BRANE - WORLDS

in

Einstein - Gauss - Bonnet THEORY

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- The universe ("bulk") is 5-dimensional
- Gravity is described by Einstein's eqns + quadratic corrections
- We live on a 4-D surface ("brane") on which matter is confined

⟨ is Newton's law recovered on the brane? YES! ⟩

(Randall - Sundrum 1999 ......)

Einstein-Gauss-Bonnet theory

● Einstein-Hilbert (1916)

- ST is curved: $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$ (D = 4) by gravity
- action: $S = \int d^4x \sqrt{-g}\,(R - 2\Lambda) + S_m$ (Hilbert)
- field eqn: $\delta_g S = 0$: $\underbrace{\Lambda g_{\mu\nu} + G_{\mu\nu} = 8\pi G\, T_{\mu\nu}}_{4\,\partial^2 g}$ (Einstein)

● Extra dimensions (Kaluza (1921); Klein (1926))

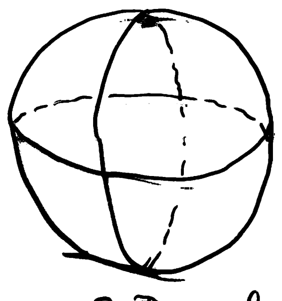
- to incorporate matter in geometry (like gravity)

electromagnetism
5D, but (A_μ, φ)
(Kaluza & Klein)

gauge theories (70's: deWitt, Kerner...)
supergravity (D = ... 11)

$$\underline{\Lambda g_{AB} + G_{AB} = 0 \qquad D > 4}$$

- Einstein-Gauss-Bonnet theory : $\boxed{D = 5}$
- Two ways of "hiding" the 4th spatial dimension:

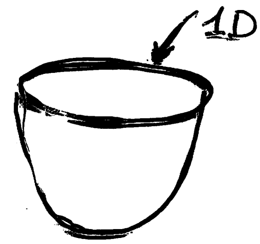


2D surface

"stretched" or "compactified"

$\simeq$ 1D
"Kaluza-Klein"

OR "cut"



"braneworld"

Quadratic corrections

Weyl (1921)
Eddington (24)

$$S_g = \int d^D x \sqrt{-g}\,(-2\Lambda + s) + \int d^D x \sqrt{-g}\,(\alpha s^2 + \beta\, ric^2 + \gamma\, Riem^2)$$

- field eqns: $\Lambda g_{AB} + \underbrace{\sigma_{AB} + \text{“}Riem^2\text{”}}_{\partial^2 g} + \underbrace{\text{“}\Box Riem\text{”}}_{\partial^4 g} = 0$

("higher derivative")

- used to:
  - renormalize gravity (in 4D. Stelle '77)
  - renormalize $\langle T_{\mu\nu}\rangle$ (4D) (see Birrell-Davies)
  - inflate universe (4D. Starobinski '80)

The G.B.-Lanczos-Lovelock term

if $S_g = \int d^5 x \sqrt{-g}\,(-2\Lambda + s) + \alpha \int d^D x \sqrt{-g}\,(\underline{\underline{s^2 - 4\, ric^2 + Riem^2}})$

then: $\Lambda g_{AB} + \underbrace{\sigma_{AB} + \alpha\, \sigma^{(2)}_{AB}}_{\partial^2 g} = 0$

$\partial^2 g$: NO higher derivative terms

properties of the Lanczos (1932). Lovelock (1971) tensor :

