

# Present status of higher order post-Newtonian calculation and its difficulties

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# Introduction

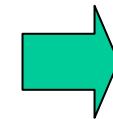
## Coalescing Compact Binaries

### Inspiralling phase

$$v_{orb} \ll c$$

weak gravity (but strong internal gravity)

small tidal effect



Post-Newton  
approximation

### Merging phase

Numerical Relativity

### Ringdown phase

BH Perturbation

# Brief History of pN equation of motion

$$m_1 \frac{dv_1^i}{dt} = - \frac{m_1 m_2}{r_{12}^2} r_{12}^i$$

18c Newton

$$+ e^2 F_{1pN}^i$$

1917 Lorentz & Droste

• EIH, Fock

$$+ e^4 F_{2pN}^i$$

~1960 Chandrasekhar

$$+ e^5 F_{2.5pN}^i$$

~1970 Chandrasekhar and others

1979 PSR1913+16

$$+ e^6 F_{3pN}^i$$

~1980 Damour, Blanchet, Schafer, Will,..

~1990 GW

Observatories

TF

~2000 Blanchet & Faya,

Jaranowski & Schafer

Pati & Will

Itoh, Asada and TF

$$+ O(e^7)$$

# Ambiguities of 3pN

Blanchet and Faye

4 parameters associated with the use of singular source

2 gauged away and 1 determined by requiring energy conservation

1 parameter  $\mathbf{I}$  remain

Jaranowski and Schafer

2 unknown parameters  $\mathbf{w}_{static}$ ,  $\mathbf{w}_{kinetic}$   
also associated with the use of singular source

Damour, Jaranowski and Schafer

Poincare Invariance       $\mathbf{w}_{kinetic} = \frac{41}{24}$

Equivalence between two approaches       $\mathbf{w}_{static} = -\frac{11}{3}\mathbf{I} - \frac{1987}{840}$

Dimensional regularization       $\mathbf{w}_{static} = 0,$

# Our approach

- No use of singular source
  - regular source
  - systematic introduction of multipole moments
- Strong internal gravity
  - strong field point particle limit
  - EIH type approach
- No assumption of regularized geodesic equation
  - use of only local conservation law

# Formulation

## Equation of motion in terms of surface integral and strong field point particle limit

Basic equation

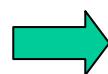
$$h^{\text{m}\text{n}} = \mathbf{h}^{\text{m}\text{n}} - \sqrt{-g} g^{\text{m}\text{n}}$$

$$h^{\text{m}\text{n}}_{,\text{n}} = 0 \quad (\text{Harmonic gauge})$$

$h^{\text{m}\text{n}} = -16 \mathbf{p} \Lambda^{\text{m}\text{n}}$  Field equation

$$\Lambda^{\text{m}\text{n}} = \Theta^{\text{m}\text{n}} + \mathbf{C}^{\text{m}\text{n}\text{a}\text{b}},_{\text{a}\text{b}} , \quad \Theta^{\text{m}\text{n}} = (-g)(T^{\text{m}\text{n}} + t^{\text{m}\text{n}}_{LL})$$

$$\Lambda^{\text{m}\text{n}}_{,\text{n}} = 0 \quad \text{local conservation law}$$



Equation of motion in the surface  
integral form

# Equation of motion in the surface integral form

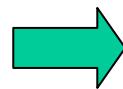
4-momentum

$$P_A^m \equiv \int_{B_A} d^3x \Lambda^{mt}$$



body zone of star A

$$\frac{dP_A^m}{dt} = - \oint_{\partial B_A} dS_k \Lambda^{km} + v_A^k \oint_{\partial B_A} dS_k \Lambda^{tm}$$



$$m_A \frac{dv_A^i}{dt} = F_A^i$$

$$P_A^t \rightarrow m_A, \quad P_A^i = m_A v_A^i$$

momentum-velocity relation

# Momentum-velocity relation

$$\Lambda^{\mathbf{m}}_{\mathbf{n}} = 0$$

$$\rightarrow \quad \Lambda^{ti} = (\Lambda^{tt} y_A^i)_{,t} + (\Lambda^{tj} y_A^i)_{,j} + v_A^i \Lambda^{tt}, \quad (y_A^i = y^i - z_A^i, \quad v_A^i = \frac{dz_A^i}{dt})$$

