Improvement on the metric reconstruction scheme in the Regge-Wheeler-Zerilli formalism

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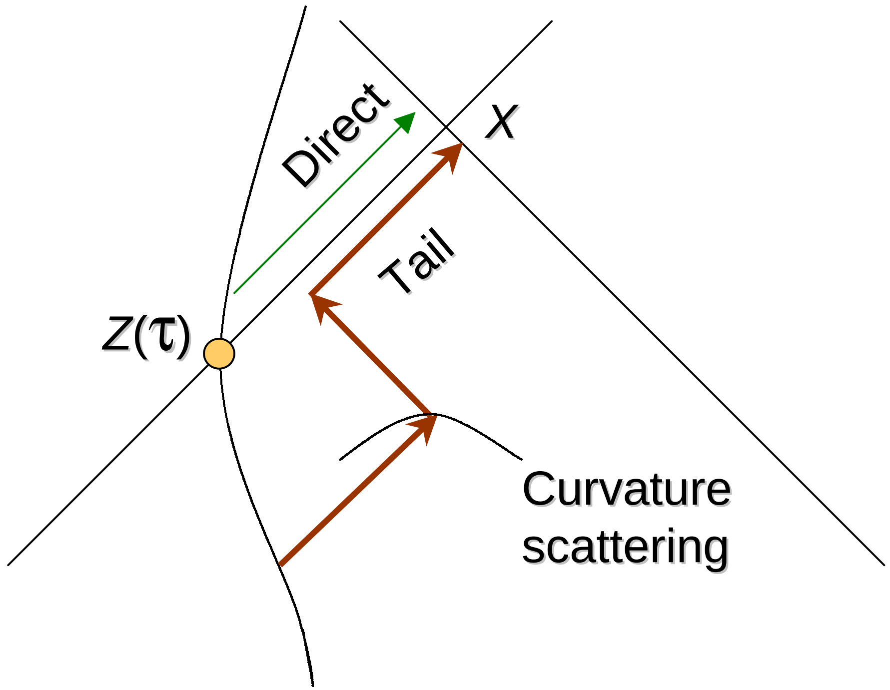
Motivation

- Gauge invariant regularisation:



- Natural choice for Master variable:

we calculate the metric perturbations starting with the original one $\tilde{h}$ through the gauge invariant variables as

$$\tilde{h}_{\mu\nu} \to \zeta \to \bar{h}_{\mu\nu}$$

Estimate the difference

$$\Delta h_i = \tilde{h}_i - \bar{h}_i$$

We will conclude that $\Delta h_i = O(\varepsilon^n)$ if $\bar{h}_i$ satisfies Einstein equations to $O(\varepsilon^n)$.

Odd parity case

1. Construct a specific master variable

$$\zeta = -\frac{r}{(l-1)(l+2)}\left[\tilde{h}_{1,t} - \tilde{h}_{0,r} + \frac{2}{r}\tilde{h}_0\right]$$

2. Metric perturbations can be written as

$$h_{\mu\nu} = h_0(e0)_{\mu\nu} + h_1(e1)_{\mu\nu} + h_2(e2)_{\mu\nu}$$

e0, e1 and e2 are, respectively, angular harmonics and their first and second derivatives

3. From the leading order behavior of mode functions, the harmonic coefficients could be expanded as

$$\tilde{h}_i \sim \tilde{h}_i^{(0)} + \tilde{h}_i^{(-1)} + \tilde{h}_i^{(-2)}$$

4. Decompose $\tilde{h}$ as

$$\tilde{h} := \tilde{h}^{trun} + \tilde{h}^{rem}$$

For this particular master variable

$$\zeta^{rem} = \mathcal{O}(l^{-4}) \text{ and } \Delta h_0^{trun} = \mathcal{O}(l^{-3}), \Delta h_1^{trun} = \mathcal{O}(l^{-3})$$

Outline

- Background on the Regge-Wheeler-Zerilli perturbation formalism

- Improved master variables
  - Odd parity modes
  - Even parity modes

- Master variables via Chandrasekhar transformation

Basics

T. Regge and J. Wheeler, Phys. Rev. **108** 1063 (1957)

Background metric - Schwarzschild

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

Odd Parity modes	h_0, h_1, h_2
Even Parity modes	$H_0, H_1, H_2, h_0^{(e)}, h_1^{(e)}, K, G$