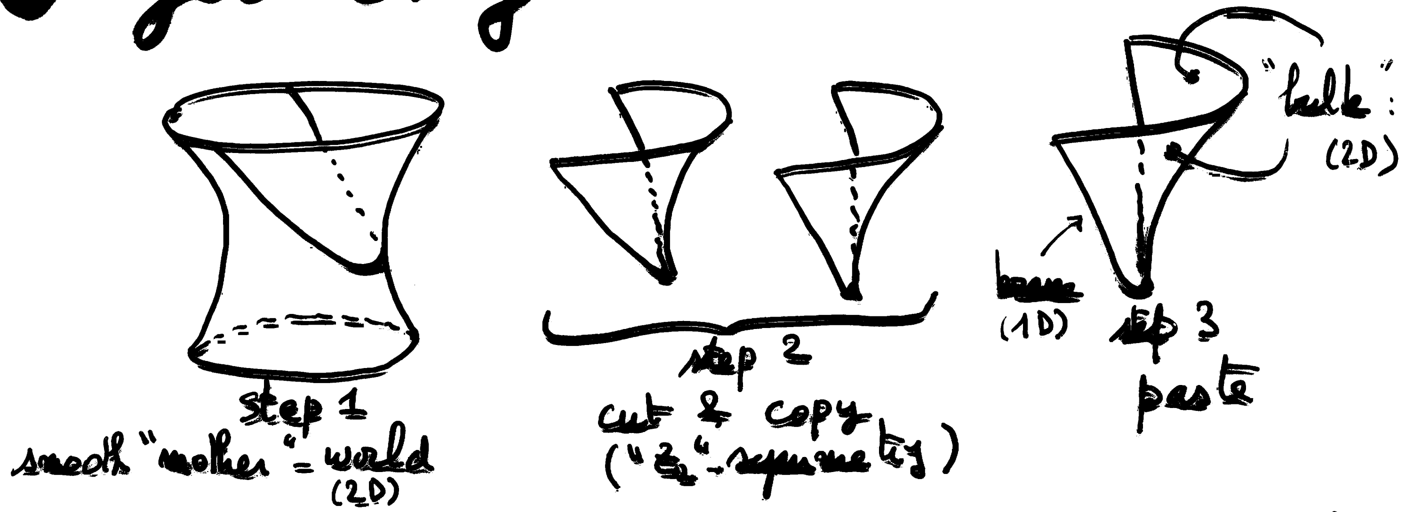
$\sigma^{(2)}_{AB}$ is
- symmetric
- conserved ($D_A \sigma_{(2)}{}^A{}_B \equiv 0$)
- quasi-linear ($\sigma_{(2)}{}^A{}_B = (\partial g)^2 \partial^2 g + \dots$)
- identically zero in $D \leq 4$

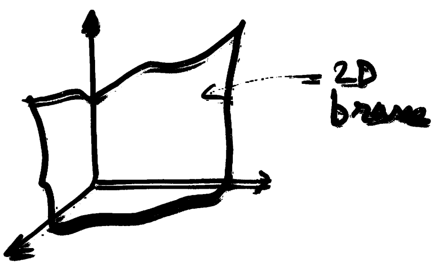
("Gauss-Bonnet")

appear in low-energy limit of string theory
(Zwiebach; Zumino 85)

Gauss-Bonnet braneworlds

• geometry.





- the geometry of bulk is determined by its Riemann tensor :
- the geometry of the brane is determined by
  its intrinsic Riemann tensor
  its extrinsic curvature

The extrinsic curvature of the brane changes sign when going from the "exterior" to the "interior" of the bulk

The intrinsic curvature of the brane is determined by the curvature of the bulk and its position in the bulk

bulk metric : $ds^2 = dw^2 + \gamma_{\mu\nu} dx^\mu dx^\nu$ } in GN coordinates

position of brane : $w = 0$

extrinsic curvature of brane : $K_{\mu\nu} = -\frac{1}{2} \frac{\partial \gamma_{\mu\nu}}{\partial w}\Big|_{w=0} \underline{\mathrm{Sign}(w)}$

curvature of bulk : $R_{w\mu w\nu} = \frac{\partial K_{\mu\nu}}{\partial w} + K \cdot K \propto \underline{\delta(w)}$

• "Physics"

– "Mother"-world curved by gravity governed by the Einstein-Gauss-Bonnet (Lanczos) equations:

$$\mathcal{G}_{[2]}{}^{A}{}_{B} \equiv \Lambda \delta^{A}_{B} + \mathcal{G}^{A}_{B} + \alpha \, \mathcal{G}_{(2)}{}^{A}{}_{B} = 0 \quad \text{(5D)} \quad \text{(everywhere)}$$

example of solutions: • AdS_5 (maximally symmetric):

$$R_{ABCD} = -\frac{1}{\ell^2}\left(g_{AC}g_{BD} - g_{AD}g_{BC}\right)$$

$$\left(\frac{4\alpha}{\ell^2} = 1 - \sqrt{1 + \frac{4\alpha\Lambda}{3}}\right)$$

• Schwarzschild AdS_5

• perturbed AdS_5 or S. AdS_5

– Brane chosen in such a way that its intrinsic geometry is, for example: • flat (M_4), or approximately flat

