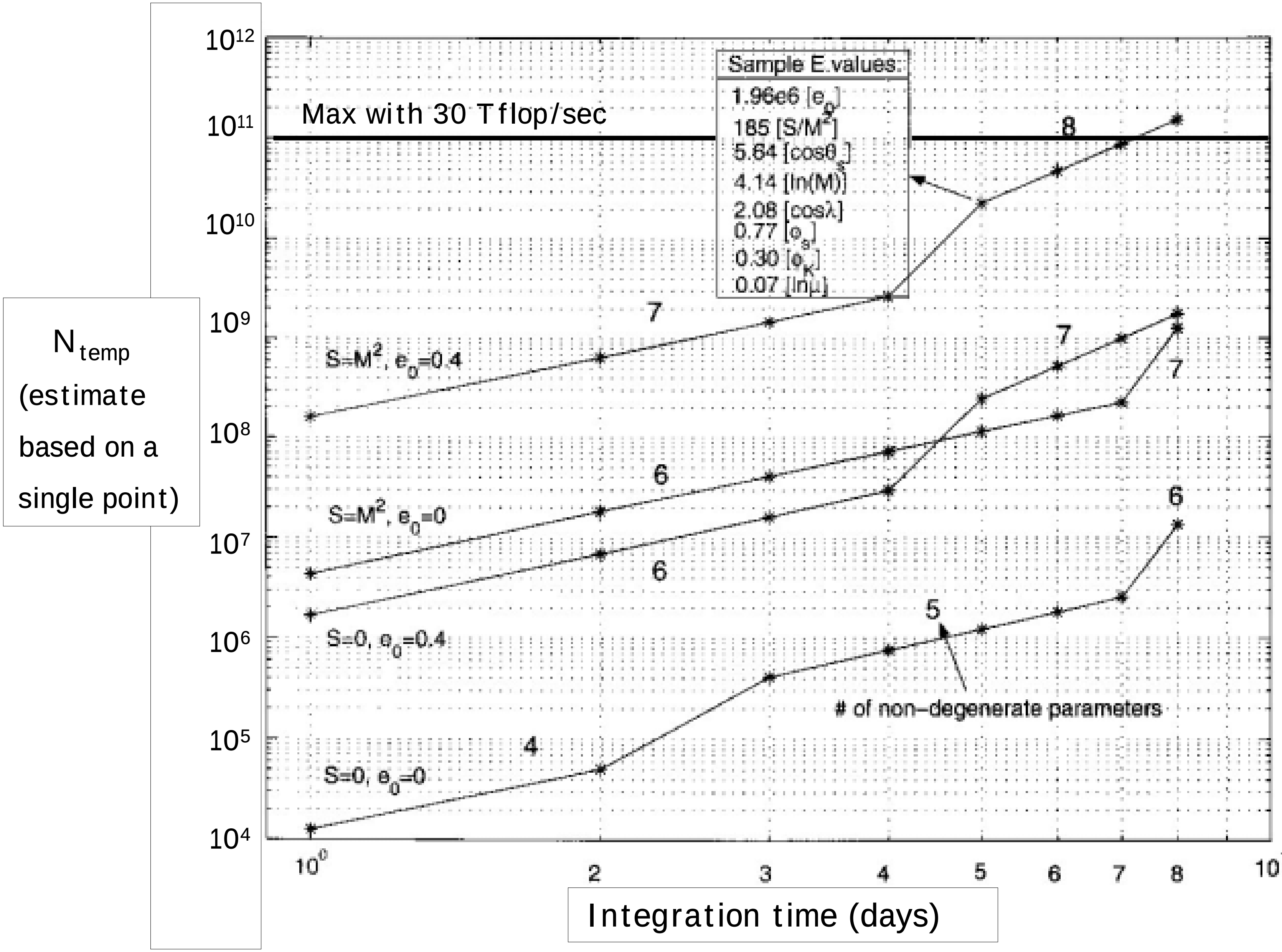
**Eigenvalues**($\gamma_{ij}$):

$\{2.83\cdot10^{6},\ 33,\ 7.07,\ 0.336,\ 0.00455,\ 0.000524,\ 0.0000524,\ 0.0000337,\ 1.52\cdot10^{-15}\}$