$$P_A^i = P_A^t v_A^i + Q_A^i + \frac{dD_A^i}{d\mathbf{t}}$$

$$Q_A^i = \oint_{\partial B_A} dS_k \left( \Lambda^{tk} - v_A^k \Lambda^{tt} \right) y_A^i$$

$$D_A^i = \int_{B_A} d^3y \Lambda^{tt} y_A^i \quad \text{Dipole moment of body A}$$

# General form of equation of motion

$$P_A^t \frac{dv_A^i}{dt} = - \oint_{\partial B_A} dS_k \Lambda^{ki} + v_A^k \oint_{\partial B_A} dS_k \Lambda^{ti} \\ + v_A^i \left( \oint_{\partial B_A} dS_k \Lambda^{kt} - v_A^k \oint_{\partial B_A} dS_k \Lambda^{tt} \right) - \frac{dQ_A^i}{dt} - \frac{d^2 D_A^i}{dt^2}$$

plus

$$\frac{dP_A^t}{dt} = - \oint_{\partial B_A} dS_k \Lambda^{kt} + v_A^k \oint_{\partial B_A} dS_k \Lambda^{tt}$$

$$\Lambda^{mn} = (-g)(T^{mn} + t_{LL}^{mn}) + C^{mna b}_{\phantom{mna b},ab}$$

Solve Field equation  $\square h^{mn} = -16 p \Lambda^{mn}$  by pN approximation

$$T^{mn} = O(\epsilon^4)$$



**pN equation of motion**

# Strong field point particle limit

$t = e \ t$  : Newtonian dynamical time

$$v_{orb}^i = \frac{dx^i}{dt} = e \frac{dx^i}{dt} : \quad e \text{ post-Newtonian expansion parameter}$$

Nearly Newtonian orbit   $m \propto e^2$

If the radius of star A scales like  $R \propto e^2$ ,

$$\frac{m}{R} \propto \text{const.}$$

$$B_A = \left\{ x^i \mid |\vec{x} - \vec{z}_A(t)| < e R_A \right\} \quad \text{Body zone}$$

Contribution from the body A at the body zone boundary can be estimated by the far zone expansion( multipole expansion)

# Scaling and ordering

Initial date

**A set of nearly stationary solutions of Einstein equation representing  
Two widely separating star each of which rotates uniformly**

Strong point particle limit ( $m \propto R \propto e^2$ )  $\rightarrow r \propto e^{-4}$  ( $t, x^k$ )

In the body zone coordinates  $(t, a^k)$   $(A^i = e^{-2} A^i)$

$$T^{tt} = O(e^{-2}), \quad T^{ti} = O(e^{-5}), \quad T^{ij} = O(e^{-8})$$

Newton	1PN	2PN	2.5PN	3PN
$(-g)t_{LL}^{tt} =$	$e^6{}_6[(-g)t_{LL}^{tt}] + e^8{}_8[(-g)t_{LL}^{tt}]$			$+ e^{10}_{10}[(-g)t_{LL}^{tt}] + O(e^{11})$
$(-g)t_{LL}^{ti} =$	$e^6{}_6[(-g)t_{LL}^{ti}] + e^8{}_8[(-g)t_{LL}^{ti}] + e^9{}_9[(-g)t_{LL}^{ti}]$	$+ e^{10}_{10}[(-g)t_{LL}^{ti}] + O(e^{11})$		
$(-g)t_{LL}^{ij} =$	$e^4{}_4[(-g)t_{LL}^{ij}] + e^6{}_6[(-g)t_{LL}^{ij}] + e^8{}_8[(-g)t_{LL}^{ij}] + e^9{}_9[(-g)t_{LL}^{ij}]$	$+ e^{10}_{10}[(-g)t_{LL}^{ij}] + O(e^{11})$		

