# Present status of higher order post-Newtonian calculation and its difficulties

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# Introduction

**Coalescing Compact Binaries**

**Inspiralling phase**

$v_{orb} << c$

weak gravity(but strong internal gravity) ⇒ **Post-Newton approximation**

small tidal effect

**Merging phase** **Numerical Relativity**

**Ringdown phase** **BH Perturbation**

# Brief History of pN equation of motion

$$m_1 \frac{dv_1^i}{dt} = -\frac{m_1 m_2}{r_{12}^2} r_{12}^i$$

$$+e^2 F_{1pN}^i$$

$$+e^4 F_{2pN}^i$$

$$+e^5 F_{2.5pN}^i$$

**1979 PSR1913+16**

**~1990 GW Observatories**

$$+e^6 F_{3pN}^i$$

$$+O(e^7)$$

18c Newton

1917 Lorentz &Droste

- EIH, Fock

~1960 Chandrasekhar

~1970 Chandrasekhar and others

~1980 Damour, Blanchet, Schafer, Will,..
TF

~2000 Blanchet & Faya,
Jaranowski & Schafer
Pati & Will
Itoh, Asada and TF

# Ambiguities of 3pN

Blanchet and Faye

4 parameters associated with the use of singular source

2 gauged away and 1 determined by requiring energy conservation

1 parameter $l$ remain

Jaranowski and Schafer

2 unknown parameters $w_{static}$, $w_{kinetic}$
also associated with the use of singular source

Damour, Jaranowski and Schafer

Poincare Invariance $w_{kinetic} = \frac{41}{24}$

Equivalence between two approches $w_{static} = -\frac{11}{3} l - \frac{1987}{840}$

Dimensional regularization $w_{static} = 0,$

# Our approach

- No use of singular source

  regular source

  systematic introduction of multipole moments

- Strong internal gravity

  strong field point particle limit

  EIH type approach

- No assumption of regularized geodesic equation

  use of only local conservation law

# Formulation

## Equation of motion in terms of surface integral and strong field point particle limit

Basic equation

$$h^{mn} = h^{mn} - \sqrt{-g}\, g^{mn}$$

$$h^{mn}{}_{,n} = 0 \qquad \text{(Harmonic gauge)}$$

$$\square\, h^{mn} = -16\, p\ \Lambda^{mn} \qquad \text{Field equation}$$

$$\Lambda^{mn} = \Theta^{mn} + c^{mnab}{}_{,ab}\ , \qquad \Theta^{mn} = (-g)(T^{mn} + t^{mn}_{LL})$$

$$\Lambda^{mn}{}_{,n} = 0 \qquad \text{local conservation law}$$

$\Rightarrow$ Equation of motion in the surface integral form

# Equation of motion in the surface integral form

4-momentum

$$P_A^m \equiv \int_{B_A} d^3x \ \Lambda^{mt}$$

body zone of star A

$$\frac{dP_A^m}{dt} = -\oint_{\partial B_A} dS_k \ \Lambda^{km} + v_A^k \oint_{\partial B_A} dS_k \ \Lambda^{tm}$$

$$\Rightarrow \qquad m_A \frac{dv_A^i}{dt} = F_A^i$$

$$P_A^t \rightarrow m_A, \quad P_A^i = m_A v_A^i$$

momentum-velocity relation

# Momentum-velocity relation

$$\Lambda^{mn}{}_{,n} = 0$$

$$\Rightarrow \quad \Lambda^{ti} = (\Lambda^{tt} y_A^i)_{,t} + (\Lambda^{tj} y_A^i)_{,j} + v_A^i \Lambda^{tt}, \quad (y_A^i = y^i - z_A^i, \quad v_A^i = \frac{dz_A^i}{dt})$$

$$P_A^i = P_A^t \, v_A^i + Q_A^i + \frac{dD_A^i}{dt}$$

$$Q_A^i = \oint_{\partial B_A} dS_k \left( \Lambda^{tk} - v_A^k \, \Lambda^{tt} \right) y_A^i$$

$$D_A^i = \int_{B_A} d^3y \; \Lambda^{tt} \; y_A^i$$

Dipole moment of body A

# General form of equation of motion

$$P_A^t \frac{dv_A^i}{dt} = -\oint_{\partial B_A} dS_k \Lambda^{ki} + v_A^k \oint_{\partial B_A} dS_k \Lambda^{ti}$$

$$+ v_A^i \left( \oint_{\partial B_A} dS_k \Lambda^{kt} - v_A^k \oint_{\partial B_A} dS_k \Lambda^{tt} \right) - \frac{dQ_A^i}{dt} - \frac{d^2 D_A^i}{dt^2}$$