• FRW (or approx. FRW)

example: • if bulk is AdS_5:

$$ds_5^2 = \left(\frac{\ell}{w}\right)^2 \eta_{AB}\, dx^A dx^B$$

$w = \ell$: brane: M_4

• " " " ":

$$ds_5^2 = -\frac{r^2 dt^2}{\ell^2} + \frac{dr^2}{r^2/\ell^2} + r^2 d\vec{x}_3^{\,2}$$

w: $r = a(\tau)$; $t = t(\tau)$; brane: FRW

– Matter on brane

• the Riemann tensor of BULK $\ni \delta(w)$

• hence: $\mathcal{G}_{[2]}{}^{A}{}_{B}$ of BULK $\ni \delta(w)$

(that is: $\mathcal{G}_{[2]}{}^{A}{}_{B}$ of "mother"-world is zero

$$\mathcal{G}_{[2]}{}^{A}{}_{B} = T^{A}_{B}\, D(w) \quad \text{in bulk}$$

computed geometrically

interpreted as Stress-energy tensor of BRANE

• The "junction conditions"

that is: for a given "mother"-world (e.g. AdS_5)
and for a given brane (e.g. M_4 or FRW)

hence: for a known (full) Riemann tensor:

$$R_{\omega\mu\omega\nu} = \frac{\partial K_{\mu\nu}}{\partial\omega} + K.K \quad \text{(in GN coord.)}$$

(which contains a $\delta(\omega)$ term)

what is the discontinuity of $\sigma_{[2]}{}^A{}_B$?

$$\sigma_{[2]}{}^\mu{}_\nu \equiv \underbrace{\Lambda\delta^\mu_\nu}_{ct} + \underbrace{\sigma^\mu{}_\nu}_{\text{"="}\frac{\partial K}{\partial\omega}+\dots} + \alpha\underbrace{\sigma_{(2)}{}^\mu{}_\nu}_{\text{"="}\frac{\partial}{\partial\omega}(K^3+K\bar{R})+\dots}$$

$$K_{\mu\nu} = K^+_{\mu\nu}\,\mathrm{Sig}(\omega)$$

$K^+_{\mu\nu}$: extrinsic curvature of brane in "mother"-world

hence: $\sigma_{[2]}{}^\mu{}_\nu$ "=" $\frac{\partial}{\partial\omega}\left[K^+\,\mathrm{Sign} + \alpha\left(K^{+3}\,\mathrm{Sign}^3 + K^+\bar{R}\,\mathrm{Sign}\right)\right]$

$\mathrm{Sign}^3 = \mathrm{Sign}^2.\mathrm{Sign} = \mathrm{Sign}$

"endless" debate (Davis - Gravanis Willison).

NET RESULT: $\boxed{\sigma_{[2]}{}^\mu{}_\nu \text{ "=" } \underbrace{2\left[K^+ + \alpha\left(KKK + K\bar{R}\right)\right]}_{\equiv T^\mu_\nu}\delta(\omega)}$

($\alpha = 0$: "Israel junction conditions").

NB: **OK with**: Davis. Gravanis - Willison. Charmousis et al. Neupane et al. Padilla et al.

at odds with: Germani - Sopuerta; Kim Lee

Summary Part 1

- Start from a smooth 5D spacetime R_{ABCD}
- Pick up a 4D hypersurface $\bar{R}_{\mu\nu\rho\sigma}$; $K^{+}_{\mu\nu}$
- "Cut, copy & Paste" and get a **BRANEWORLD**

$$K_{\mu\nu} = K^{+}_{\mu\nu}\,\mathrm{Sign}(w)$$

$$R_{\mu w \nu w} = 2\frac{\partial K^{+}_{\mu\nu}}{\partial w}\,\delta(w) + \cdots$$

- Describe gravity by the EGB equations:

smooth 5D "mother" world : $G_{[2]}{}^{A}{}_{B} \equiv \Lambda\delta^{A}_{B} + G^{A}_{B} + \alpha G_{(2)}{}^{A}{}_{B} = 0$

choose a solution : e.g. AdS_5

- then : metric & intrinsic curvature $\bar{R}_{\mu\nu\rho\sigma}$ known

extrinsic curvature of brane $K_{\mu\nu}$ also known

- hence the $\delta(w)$ part of $G_{[2]}{}^{A}{}_{B}$ can be calculated :

$$\begin{cases} G_{[2]}{}^{\mu}{}_{\nu} = T^{\mu}_{\nu}\,\delta(w) \\ T^{\mu}_{\nu} = K^{\mu}_{\nu} - \delta^{\mu}_{\nu}K + \alpha\left[KKK + \bar{R}K\right] \end{cases}$$