(n-1)- pN EOM and field h

$$\frac{dP_A^t}{dt} = - \oint_{\partial B_A} dS_k \Lambda^{kt} + v_A^k \oint_{\partial B_A} dS_k \Lambda^{tt} \quad \text{n-pN}$$

$$Q_A^i = \oint_{\partial B_A} dS_k \left( \Lambda^{tk} - v_A^k \Lambda^{tt} \right) y_A^i$$

$$P_A^i = P_A^t v_A^i + Q_A^i + \frac{dD_A^i}{dt}$$



n-th order field and EOM

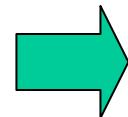
Newtonian order

$$h^{tt} = \epsilon^4 {}_4 h^{tt} = 4\epsilon^4 \sum_A \frac{m_A}{r_A} + O(\epsilon^6)$$

$$(Q_A^i)_{\text{Newton}} = 0$$

$$\frac{dP_A^t}{dt} = O(\epsilon^2) \Rightarrow m_A = \lim_{\epsilon \rightarrow 0} P_A^t$$

$$P_A^i = m_A v_A^i + O(\epsilon^2)$$



$$\begin{aligned} m_1 \frac{dv_1^i}{dt} &= - \int_{\partial B_1} dS_k [(-g) t_{LL}^{ik}]_{\text{Newton}} \\ &= - \frac{m_1 m_2}{r_{12}^3} r_{12}^i \end{aligned}$$

# Details of calculation

$$h^{\text{mm}}(t, \vec{x}) = 4 \int d^3y \frac{\Lambda^{\text{mm}}(\mathbf{t} - \mathbf{e} | \vec{x} - \vec{y}|, \vec{y})}{| \vec{x} - \vec{y}|}$$

$$= 4\Sigma_A \int_{B_A} + \int_{N/B}$$

Multipole  
expansion

Super potential + other  
method

Super potential       $\Delta g(\vec{x}) = f(\vec{x})$

$$\int_{B_A} d^3y \frac{f(\vec{y})}{| \vec{x} - \vec{y}|} = -4p g(\vec{x}) + \oint_{\partial(N/B)} dS_k \left[ \frac{1}{| \vec{x} - \vec{y}|} \frac{\partial g(\vec{y})}{\partial y^k} - g(\vec{y}) \frac{\partial}{\partial y^k} \left( \frac{1}{| \vec{x} - \vec{y}|} \right) \right]$$

Example:

$$\Delta g = \frac{1}{r_1 r_2} \Rightarrow g(\vec{x}) = \ln S \quad \text{with} \quad S = r_1 + r_2 + r_{12}$$

# Newtonian order

$$h_B^{\text{mm}} = O(\epsilon^4)$$

$$\Lambda^{tt} = \Lambda^{ti} = O(\epsilon^6)$$

$$\Lambda^{ij} = \epsilon^4 \frac{1}{64p} \left( \mathbf{d}_k^i \mathbf{d}_l^j - \frac{1}{2} \mathbf{d}^{ij} \mathbf{d}_{kl} \right)_4 h_B^{tt,k} {}_4 h_B^{tt,l}$$

$$\xrightarrow{} \quad \frac{dP_A^t}{dt} = O(\epsilon^2) \quad \xrightarrow[r_A = |\vec{x} - \vec{z}_A(t)|]{} m_A = \lim_{\epsilon \rightarrow 0} P_A^t \quad (\text{ADM mass})$$

$$h^{tt}_B(t, x^k) = 4\epsilon^6 \int_B d^3 \mathbf{a} \frac{\Lambda^{tt}}{|x^i - \epsilon^2 \mathbf{a}^i|} = 4\epsilon^4 \sum_A \frac{m_A}{r_A}$$