plus

$$\frac{dP_A^t}{dt} = -\oint_{\partial B_A} dS_k \Lambda^{kt} + v_A^k \oint_{\partial B_A} dS_k \Lambda^{tt}$$

$$\Lambda^{mn} = (-g)(T^{mn} + t_{LL}^{mn}) + c^{mnab}{}_{,ab}$$

Solve Field equation $\Box h^{mn} = -16 p \Lambda^{mn}$ by pN approximation

$$T^{mn} = O(e^4)$$

**pN equation of motion**

# Strong field point particle limit

$t = e\ t$ : Newtonian dynamical time

$$v^i_{orb} = \frac{dx^i}{dt} = e\,\frac{dx^i}{dt} : \qquad e \ \text{post-Newtonian expansion parameter}$$

Nearly Newtonian orbit ⇒ $m \propto e^2$

If the radius of star A scales like $R \propto e^2$,

$$\frac{m}{R} \propto const.$$

$$B_A = \left\{ \ x^i \mid \ | \vec{x} - \vec{z}_A(t) | < e \ \ R_A \right\} \qquad \text{Body zone}$$

Contribution from the body A at the body zone boundary can be estimated by the far zone expansion( multipole expansion)

# Scaling and ordering

Initial date

**A set of nearly stationary solutions of Einstein equation representing**

**Two widely separating star each of which rotates uniformly**

Strong point particle limit $(m \propto R \propto e^2) \Longrightarrow r \propto e^{-4} \quad (t, x^k)$

In the body zone coordinates $(t, a^{\underline{k}}) \qquad (A^{\underline{i}} = e^{-2} A^i)$

$$T^{tt} = O(e^{-2}), \quad T^{t\underline{i}} = O(e^{-5}), \quad T^{\underline{ij}} = O(e^{-8})$$

**Newton** **1PN** **2PN** **2.5PN** **3PN**

$$(-g)t^{tt}_{LL} = e^6{}_6[(-g)t^{tt}_{LL}] + e^8{}_8[(-g)t^{tt}_{LL}] + e^{10}{}_{10}[(-g)t^{tt}_{LL}] + O(e^{11})$$

$$(-g)t^{ti}_{LL} = e^6{}_6[(-g)t^{ti}_{LL}] + e^8{}_8[(-g)t^{ti}_{LL}] + e^9{}_9[(-g)t^{ti}_{LL}] + e^{10}{}_{10}[(-g)t^{ti}_{LL}] + O(e^{11})$$

$$(-g)t^{ij}_{LL} = e^4{}_4[(-g)t^{ij}_{LL}] + e^6{}_6[(-g)t^{ij}_{LL}] + e^8{}_8[(-g)t^{ij}_{LL}] + e^9{}_9[(-g)t^{ij}_{LL}] + e^{10}{}_{10}[(-g)t^{ij}_{LL}] + O(e^{11})$$

# (n-1)- pN EOM and field h

$$\frac{dP_A^t}{dt} = -\oint_{\partial B_A} dS_k \Lambda^{kt} + v_A^k \oint_{\partial B_A} dS_k \Lambda^{tt}$$

n-pN

$$Q_A^i = \oint_{\partial B_A} dS_k \left( \Lambda^{tk} - v_A^k \Lambda^{tt} \right) y_A^i$$

$$P_A^i = P_A^t v_A^i + Q_A^i + \frac{dD_A^i}{dt}$$

n-th order field and EOM

## Newtonian order

$$h^{tt} = e^4 {}_4h^{tt} = 4e^4 \sum_A \frac{m_A}{r_A} + O(e^6)$$

$$(Q_A^i)_{Newton} = 0$$

$$\frac{dP_A^t}{dt} = O(e^2) \Rightarrow m_A = \lim_{e\to 0} P_A^t$$

$$P_A^i = m_A v_A^i + O(e^2)$$

$$m_1 \frac{dv_1^i}{dt} = -\int_{\partial B_1} dS_k \left[(-g) t_{LL}^{ik}\right]_{Newton}$$

$$= -\frac{m_1 m_2}{r_{12}^3} r_{12}^i$$

# Details of calculation

$$h^{mn}(t,\vec{x}) = 4\int d^3y \; \frac{\Lambda^{mn}(t-e\,|\vec{x}-\vec{y}|,\;\vec{y})}{|\vec{x}-\vec{y}|}$$

$$= 4\Sigma_A \int_{B_A} \qquad\qquad + \int_{N/B}$$

Multipole expansion

Super potential + other method

Super potential $\Delta\; g(\vec{x}) = f(\vec{x})$

$$\int_{B_A} d^3y \frac{f(\vec{y})}{|\vec{x}-\vec{y}|} = -4p\;\; g(\vec{x}) + \oint_{\partial(N/B)} dS_k \left[ \frac{1}{|\vec{x}-\vec{y}|} \frac{\partial g(\vec{y})}{\partial y^k} - g(\vec{y}) \frac{\partial}{\partial y^k} \left( \frac{1}{|\vec{x}-\vec{y}|} \right) \right]$$