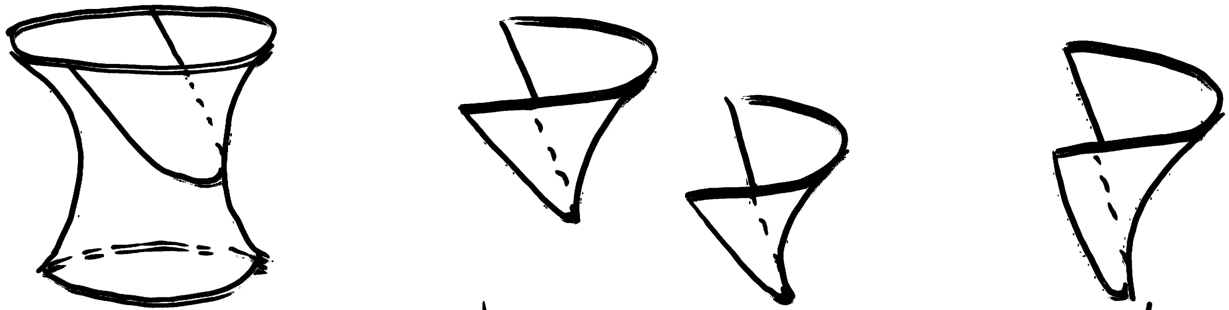
Question : • These equations define a tensor T^{μ}_{ν} (interpreted as the stress-energy tensor of matter)

• Einstein tensor G^{μ}_{ν} is built with the curvature of the brane; Relations between G & T?

HOW DO THEY COMPARE TO EINSTEIN'S?

(1) "Newton's law" on ~M_4 Einstein brane

{Randall-Sundrum 99
{Garriga-Tanaka 00



$$(AdS_5)\quad \begin{cases} ds_5^2 = dw^2 + e^{-2|w|/\ell}\, \eta_{\mu\nu}\, dx^\mu dx^\nu \\ \Lambda \delta^A_B + \sigma^A_B = T^A_B\, \delta(w) \end{cases}$$

brane : $w = 0$

$$\left(\Lambda = -\frac{6}{\ell^2}\right)$$

$$\begin{cases} T^w_{\ w} = 0\ ;\ T^w_{\ \mu} = 0\ ; \\ T^\mu_{\ \nu} = 2\left[K^\mu_{\ \nu} - \delta^\mu_\nu K\right]^+ = -\frac{6}{\ell}\,\delta^\mu_\nu \end{cases}$$

- perturb the position of the brane in bulk

- perturb the bulk

$$\begin{cases} \Rightarrow\ ds_5^2 = dw^2 + e^{-2w/\ell}\, \eta_{\mu\nu}\, dx^\mu dx^\nu + \gamma_{AB}\, dx^A dx^B \\ \text{and:}\quad w = \zeta(x^A) + \ell \end{cases}$$

$\Rightarrow$ $\boxed{\begin{matrix} \eta_{\mu\nu} + h_{\mu\nu} \\ \eta_{\mu\nu} + S_{\mu\nu} \end{matrix}}$

- <u>question</u> • what is the relationship between $h_{\mu\nu}$ & $S_{\mu\nu}$?
  - how does it compare to the Einstein eqns for linearized gravity?

$$\Box_4 h^{(h)}_{\mu\nu} = -16\pi G\left(S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} S\right)$$

  - how does it compare with Newton's law?

$$h^{(h)}_{00} \xrightarrow[r\to\infty]{\text{static}} \frac{2GM}{r} \qquad (S_{00} = M\delta(\vec{r}))$$

▲ 1st step: the bulk "gravitons"

$$\begin{cases} ds^2\big|_5 = \left(\frac{\ell}{\omega}\right)^2 (\eta_{AB} + \gamma_{AB})\, dX^A dX^B \\ -\frac{6}{\ell^2}\delta^A_B + \sigma^A_B = 0 \end{cases} \qquad \sigma^A_B = \frac{6}{\ell^2}\delta^A_B + \underbrace{\delta\sigma^A_B(\gamma_{CD})}_{(\text{compute})}$$

- choose $\gamma_{A\omega} = 0$ gauge.
- then the $(\omega\text{-}\omega)$ & $(\omega\text{-}\mu)$ components of e.o.m give $\underline{\gamma^\rho_\rho = 0}$, $\underline{\partial_\rho \gamma^\rho_\mu = 0}$
- and $(\mu\text{-}\nu)$ components of e.o.m reduce to:

$$\boxed{\Box_4 \gamma_{\mu\nu} + \partial_{\omega\omega}\gamma_{\mu\nu} - \frac{3}{\omega}\partial_\omega \gamma_{\mu\nu} = 0}$$