RW GAUGE $\quad h_2^{RW} = 0, \quad h_0^{(e),RW} = h_1^{(e),RW} = G^{RW} = 0$

Assuming time dependence $\quad \exp(-i\omega t)$

Master equation $\quad \dfrac{d^2\zeta}{dr^{*2}} + (\omega^2 - V)\zeta = \mathcal{S}(T_{\mu\nu})$

Reconstructed Metric Components

$$h^{RW} = \hat{h}^{(M)}(\zeta) + \hat{h}^{(T)}(T_{\mu\nu})$$

Odd Parity Modes

$$h^{RW}_{0,rr} + i\omega h^{RW}_{1,r} + i\omega\frac{2}{r}h^{RW}_1 + \left[\frac{4M}{r} - 2(1+\lambda)\right]\frac{h^{RW}_0}{r(r-2M)} = \frac{8\pi}{\sqrt{1+\lambda}}\frac{r^2}{r-2M}Q^{(0)}$$

$$-\omega^2 h^{RW}_1 + i\omega h^{RW}_{0,r} + 2\lambda(r-2M)\frac{h^{RW}_1}{r^3} - i\omega\frac{2}{r}h^{RW}_0 = -\frac{8\pi i}{\sqrt{1+\lambda}}(r-2M)Q$$

$$\left(1-\frac{2M}{r}\right)h^{RW}_{1,r} + i\omega\left(1-\frac{2M}{r}\right)^{-1}h^{RW}_0 + \frac{2M}{r^2}h^{RW}_1 = -\frac{8\pi i}{\sqrt{2\lambda(1+\lambda)}}r^2 D$$

Master Variable $\quad {}^{(o)}\chi = \frac{r-2M}{r^2}h^{RW}_1$

Master equation $\quad \left[\partial^2_{r*} + \omega^2 - V^{RW}_l(r)\right]{}^{(o)}\chi = \mathcal{S}^{(o)\chi}$

Potential $\quad V^{RW}_l = \left(1-\frac{2M}{r}\right)\left(\frac{2(\lambda+1)}{r^2} - \frac{6M}{r^3}\right)$

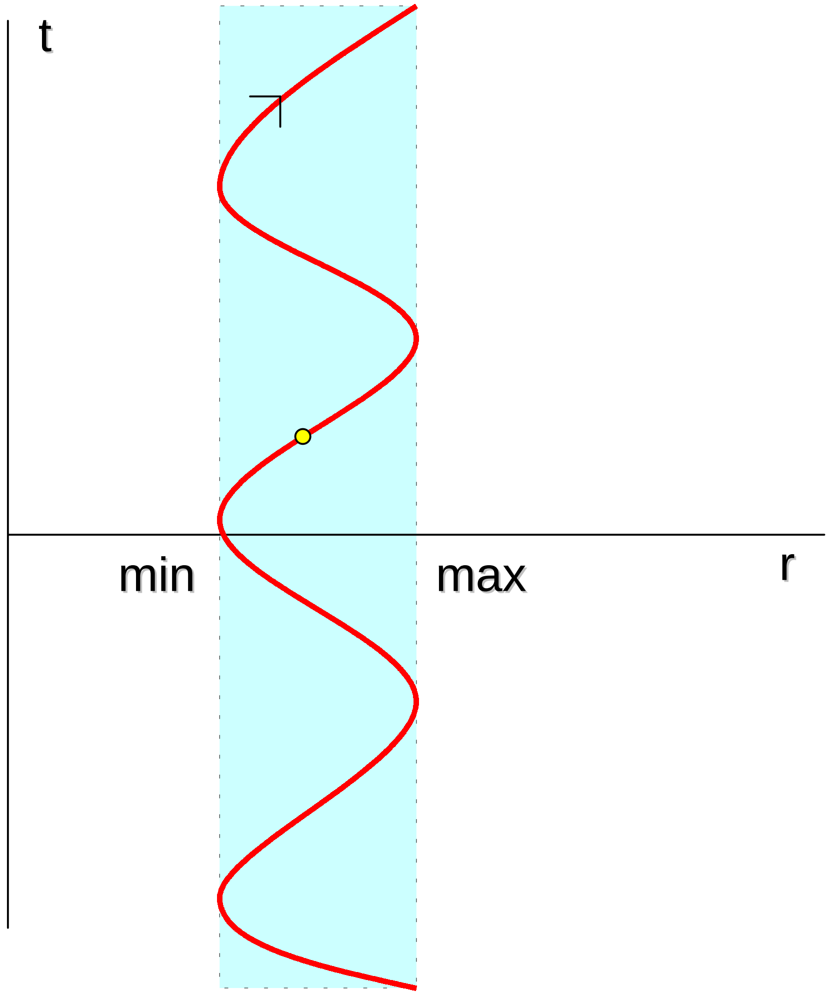
Source $\quad \mathcal{S}^{(o)\chi} = \frac{8\pi i}{\sqrt{\lambda+1}}\left(1-\frac{2M}{r}\right)\left[\left(1-\frac{2M}{r}\right)Q + \frac{r}{\sqrt{2\lambda}}\partial_r\left(\frac{r-2M}{r}D\right)\right]$