**Eigenvectors**($\gamma_{ij}$):

$$\begin{array}{lllllllll} \{0.000231 & -0.575 & -0.818 & -0.000321 & 0.000113 & -6.7\cdot10^{-6} & -0.000384 & -0.00145 & 0.000181\} \\ \{-0.185 & 0.15 & -0.105 & 0.000692 & -0.000242 & 0.953 & -0.14 & 0.0668 & 0.0315\} \\ \{0.206 & -0.779 & 0.547 & -0.0118 & 0.00412 & 0.219 & -0.0358 & -0.0371 & 0.0148\} \\ \{-0.939 & -0.201 & 0.14 & 0.0441 & -0.0154 & -0.142 & 0.032 & 0.185 & -0.0328\} \\ \{0.201 & -0.0028 & 0.000175 & 0.132 & -0.0464 & 0.0158 & 0.263 & 0.929 & -0.0911\} \\ \{-0.0309 & -0.0000429 & 1.66\cdot10^{-6} & 0.113 & -0.0399 & 0.151 & 0.828 & -0.292 & -0.437\} \\ \{0.0327 & -0.000295 & 0.000071 & 0.828 & -0.291 & -0.0258 & -0.327 & -0.0792 & -0.338\} \\ \{0.0147 & 6.67\cdot10^{-6} & 0.0000191 & -0.414 & 0.146 & -0.025 & -0.34 & 0.0784 & -0.828\} \\ \{0.000123 & 3.06\cdot10^{-8} & -5.96\cdot10^{-10} & 0.332 & 0.943 & 0.000021 & 2.10\cdot10^{-6} & 8.71\cdot10^{-6} & 0.000036\} \end{array}$$

# Applications: I. counting templates



# How much S/N needed for 1yr of data?

For first, coherent, step of stacked search, probably need

$$S/N \approx 4-5$$

(from experience gained in NS search for LIGO)