Example:

$$\Delta g = \frac{1}{r_1 r_2} \Rightarrow g(\vec{x}) = \ln S \quad \text{with} \quad S = r_1 + r_2 + r_{12}$$

# Newtonian order

$$h_B^{mn} = O(e^4)$$

$$\Lambda^{tt} = \Lambda^{ti} = O(e^6)$$

$$\Lambda^{ij} = e^4 \frac{1}{64p}\left( d_k^i d_l^j - \frac{1}{2} d^{ij} d_{kl} \right) 4 h_B^{tt,k}\, 4 h_B^{tt,l}$$

$$\Rightarrow \quad \frac{dP_A^t}{dt} = O(e^2) \quad \Rightarrow \quad m_A = \lim_{e \to 0} P_A^t \qquad \text{(ADM mass)}$$

$r_A = | \vec{x} - \vec{z}_A(t) |$

$$h^{tt}{}_B(t, x^k) = 4e^6 \int_B d^3a \, \frac{\Lambda^{tt}}{| x^i - e^2 a^i |} = 4e^4 \sum_A \frac{m_A}{r_A}$$

Up to 2.5 pN order we can find appropriate super potential to calculate near zone field

$$\int_{N/B} \frac{d^3y}{|\vec{x}-\vec{y}|}\frac{r_A^i r_A^j}{r_A^6} = \frac{1}{8}\left(d_k^i d_l^j + d^{ij} d_{kl}\right)\int_{N/B} d^3y \frac{(\Delta \ln r_A)^{,kl}}{|\vec{x}-\vec{y}|}$$

$$\int_{N/B} \frac{d^3y}{|\vec{x}-\vec{y}|}\frac{r_1^i r_2^j}{r_1^3 r_2^3} = \int_{N/B} d^3y \frac{1}{|\vec{x}-\vec{y}|}\frac{\partial}{\partial z_1^i}\frac{\partial}{\partial z_2^j}(\Delta \ln S)$$

$$\begin{aligned}4\int_{N/B} d^3y \frac{{}_6\left[-g t_{LL}^{ti}\right]}{|\vec{x}-\vec{y}|} &= \sum_A \frac{(P_A^t)^2}{r_A^2}\left\{(\vec{n}_A \cdot \vec{v}_A) n_A^i + 7 v_A^i\right\} + \frac{4 P_1^t P_2^t}{S r_{12}}(v_{1k}+v_{2k})(d^{ki} - n_{12}^i n_{12}^k) \\ &+ 8 P_1^t P_2^t (v_1^i + v_2^i)\left(-\frac{1}{r_{12}}\sum_A \frac{1}{r_A} + \frac{1}{r_1 r_2}\right) \\ &- 16\frac{P_1^t P_2^t}{S^2}\left\{v_1^k (n_{12}^i - n_1^i)(n_{12}^k + n_2^k) + v_2^k (n_{12}^k - n_1^k)(n_{12}^i + n_2^i)\right\} \\ &+ 12\frac{P_1^t P_2^t}{S^2}\left\{v_1^k (n_{12}^k - n_1^k)(n_{12}^i + n_2^i) + v_2^k (n_{12}^i - n_1^i)(n_{12}^k + n_2^k)\right\}\end{aligned}$$

# 3pN EOM

$$\left(\frac{dP_1^t}{dt}\right)_{\le 3PN} = \left(\frac{dP_1^t}{dt}\right)_{\le 2.5N} + e^6\left[-\oint_{\partial B_1} dS_{k\,10}\Lambda_N^{tk} + v_1^k \oint_{\partial B_1} dS_{k\,10}\Lambda_N^{tt}\right]$$

$$m_1\left(\frac{dv_1^i}{dt}\right)_{\le 3PN} = \left(m_1\frac{dv_1^i}{dt}\right)_{\le 2.5N} + e^6\left[-\oint_{\partial B_1} dS_{k\,10}\Lambda_N^{ki} + v_1^k \oint_{\partial B_1} dS_{k\,10}\Lambda_N^{ti}\right]$$

$$+ e^6\left(\frac{dP_1^t}{dt}\right)_{3PN} v_1^i + e^6\left((m_1 - P_1^t\,\frac{dv_1^i}{dt}\right)_{3PN} - e^6\frac{d({}_6Q_1^i)}{dt} - e^6\frac{d^2({}_4D_1^i)}{dt^2}$$