<u>General solution in bulk, i.e for $\omega \in [0, +\infty[$</u>

superposition of all modes with <u>converge at $\omega \to \infty$</u>

hence more allowed modes than in full AdS_5

$\exists$ in particular <u>STATIC</u> modes

$$\gamma^{(s)}_{\mu\nu}(\omega, x^i) = 2e \int \frac{d^3k}{(2\pi)^{3/2}}\, e^{i\vec{k}\cdot\vec{x}}\, \underline{e_{\mu\nu}(k^i)}\, \omega^2 H^{(1)}_2(ik\omega)$$

▲ 2nd step: metric on brane ($h_{\mu\nu}$)

$$ds^2|_5 = \left(\frac{\ell}{\omega}\right)^2 (\eta_{AB} + \gamma_{AB})\,dX^A dX^B \quad ; \quad \omega = \ell + \zeta(x^\mu)$$

$$\Rightarrow ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})\,dx^\mu dx^\nu \qquad \boxed{h_{\mu\nu} = \gamma_{\mu\nu}\Big|_{\omega=\ell} - \frac{2\zeta}{\ell}\eta_{\mu\nu}} \quad (1)$$

▲ 3rd step: matter on brane ($S_{\mu\nu}$)

$$K_\mu^\nu = -\frac{1}{\ell}\delta_\mu^\nu + \delta K_\mu^\nu \qquad \left(\delta K^\mu{}_\nu = -\frac{1}{2}\partial_\omega \gamma^\mu_\nu\Big|_\Sigma + \partial^\mu_\nu \zeta\right)$$

junction conditions: $2\left[K_\mu^\nu - \delta_\mu^\nu K\right]^+ = T_\mu^\nu$

hence $\quad T_\mu^\nu = -\frac{6}{\ell}\delta_\mu^\nu + \kappa S_\mu^\nu$

with $\quad \boxed{\frac{\kappa}{2}\left(S_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu} S\right) = \partial_{\mu\nu}\zeta - \frac{1}{2}\left(\partial_\omega \gamma_{\mu\nu}\right)\Big|_{\omega=\ell}} \quad (2)$

If the geometry of the bulk is known (ie $\gamma_{\mu\nu}$ & ζ) then the metric on brane & matter on brane are known

▲ Compare with Einstein's eqns

A consequence of (1) & (2) is that

$$\boxed{\Box_4 h_{\mu\nu}^{(h)} = -16\pi G\left(S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} S\right) - \left(\partial_{\omega\omega}\gamma_{\mu\nu}\right)_\Sigma + \frac{1}{\ell}\left(\partial_\omega \gamma_{\mu\nu}\right)\Big|_\Sigma}$$

(with $\frac{\kappa}{\ell} = 8\pi G$)

▲ Correction to Newton's law

$$\begin{cases} (1) & h_{\mu\nu} = \gamma_{\mu\nu}\big|_\Sigma - \dfrac{2\zeta}{\ell}\eta_{\mu\nu} \\ (2) & \dfrac{\kappa}{2}\left(S_{\mu\nu} - \dfrac{1}{3}\eta_{\mu\nu} S\right) = \partial_{\mu\nu}\zeta - \dfrac{1}{2}\left(\partial_w \gamma_{\mu\nu}\right)\big|_\Sigma \end{cases}$$

- Choose $S_{00} = M\,\delta(\vec{r})$; $S_{0j} = 0$.
- then, from (2) : $\begin{cases} \zeta = -\dfrac{\kappa M}{24\pi}\dfrac{1}{r} \\ \partial_w \gamma_{\mu\nu}\big|_\Sigma \text{ also known} \end{cases}$ (3)
- Now the static bulk modes are

$$\hat{\gamma}_{\mu\nu}(w,\vec{k}) = e_{\mu\nu}(\vec{k})\, w^2 H_2^{(1)}(ikw)$$

hence, from (3), $e_{\mu\nu}(\vec{k})$ is known.

$$\left(\text{e.g: } e_{00}(\vec{k}) = -\frac{2\kappa M}{3}\frac{1}{(2\pi)^{3/2}}\frac{1}{ik\ell^2}\frac{1}{H_1^{(1)}(ik\ell)}\right)$$

Therefore : <u>$h_{\mu\nu}$ is known</u>

$$h_{00}^{(h)}(\vec{r}) = \frac{2GM}{r}\left(1 + \frac{4}{3\pi}K_\beta\right) \qquad \left(\beta \equiv \frac{r}{\ell}\right)$$

$$K_\beta = \lim_{\varepsilon\to 0}\int_0^\infty du\, \sin(\beta u)\frac{K_0(u)}{K_1(u)}e^{-\varepsilon u}$$