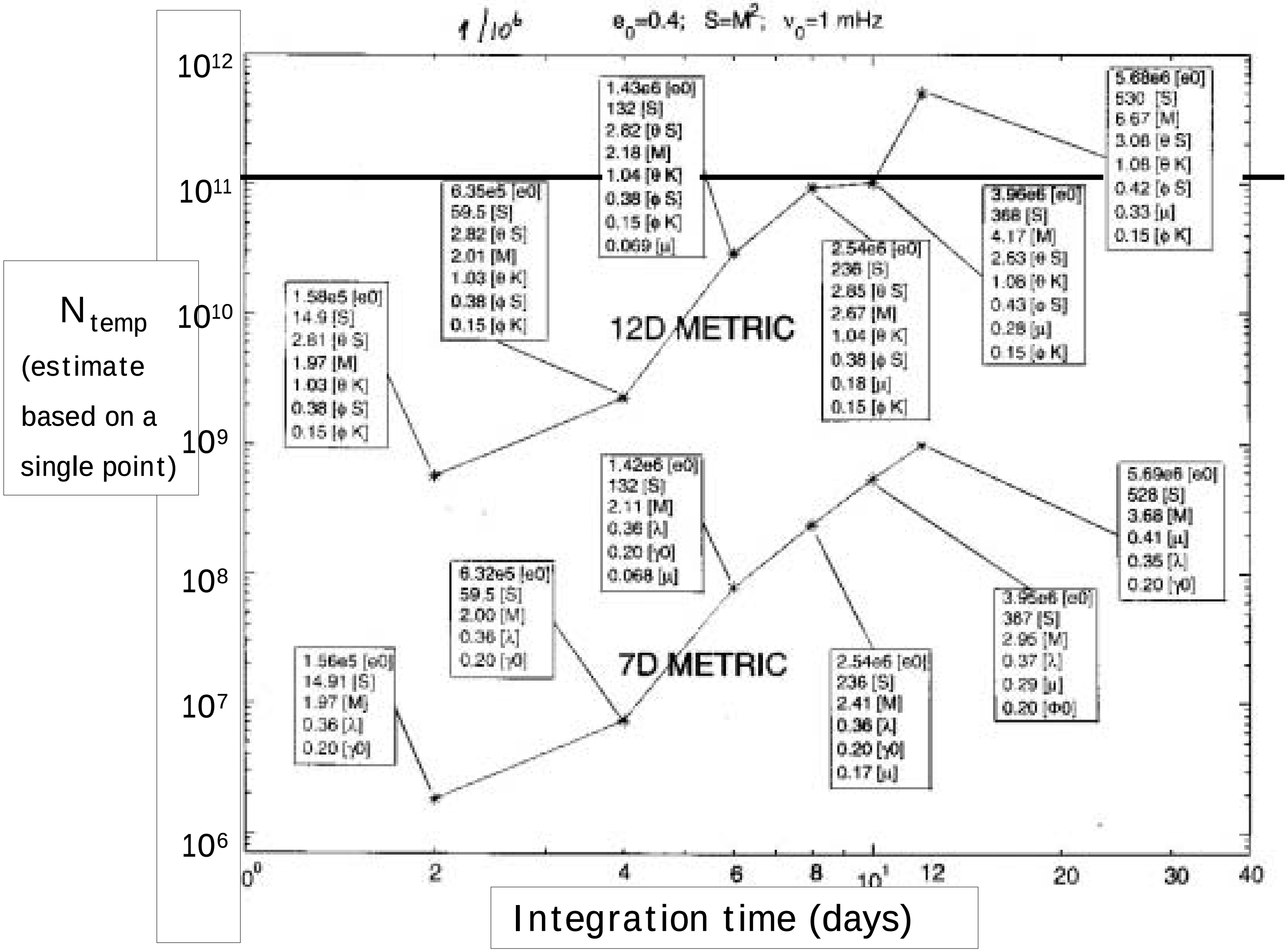
ﬂ

If first coherent integration is 1 week long, then
for a year-long integration need

$$S/N \approx 5 \times \sqrt{\text{year/week}} \approx 36$$

# Buonanno-Chen-Vallisneri reduction of P.S.

"Extrinsic" parameters easier to search over -> reduce PS to 7D




fi

With BCV method can integrate coherently for as long as a month!