Up to 2.5 pN order we can find appropriate super potential to calculate near zone field

$$\int_{N/B} \frac{d^3 y}{|\vec{x} - \vec{y}|} \frac{r_A^i r_A^j}{r_A^6} = \frac{1}{8} (\mathbf{d}_k^i \mathbf{d}_l^j + \mathbf{d}^{ij} \mathbf{d}_{kl}) \int_{N/B} d^3 y \frac{(\Delta \ln r_A)^{,kl}}{|\vec{x} - \vec{y}|}$$

$$\int_{N/B} \frac{d^3 y}{|\vec{x} - \vec{y}|} \frac{r_1^i r_2^j}{r_1^3 r_2^3} = \int_{N/B} d^3 y \frac{1}{|\vec{x} - \vec{y}|} \frac{\partial}{\partial z_1^i} \frac{\partial}{\partial z_2^j} (\Delta \ln S)$$

$$\begin{aligned} 4 \int_{N/B} d^3 y \frac{6 \left[ -gt_{LL}^{ti} \right]}{|\vec{x} - \vec{y}|} &= \sum_A \frac{(P_A^t)^2}{r_A^2} \left\{ (\vec{n}_A \cdot \vec{v}_A) n_A^i + 7 v_A^i \right\} + \frac{4 P_1^t P_2^t}{S r_{12}} (v_{1k} + v_{2k}) (\mathbf{d}^{ki} - n_{12}^i n_{12}^k) \\ &+ 8 P_1^t P_2^t (v_1^i + v_2^i) \left( -\frac{1}{r_{12}} \sum_A \frac{1}{r_A} + \frac{1}{r_1 r_2} \right) \\ &- 16 \frac{P_1^t P_2^t}{S^2} \left\{ v_1^k (n_{12}^i - n_1^i) (n_{12}^k + n_2^k) + v_2^k (n_{12}^k - n_1^k) (n_{12}^i + n_2^i) \right\} \\ &+ 12 \frac{P_1^t P_2^t}{S^2} \left\{ v_1^k (n_{12}^k - n_1^k) (n_{12}^i + n_2^i) + v_2^k (n_{12}^i - n_1^i) (n_{12}^k + n_2^k) \right\} \end{aligned}$$

# 3pN EOM

$$\begin{aligned}
 \left( \frac{dP_1^t}{dt} \right)_{\leq 3PN} &= \left( \frac{dP_1^t}{dt} \right)_{\leq 2.5N} + \mathbf{e}^6 \left[ - \oint_{\partial B_1} dS_{k10} \Lambda_N^{tk} + v_1^k \oint_{\partial B_1} dS_{k10} \Lambda_N^{tt} \right] \\
 m_1 \left( \frac{dv_1^i}{dt} \right)_{\leq 3PN} &= \left( m_1 \frac{dv_1^i}{dt} \right)_{\leq 2.5N} + \mathbf{e}^6 \left[ - \oint_{\partial B_1} dS_{k10} \Lambda_N^{ki} + v_1^k \oint_{\partial B_1} dS_{k10} \Lambda_N^{ti} \right] \\
 &\quad + \mathbf{e}^6 \left( \frac{dP_1^t}{dt} \right)_{3PN} v_1^i + \mathbf{e}^6 \left( (m_1 - P_1^t) \frac{dv_1^i}{dt} \right)_{3PN} - \mathbf{e}^6 \frac{d( {}_6 Q_1^i )}{dt} - \mathbf{e}^6 \frac{d^2 ( {}_4 D_1^i )}{dt^2}
 \end{aligned}$$

Problem:

All of necessary super potential can not be obtained

We only need surface integral at body zone boundary

Expand the field around the boundary

$$\frac{1}{|\vec{r}_1 - \vec{y}_1|} = \sum_n \frac{1}{r_>} \left( \frac{r_<}{r_>} \right)^n P_n \left( \frac{\vec{r}_1 \cdot \vec{y}_1}{r_1 y_1} \right)$$