- $\beta \to 0 \quad K_\beta \to \dfrac{\ell}{r}$

  $\beta \to \infty \quad K_\beta \to \dfrac{\pi}{2\beta^2}$

$$\left(h_{00}^{(h)} \xrightarrow{r\to\infty} \frac{2GM}{r}\left[1 + \frac{2}{3}\left(\frac{\ell}{r}\right)^2\right]\right)$$

Randall - Sundrum
Garriga - Tanaka

2) "Newton's law" on ~ M_4 Einstein-Gauss-Bonnet brane

Kim & Lee '00
Cho & Neupane '01
Meissner-Olechowski '02

Follow step by step what was done in the Einstein theory.

$$(AdS_5)\begin{cases} ds^2|_5 = dw^2 + e^{-\frac{2|w|}{\ell}}\eta_{\mu\nu}\,dx^\mu dx^\nu & \text{brane}: w \equiv 0 \\ \Lambda\delta^A_B + G^A_B + \alpha\, G_{(2)}{}^A_B = T^A_B\,\delta(w) & \Lambda = -\frac{6}{\ell^2} + \frac{12\alpha}{\ell^4} \end{cases}$$

$$\begin{cases} T^w_w = 0 \quad ; \quad T^w_\mu = 0 \\ T^\mu_\nu = 2[K^\mu_\nu - K\delta^\mu_\nu + 4\alpha\,(\text{"}K\cdot K\cdot K + K\bar{R}\text{"})] = -\frac{6}{\ell}\delta^\mu_\nu\left(1 - \frac{4\alpha}{3\ell^2}\right) \end{cases}$$

- perturb the position of the brane in the bulk
- perturb the bulk

$$\begin{cases} ds^2|_5 = dw^2 + e^{-\frac{2w}{\ell}}\eta_{\mu\nu}\,dx^\mu dx^\nu + \gamma_{AB}\,dx^A dx^B \\ \text{and}: \quad w = \ell + \zeta(x^\mu) \end{cases}$$

$\Rightarrow ds^2|_4 = \eta_{\mu\nu} + h_{\mu\nu}$; $\quad T_{\mu\nu} \Rightarrow T_{\mu\nu} + S_{\mu\nu}$

and, AS BEFORE:

- what is the relationship between $h_{\mu\nu}$ & $S_{\mu\nu}$?
- how does it compare to linearized Einstein eqns?
- how does it compare to Newton's law?

▲ 1st step: the bulk "gravitons"

$$\begin{cases} ds^2\big|_5 = \left(\frac{\ell}{\omega}\right)^2 (\eta_{AB} + \gamma_{AB})\, dx^A dx^B \\ \Lambda \delta^A_B + \sigma^A_B + \alpha\, \sigma_{(2)}{}^A_B = 0 \end{cases}$$

$$\sigma^A_B = \frac{6}{\ell^2}\delta^A_B + \delta\sigma^A_B \quad ; \quad \sigma_{(2)}{}^A_B = -\frac{12}{\ell^4}\delta^A_B + \underbrace{\delta\sigma_{(2)}{}^A_B}_{\text{compute}}$$

- choose $\underline{\gamma_{A\omega} = 0}$ gauge
- FIND that the $(\omega\text{-}\omega)$ & $(\omega\text{-}\mu)$ components of e.o.m give $\underline{\gamma^\rho_\rho = 0}$, $\underline{\partial_\rho \gamma^\mu_\rho = 0}$
- FIND that the $(\mu\text{-}\nu)$ components of e.o.m reduce to:

$$\left(1 + \frac{4\alpha\Lambda}{3}\right)\left[\Box_4 \gamma_{\mu\nu} + \partial_{\omega\omega}\gamma_{\mu\nu} - \frac{3}{\omega}\partial_\omega \gamma_{\mu\nu}\right] = 0$$

((hence : unless $1 + \frac{4\alpha\Lambda}{3} = 0$ ("crack of doom") Einstein's & E-G-B gravitons ARE THE SAME))
(Kim & Lee, Cho & Neupane ....)

Hence : the STATIC modes (which converge at $\omega \to \oplus\infty$) are: (just as in Einstein's, $\alpha = 0$, theory):

$$\gamma^{(s)}_{\mu\nu}(\omega, x^i) = \mathrm{Re}\int \frac{d^3k}{(2\pi)^{3/2}}\, e^{i\vec{k}\cdot\vec{x}}\, \underline{e_{\mu\nu}(\vec{k})}\, \omega^2 H_2^{(1)}(ik\omega)$$

(NB): if the background is NOT AdS_5, then the e.o.m for the bulk gravitons are NOT the same as in Einstein's theory

▲ 2nd step: metric on brane ($h_{\mu\nu}$)

$$ds^2|_5 = \left(\frac{\ell}{\omega}\right)^2 (\eta_{AB} + \gamma_{AB})\, dx^A dx^B ; \quad \omega = \ell + \xi(x^\mu)$$

$$\Rightarrow \quad ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})\, dx^\mu dx^\nu .$$

$$h_{\mu\nu} = \gamma_{\mu\nu}|_{\omega=\ell} - \frac{2\xi}{\ell}\eta_{\mu\nu}$$

(Remark: since Einstein's & EGB 5-D gravitons are solutions of the SAME eqns of motion, the metric on the brane is given by the same formula in Einstein & EGB Th.)