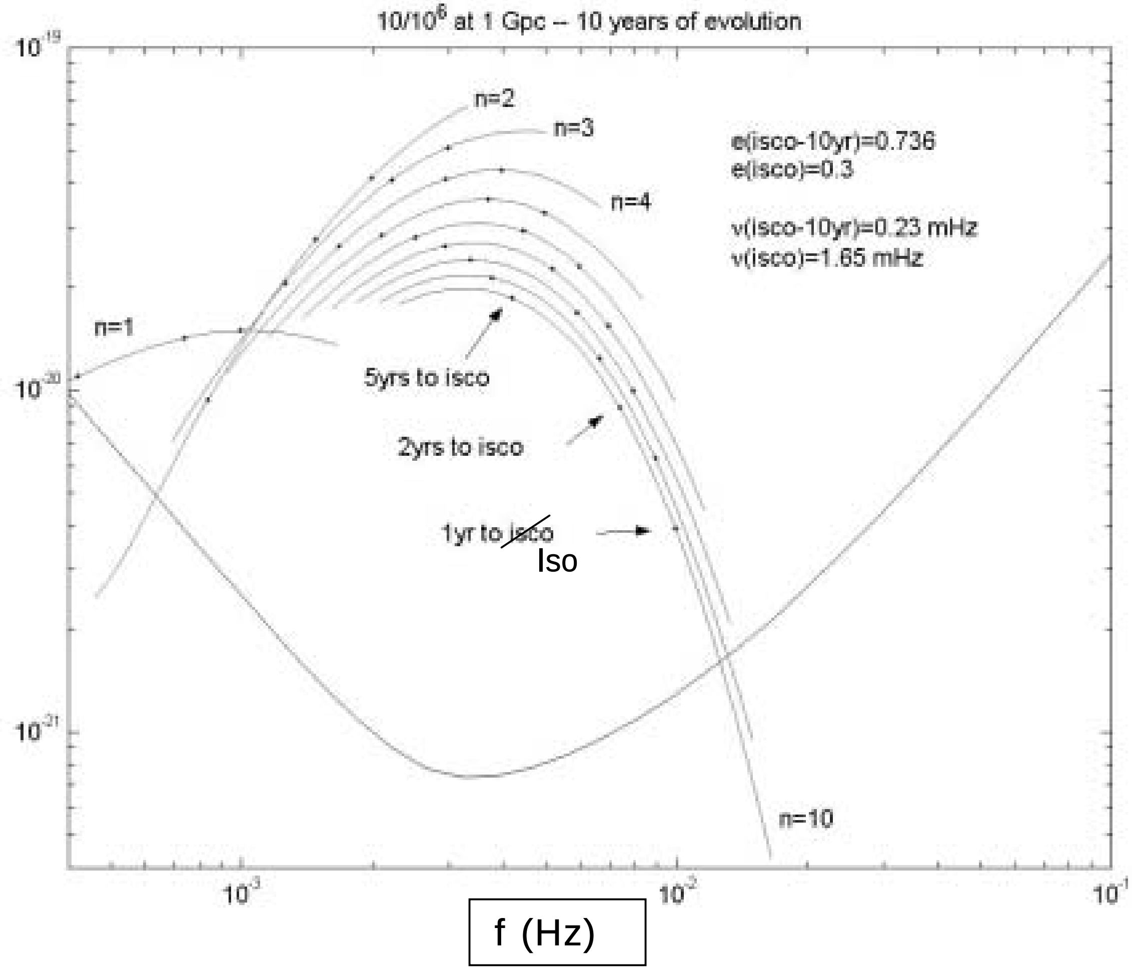
fl

For 1yr of data May Need only

S/N~20

# Applications: II. S/N estimates

Analysis like Finn & Thorne (2000), but with eccentric orbits




$$(S/N)^2 = \sum_n \int \frac{h_{c,n}^2(f_n)}{5S(f_n)} d\ln f_n$$

(Average over source orientations)

$$h_{c,n} = (\pi D)^{-1}\sqrt{2\dot{E}_n / \dot{f}_n}$$

($\dot{E}_n, \dot{f}_n$ derived from AK model)

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- Signal Cut-off at (S=0) Iso frequency

# Applications: III. LISA's resolution

$$\delta\lambda^i = \sqrt{(\Gamma^{-1})^{ii}} \times (S/N)^{-1}$$

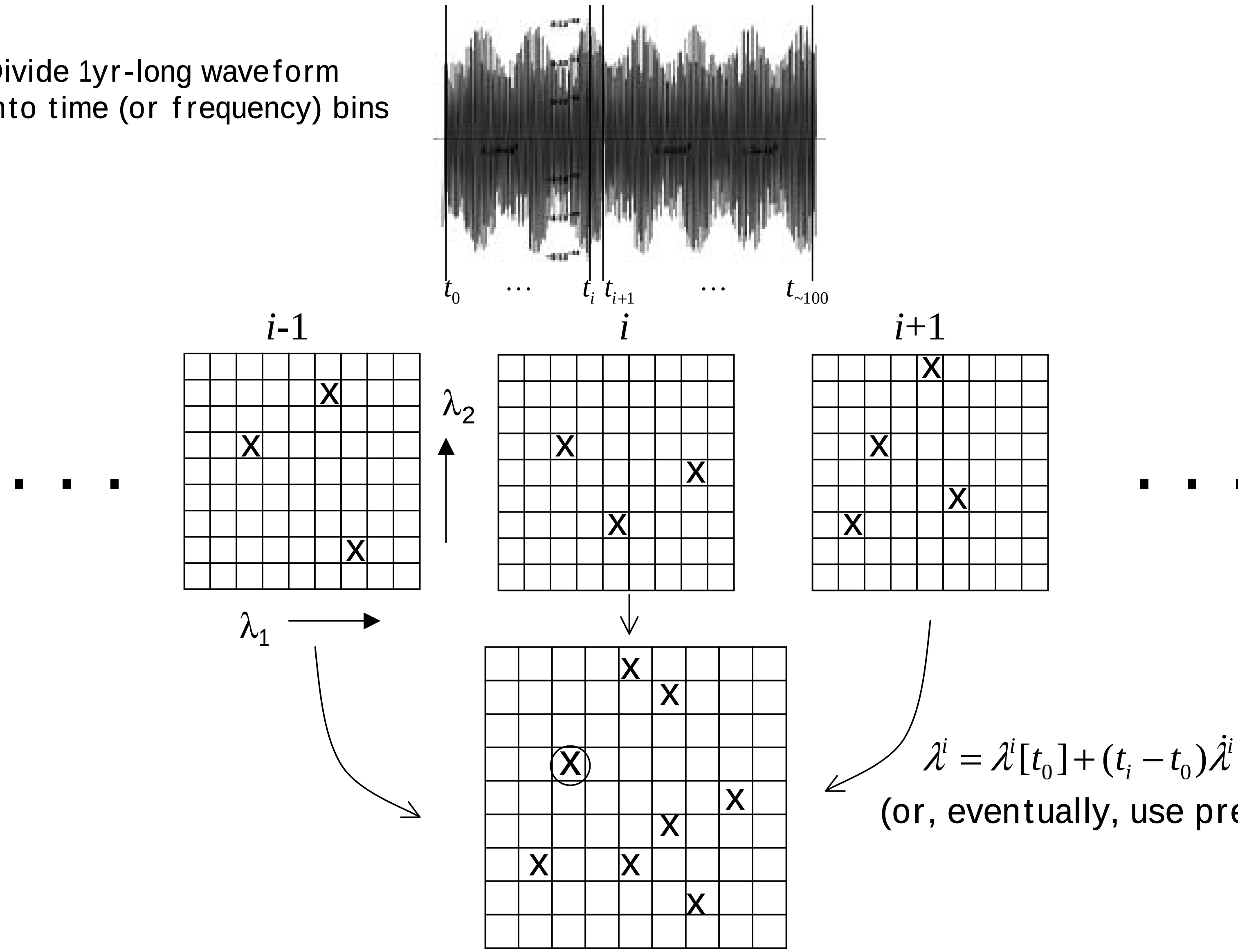
$$\delta\Omega = 2\pi\left\{\sqrt{(\Gamma^{-1})^{\cos\theta,\cos\theta}(\Gamma^{-1})^{\phi\phi}} - (\Gamma^{-1})^{\cos\theta,\phi}\right\} \times (S/N)^{-2}$$