Evaluate surface integral directly

$$\oint_{\partial B_1} dS_k \int_{N/B} h^{tt,l} \int \frac{d^3 y}{|\vec{x} - \vec{y}|} {}^8\Lambda^{ti}$$

$$\oint_{\partial B_1} dS_k \frac{r_2^l}{r_2^3} \int_{N/B} d^3 y \frac{f(\vec{y}_1)}{|\vec{x}_1 - \vec{y}_1|} \approx \int_{N/B} d^3 y_1 f(\vec{y}) \oint_{\partial B_1} dS_k \frac{1}{|\vec{y}_1 - r_1 \vec{n}_{1l}|} \partial_{z_2^l} \frac{1}{|\vec{r}_{12} - r_1 \vec{n}_{1l}|}$$

# 3pN mass-energy relation

# 3pN momentum-velocity relation

$$Q_1^i = \mathbf{e}^6 \frac{d}{dt} \left( \frac{1}{6} m_1^3 a_1^i \right)$$

$$P_1^i = P_1^t v_1^i + \mathbf{e}^6 \frac{d}{dt} \left( \frac{1}{6} m_1^3 a_1^i \right) + \mathbf{e}^2 \frac{dD_1^i}{dt}$$

Possible choice  $D_A^i = -\mathbf{e}^4 \frac{1}{6} m_A^3 a_A^i \equiv \mathbf{e}^4 \mathbf{d}_A^i$

Then metric changes as

$$h^{tt} |_{\mathbf{d}_A} = 4\mathbf{e}^{10} \sum_A \frac{\mathbf{d}_A^k r_A^k}{r_A^3} + O(\mathbf{e}^{11})$$

Corresponding acceleration

$$m_1 a_1^i |_{\mathbf{d}_A} = -\mathbf{e}^6 \frac{3m_1 \mathbf{d}_2^i}{r_{12}^3} n_{12}^{<ik>} + \mathbf{e}^6 \frac{3m_2 \mathbf{d}_1^i}{r_{12}^3} n_{12}^{<ik>}$$

# Logarithmic dependence of 3pN EOM

$$\begin{aligned} m_1 \left( \frac{dv_1^i}{dt} \right) &= m_1 \left( \frac{dv_1^i}{dt} \right)_{\leq 2.5 \text{ pN}} \\ &+ e^6 \frac{m_1^2 m_2}{r_{12}^3} \left[ \frac{44m_1 m_2}{3r_{12}^2} n_{12}^i \ln \left( \frac{r_{12}}{eR_1} \right) - \frac{44m_2^2}{3r_{12}^2} n_{12}^i \ln \left( \frac{r_{12}}{eR_2} \right) \right] \\ &+ e^6 \frac{m_1^3 m_2}{r_{12}^4} \left( 5(\vec{n}_{12} \cdot \vec{V})^2 n_{12}^i - V^2 n_{12}^i - 2(\vec{n}_{12} \cdot \vec{V})V^i \right) \ln \left( \frac{r_{12}}{eR_1} \right) \\ &+ \text{other terms} \end{aligned}$$

If we choose

$$D_A^i = e^4 d_A - e^4 \frac{22}{3} m_A^3 a_A^i \ln\left(\frac{r_{12}}{eR_A}\right)$$

Then the correspond acceleration cancel the logarithmic dependence of 3pN EOM

## 3pN EOM

Our equation has no ambiguous parameters, admits automatically conservation of an orbital energy of the binary system and gives a correct geodesic equation in the test particle limit, is Lorentz invariant.

### Relation with BF

$$m_1 a_1^{Our} = m_1 (a_1^{BF})_{I=-1987/3080} + m_1 a_1 |_{\mathbf{d}_{A,BF}}$$

Acceleration induced by

$$\mathbf{d}_{A,BF}^i = -\frac{3709}{1260} m_A^3 a_A^i$$

$$I = -\frac{1987}{3080} \quad \text{corresponds to} \quad \mathbf{w}_{static} = 0$$

# Is 4pN calculation possible?

**NO**, because

We do not have explicit expression  
for 3 pN field in near zone

~100,000 terms in 3pN order

~10,000,000 terms in 4pN order

New approximation scheme??