▲ 3rd step: matter on brane ($S_{\mu\nu}$)

$$K^\mu{}_\nu = -\frac{1}{\ell}\delta^\mu_\nu + \delta K^\mu{}_\nu ; \quad \delta K^\mu{}_\nu = -\frac{1}{2}\partial_\omega \gamma^\mu_\nu|_\Sigma + \partial^\mu_\nu \xi \quad \text{as before}$$

junction conditions $\quad T^\mu{}_\nu = 2\left[K^\mu{}_\nu - K\delta^\mu_\nu + 4\alpha\left(\text{"}KKK + K\bar{P}\text{"}\right)\right]$ Davis

FIND: $\quad T^\mu{}_\nu = -\frac{6}{\ell}\left(1 - \frac{4\alpha}{3\ell^2}\right) + \kappa S^\mu{}_\nu$

$$\frac{\kappa}{2}\left(S^\mu{}_\nu - \frac{1}{3}\eta^\mu{}_\nu S\right) = \partial^\mu_\nu \xi \left(1 + \frac{4\alpha}{\ell^2}\right) - \frac{1}{2}\left(1 - \frac{4\alpha}{\ell^2}\right)\partial_\omega \gamma^\mu_\nu|_\Sigma - \frac{2\alpha}{\ell^2}\Box \gamma_{\mu\nu}|_\Sigma$$

((If the geometry of the bulk is known (ie $\gamma_{\mu\nu}$ & ξ known) then the metric on brane & matter on brane are known))

(The $\neq$ between E & EGB is in the expression for $S_{\mu\nu}$ ONLY.)

▲ Compare with Einstein's 4D eqns

metric on brane : $ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu$; $h_{\mu\nu} = \gamma_{\mu\nu}|_\Sigma - \frac{2\zeta}{\mathcal{L}}\eta_{\mu\nu}$

go to harmonic gauge & compute $\Box_4 h^{(h)}_{\mu\nu} \equiv \Box_4 \gamma_{\mu\nu}|_\Sigma - \frac{2}{\mathcal{L}}\eta_{\mu\nu}\Box\zeta - \frac{4}{\mathcal{L}}\partial_{\mu\nu}\zeta$

matter on brane : $\left[S_{\mu\nu} = 2\left(1+\frac{4\alpha}{\mathcal{L}^2}\right)(\partial_{\mu\nu}\zeta - \eta_{\mu\nu}\Box_4\zeta) - \left(1-\frac{4\alpha}{\mathcal{L}^2}\right)\partial_w\gamma_{\mu\nu}|_\Sigma - 4\mathcal{L}\frac{\alpha}{\mathcal{L}^2}\Box_4\gamma_{\mu\nu}|_\Sigma \right]$

give $\begin{cases} \Box\zeta \text{ in fction of } S \\ \partial_{\mu\nu}\zeta \text{ in fction of } S_{\mu\nu},\ \partial_w\gamma_{\mu\nu}|_\Sigma \ \&\ \Box\gamma_{\mu\nu}|_\Sigma \end{cases}$

eqn of motion for bulk gravitons $0 = \Box_4\gamma_{\mu\nu}|_\Sigma + \partial_{ww}\gamma_{\mu\nu}|_\Sigma - \frac{3}{\mathcal{L}}\partial_w\gamma_{\mu\nu}$

gives $\Box_4\gamma_{\mu\nu}|_\Sigma$ in fction of $\partial_{ww}\gamma_{\mu\nu}$ & $\partial_w\gamma_{\mu\nu}$

$$\Box_4 h^{(h)}_{\mu\nu} = -\frac{2\kappa}{\mathcal{L}(1+4\alpha/\mathcal{L}^2)}\left(S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}S\right) + \frac{(1-4\alpha/\mathcal{L}^2)}{(1+4\alpha/\mathcal{L}^2)}\left(-\partial_{ww}\gamma_{\mu\nu}|_\Sigma + \frac{1}{\mathcal{L}}\partial_w\gamma_{\mu\nu}|_\Sigma\right)$$