| | | |
|---|---|---|
| $\delta(t_0/\nu_0)$ | $9.94 \cdot 10^{-3}$ | |
| $\delta(\ln\mu)$ | $1.29 \cdot 10^{-5}$ | |
| $\delta(\ln M)$ | $6.40 \cdot 10^{-5}$ | (Ryan $2.6 * 10^{-4}$) |
| $\delta(e_0)$ | $6.28 \cdot 10^{-6}$ | |
| $\delta(\gamma_0)$ | 0.44 | |
| $\delta(\Phi_0)$ | 0.062 | |
| $\delta(\Omega_s)$ | $5.68 \cdot 10^{-4}$ | |
| $\delta(\lambda)$ | $3.66 \cdot 10^{-4}$ | |
| $\delta(\alpha_0)$ | 0.79 | |
| $\delta(S/M^2)$ | $6.94 \cdot 10^{-5}$ | (Ryan $7 * 10^{-4}$) |
| $\delta(\Omega_K)$ | 0.091 | |
| $\delta[\ln(\mu/D)]$ | 0.057 | |

Table 1: Resolution at S/N=30 for last of year of $10M_\odot/10^6M_\odot$ inspiral. A year before the last stable orbit $e = 0.48$ and $\nu = 1.02$ mHz. At the last stable orbit $e = 0.4$ and $\nu = 1.40$ mHz. Noise model includes WD confusion noise.

# Stacked Search Strategy: a sketch

- Divide 1yr-long waveform into time (or frequency) bins




Master Grid

$$\lambda^i = \lambda^i[t_0] + (t_i - t_0)\dot{\lambda}^i + \cdots$$

(or, eventually, use precise EOM)

# Where Next?

- Finish validation of codes
- Explore parameter space; Monte-Carlo integration (project now initiated at UTB)
- Compare with NK, TB
- Develop search strategy
- Study problem of self-confusion

Census of Supermassive black holes as of March 2001

(Kormendy & Gebhardt 2001)