Problem:

All of necessary super potential can not be obtained

We only need surface integral at body zone boundary

Expand the field around the boundary

$$\frac{1}{|\vec{r}_1-\vec{y}_1|}=\sum_n \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^n P_n\left(\frac{\vec{r}_1\cdot\vec{y}_1}{r_1 y_1}\right)$$

Evaluate surface integral directly

$$\oint_{\partial B_1} dS_k \quad {}_4h^{tt,l}\int_{N/B}\frac{d^3y}{|\vec{x}-\vec{y}|}\quad {}_8\Lambda^{ti}$$

$$\oint_{\partial B_1} dS_k \frac{r_2^l}{r_2^3}\int_{N/B} d^3y \frac{f(\vec{y}_1)}{|\vec{x}_1-\vec{y}_1|} \approx \int_{N/B} d^3y_1 f(\vec{y}) \oint_{\partial B_1} dS_k \frac{1}{|\vec{y}_1-r_1\vec{n}_{1|}}\partial_{z_2^l}\frac{1}{|\vec{r}_{12}-r_1\vec{n}_{1|}|}$$

# 3pN mass-energy relation

# 3pN momentum-velocity relation

$$Q_1^i = e^6 \frac{d}{dt}\left(\frac{1}{6} m_1^3 a_1^i\right)$$

$$P_1^i = P_1^t v_1^i + e^6 \frac{d}{dt}\left(\frac{1}{6} m_1^3 a_1^i\right) + e^2 \frac{dD_1^i}{dt}$$

Possible choice $\quad D_A^i = -e^4 \frac{1}{6} m_A^3 a_A^i \equiv e^4 d_A^i$

Then metric changes as

$$h^{tt}\,|_{d_A} = 4e^{10} \sum_A \frac{d_A^k r_A^k}{r_A^3} + O(e^{11})$$

Corresponding acceleration

$$m_1 a_1^i\,|_{d_A} = -e^6 \frac{3 m_1 d_2^i}{r_{12}^3} n_{12}^{<ik>} + e^6 \frac{3 m_2 d_1^i}{r_{12}^3} n_{12}^{<ik>}$$

# Logarithmic dependence of 3pN EOM

$$m_1\left(\frac{dv_1^i}{dt}\right)=m_1\left(\frac{dv_1^i}{dt}\right)_{\leq 2.5\,pN}$$

$$+e^6\,\frac{m_1^2 m_2}{r_{12}^3}\left[\frac{44 m_1 m_2}{3 r_{12}^2}\,n_{12}^i\,\ln\left(\frac{r_{12}}{eR_1}\right)-\frac{44 m_2^2}{3 r_{12}^2}\,n_{12}^i\,\ln\left(\frac{r_{12}}{eR_2}\right)\right]$$

$$+e^6\,\frac{m_1^3 m_2}{r_{12}^4}\left(5(\vec{n}_{12}\cdot\vec{V})^2 n_{12}^i - V^2 n_{12}^i - 2(\vec{n}_{12}\cdot\vec{V})V^i\right)\ln\left(\frac{r_{12}}{eR_1}\right)$$

+ other terms

If we choose

$$D_A^i = e^4 d_A - e^4 \frac{22}{3} m_A^3 a_A^i \ln\left( \frac{r_{12}}{eR_A} \right)$$

Then the correspond acceleration cancel the logarithmic dependence of 3pN EOM

# 3pN EOM

**Our equation has no ambiguous parameters, admits automatically conservation of an orbital energy of the binary system and gives a correct geodesic equation in the test particle limit, is Lorentz invariant.**

## Relation with BF

$$m_1 a_1^{Our} = m_1 (a_1^{BF})_{l=-1987/3080} + m_1 a_1 |_{d_{A,BF}}$$

Acceleration induced by

$$d^i_{A,BF} = -\frac{3709}{1260} m_A^3 a^i_A$$

$l = -\frac{1987}{3080}$ corresponds to $w_{static} = 0$

# Is 4pN calculation possible?

**NO**, because

We do not have explicit expression
for 3 pN field in near zone

~100,000 terms in 3pN order

~10,000,000 terms in 4pN order

**New approximation scheme??**