Einstein's eqns $\Box_4 h^{(h)}_{\mu\nu} = -16\pi G\left(S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}S\right)$

hence identify $\quad 8\pi G = \frac{\kappa}{\mathcal{L}(1+4\alpha/\mathcal{L}^2)} \quad (*)$

hence Just like in Einstein's ($\alpha = 0$) theory, if there are no "zero modes" in the bulk (ie $\partial_w\gamma_{\mu\nu}|_\Sigma = \partial_{ww}\gamma_{\mu\nu}|_\Sigma = 0$) the metric & matter on brane satisfy the linearized Einstein equations (with $8\pi G$ given above $(*)$)

Correction to Newton's law

- matter on brane: $\frac{\kappa}{2}\left(S_{\mu\nu}-\frac{1}{3}\eta_{\mu\nu}S\right)=\partial_{\mu\nu}\xi\left(1+\frac{4\alpha}{\ell^2}\right)-\frac{1}{2}\left(1-\frac{4\alpha}{\ell^2}\right)\partial_w\gamma_{\mu\nu}|_\Sigma=$
$=\frac{2\alpha}{\ell^2}\Box\gamma_{\mu\nu}|_\Sigma$

- choose $\boxed{S_{00}=M\delta(\vec{r})}$; $S_{0i}\equiv 0$

- then: (1) $\xi=-\frac{\kappa M}{24\pi(1+4\alpha/\ell^2)}\frac{1}{r}$ is known

(2) so that $-\frac{1}{2}\left(1-\frac{4\alpha}{\ell^2}\right)\partial_w\gamma_{\mu\nu}|_\Sigma-\frac{2\alpha}{\ell^2}\Box\gamma_{\mu\nu}|_\Sigma$ is also known in terms of M

- Now the static bulk modes are also known:
$$\gamma_{\mu\nu}(w,\vec{k})=e_{\mu\nu}(\vec{k})\,w^2H_2^{(1)}(ikw)$$
hence the polarization $e_{\mu\nu}(\vec{k})$ is known in terms of M

- metric on brane $h_{\mu\nu}=\gamma_{\mu\nu}|_\Sigma-\frac{2\xi}{\ell}\eta_{\mu\nu}$

therefore: $h_{\mu\nu}$ is known in terms of M

one finds
$$\boxed{h_{00}^{(1)}(\vec{r})=\frac{2GM}{r}\left[1+\frac{4}{3\pi}K_\beta\right] \qquad K_\beta=\lim_{\varepsilon\to0}\int_0^\infty du\,\sin(\beta u)\frac{K_0(u)}{K_1(u)\left(1+\frac{4\alpha}{\ell^2}\right)+\frac{4\alpha}{\ell^2}uK_0(u)}e^{-\varepsilon u}}$$
$(\beta\equiv r/\ell)$

hence
$\alpha\to\infty$: $\simeq$ Same as in Einstein's theory
$\alpha\to0$: Gravity less attractive than $1/r^2$

Conclusion

- **The geometer view-point:**
  - Construct a brane world
  - Compute the extrinsic curvature of brane in bulk and get $\underline{T^\mu_\nu = K^\mu_\nu - \delta^\mu_\nu K + \alpha(KKK + \bar{R}K)}$
  - Compute the intrinsic curvature of brane and get $\underline{\sigma^\mu_\nu}$ (brane Einstein tensor)

  <u>What is the relation between σ^μ_ν & T^μ_ν</u> ?

Answer: for an almost flat brane in an almost quasi AdS_5 bulk:

$$\boxed{\Box_4 h^{(4)}_{\mu\nu} = -16\pi G\left(S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} S\right) + \frac{(1-4\alpha/\ell^2)}{(1+4\alpha/\ell^2)}\left(-\partial_{ww}\gamma_{\mu\nu}|_\Sigma + \frac{\partial_w \gamma_{\mu\nu}}{\ell}\Big|_\Sigma\right)}$$

Randall-Sundrum; Garriga-Tanaka; Neupane & al

- **The physicist view-point:**
  - I choose a source, hence a $T_{\mu\nu}$ (eg point source)
  - The "junction conditions" then give the bending of the brane and the perturbation of bulk compatible with this $T_{\mu\nu}$
  - I hence get the induced metric on brane and a modified Newton's potential:

$$\boxed{U = -\frac{GM}{r}\left(1 + \frac{4}{3\pi}K_3\right)}$$

Explicit form of $K(r)$ for all r: Deruelle-Sasaki.