**TABLE 1**

Census of Supermassive Black Holes (2001 March)

| Galaxy | Type | $M_{B,\mathrm{bulge}}$ | $M_\bullet$ ($M_{\mathrm{low}}$, $M_{\mathrm{high}}$) ($M_\odot$) | $\sigma_e$ km/s) | $D$ (Mpc) | $r_{\mathrm{cusp}}$ (arcsec) | Reference |
|---|---|---|---|---|---|---|---|
| Galaxy | Sbc | −17.65 | 2.6 (2.4 − 2.8) e6 | 75 | 0.008 | 51.40 | See notes |
| M 31 | Sb | −19.00 | 4.5 (2.0 − 8.5) e7 | 160 | 0.76 | 2.06 | Dressler + 1988; Kormendy 1988a |
| M 32 | E2 | −15.83 | 3.9 (3.1 − 4.7) e6 | 75 | 0.81 | 0.76 | Tonry 1984, 1987 |
| M 81 | Sb | −18.16 | 6.8 (5.5 − 7.5) e7 | 143 | 3.9 | 0.76 | Bower + 2001b |
| NGC 821 | E4 | −20.41 | 3.9 (2.4 − 5.6) e7 | 209 | 24.1 | 0.03 | Gebhardt + 2001 |
| NGC 1023 | S0 | −18.40 | 4.4 (3.8 − 5.0) e7 | 205 | 11.4 | 0.08 | Bower + 2001a |
| NGC 2778 | E2 | −18.59 | 1.3 (0.5 − 2.9) e7 | 175 | 22.9 | 0.02 | Gebhardt + 2001 |
| NGC 3115 | S0 | −20.21 | 1.0 (0.4 − 2.0) e9 | 230 | 9.7 | 1.73 | Kormendy + 1992 |
| NGC 3377 | E5 | −19.05 | 1.1 (0.6 − 2.5) e8 | 145 | 11.2 | 0.42 | Kormendy + 1998 |
| NGC 3379 | E1 | −19.94 | 1.0 (0.5 − 1.6) e8 | 206 | 10.6 | 0.20 | Gebhardt + 2000a |
| NGC 3384 | S0 | −18.99 | 1.4 (1.0 − 1.9) e7 | 143 | 11.6 | 0.05 | Gebhardt + 2001 |
| NGC 3608 | E2 | −19.86 | 1.1 (0.8 − 2.5) e8 | 182 | 23.0 | 0.13 | Gebhardt + 2001 |
| NGC 4291 | E2 | −19.63 | 1.9 (0.8 − 3.2) e8 | 242 | 26.2 | 0.11 | Gebhardt + 2001 |
| NGC 4342 | S0 | −17.04 | 3.0 (2.0 − 4.7) e8 | 225 | 15.3 | 0.34 | Cretton + 1999a |
| NGC 4473 | E5 | −19.89 | 0.8 (0.4 − 1.8) e8 | 190 | 15.7 | 0.13 | Gebhardt + 2001 |
| NGC 4486B | E1 | −16.77 | 5.0 (0.2 − 9.9) e8 | 185 | 16.1 | 0.81 | Kormendy + 1997 |
| NGC 4564 | E3 | −18.92 | 5.7 (4.0 − 7.0) e7 | 162 | 15.0 | 0.13 | Gebhardt + 2001 |
| NGC 4594 | Sa | −21.35 | 1.0 (0.3 − 2.0) e9 | 240 | 9.8 | 1.58 | Kormendy + 1988b |
| NGC 4649 | E1 | −21.30 | 2.0 (1.0 − 2.5) e9 | 375 | 16.8 | 0.75 | Gebhardt + 2001 |
| NGC 4697 | E4 | −20.24 | 1.7 (1.4 − 1.9) e8 | 177 | 11.7 | 0.41 | Gebhardt + 2001 |
| NGC 4742 | E4 | −18.94 | 1.4 (0.9 − 1.8) e7 | 90 | 15.5 | 0.10 | Kaiser + 2001 |
| NGC 5845 | E | −18.72 | 2.9 (0.2 − 4.6) e8 | 234 | 25.9 | 0.18 | Gebhardt + 2001 |
| NGC 7457 | S0 | −17.69 | 3.6 (2.5 − 4.5) e6 | 67 | 13.2 | 0.05 | Gebhardt + 2001 |
| NGC 2787 | SB0 | −17.28 | 4.1 (3.6 − 4.5) e7 | 185 | 7.5 | 0.14 | Sarzi + 2001 |
| NGC 3245 | S0 | −19.65 | 2.1 (1.6 − 2.6) e8 | 205 | 20.9 | 0.21 | Barth + 2001 |
| NGC 4261 | E2 | −21.09 | 5.2 (4.1 − 6.2) e8 | 315 | 31.6 | 0.15 | Ferrarese + 1996 |
| NGC 4374 | E1 | −21.36 | 4.3 (2.6 − 7.5) e8 | 296 | 18.4 | 0.24 | Bower + 1998 |
| NGC 4459 | SA0 | −19.15 | 7.0 (5.7 − 8.3) e7 | 167 | 16.1 | 0.14 | Sarzi + 2001 |
| M 87 | E0 | −21.53 | 3.0 (2.0 − 4.0) e9 | 375 | 16.1 | 1.18 | Harms + 1994 |
| NGC 4596 | SB0 | −19.48 | 0.8 (0.5 − 1.2) e8 | 136 | 16.8 | 0.22 | Sarzi + 2001 |
| NGC 5128 | S0 | −20.80 | 2.4 (0.7 − 6.0) e8 | 150 | 4.2 | 2.26 | Marconi + 2001 |
| NGC 6251 | E2 | −21.81 | 6.0 (2.0 − 8.0) e8 | 290 | 106 | 0.06 | Ferrarese + 1999 |
| NGC 7052 | E4 | −21.31 | 3.3 (2.0 − 5.6) e8 | 266 | 58.7 | 0.07 | van der Marel + 1998 |
| IC 1459 | E3 | −21.39 | 2.0 (1.2 − 5.7) e8 | 323 | 29.2 | 0.06 | Verdoes Kleijn + 2001 |
| NGC 1068 | Sb | −18.82 | 1.7 (1.0 − 3.0) e7 | 151 | 15 | 0.04 | Greenhill + 1996 |
| NGC 4258 | Sbc | −17.19 | 4.0 (3.9 − 4.1) e7 | 120 | 7.2 | 0.36 | Miyoshi + 1995 |
| NGC 4945 | Scd | −15.14 | 1.4 (0.9 − 2.1) e6 | | 3.7 | | Greenhill + 1997 |

# Mode distribution of power emitted

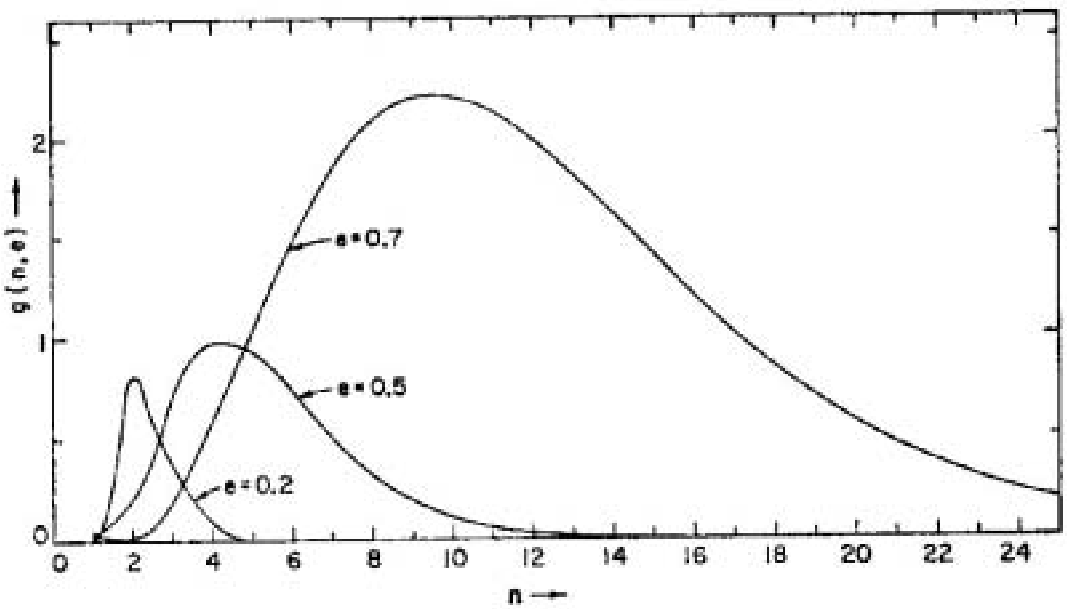



FIG. 3. $g(n,e)$, the relative power radiated into the $n$th harmonic for $e=0.2$, 0.5, and 0.7.

[Peters & Mathews (1963)]