Metric Reconstruction

$$h^{RW}_1 = \frac{r^2}{r-2M}{}^{(o)}\chi, \qquad h^{RW}_0 = -\frac{1}{i\omega}\left(1-\frac{2M}{r}\right)\left[(r\,{}^{(o)}\chi)_{,r} + \frac{8\pi i r^2}{\sqrt{2\lambda(\lambda+1)}}D\right]$$

trouble

The appearance of ω in the denominator means that $\hat{h}^{(T)}$ is no longer localized on the radial shell where the particle orbit lies. The source is distributed in the region $r_{\min}$ and $r_{\max}$.



Improved Master variable

V. Moncrief,
Ann. Phys. **88** 323 (1974)

$$^{(o)}\zeta = -\frac{r}{2\lambda}\left[-i\omega h_1^{RW} - h_{0,r}^{RW} + \frac{2}{r}h_0^{RW}\right]$$

Reconstructed Components

$$\begin{aligned} \hat{h}_1^{RW} &= -\frac{i\omega r^2}{r-2M}{}^{(o)}\zeta + \frac{4\pi i r^3}{\lambda\sqrt{1+\lambda}}Q \\ \hat{h}_0^{RW} &= (r-2M)\left({}^{(o)}\zeta_{,r} + \frac{1}{r}{}^{(o)}\zeta + \frac{4\pi r^2}{\lambda\sqrt{1+\lambda}}Q^{(0)}\right) \end{aligned}$$

Source
$$S^{(o)\zeta} = \frac{8\pi(r-2M)}{2\lambda\sqrt{1+\lambda}}\left[\omega r Q - \partial_r(rQ^{(0)})\right]$$

Conservation Law
$$\sqrt{2\lambda}D = \frac{\omega r^2}{r-2M}Q^{(0)} + \left(3-\frac{4M}{r}\right)Q + (r-2M)Q_{,r}$$

Even Parity Modes

$$\left(1-\frac{2M}{r}\right)\left[\left(1-\frac{2M}{r}\right)\left(K^{RW}_{,rr}-\frac{1}{r}H^{RW}_{2,r}\right)+\left(3-\frac{5M}{r}\right)\frac{1}{r}K^{RW}_{,r}-\frac{1}{r^2}\left(H^{RW}_2-K^{RW}\right)\right.$$
$$\left.-\frac{\lambda}{r^2}\left(H^{RW}_2+K^{RW}\right)\right]=-8\pi A^{(0)}$$

$$-i\omega K^{RW}_{,r}-i\omega\frac{1}{r}\left(K^{RW}-H^{RW}_2\right)+i\omega\frac{M}{r(r-2M)}K^{RW}-\frac{(1+\lambda)}{r^2}H^{RW}_1=-\frac{8\pi i}{\sqrt{2}}A^{(1)}$$

$$\frac{1}{(r-2M)}\left[-\omega^2\frac{r^2}{(r-2M)}K^{RW}-\left(1-\frac{M}{r}\right)K^{RW}_{,r}+2i\omega H^{RW}_1+\frac{(r-2M)}{r}H^{RW}_{0,r}+\frac{1}{r}\left(H^{RW}_2-K^{RW}\right)\right.$$
$$\left.+\frac{(1+\lambda)}{r}\left(K^{RW}-H^{RW}_0\right)\right]=-8\pi A$$

$$\left[\left(1-\frac{2M}{r}\right)H^{RW}_1\right]_{,r}+i\omega\left(H^{RW}+K^{RW}\right)=\frac{8\pi i}{\sqrt{1+\lambda}}rB^{(0)}$$

$$i\omega H^{RW}_1+\left(1-\frac{2M}{r}\right)\left(H^{RW}_0-K^{RW}\right)_{,r}+\frac{2M}{r^2}H^{RW}_0+\frac{1}{r}\left(1-\frac{M}{r}\right)\left(H^{RW}_2-H^{RW}_0\right)=\frac{8\pi}{\sqrt{1+\lambda}}(r-2M)B$$

$$\omega^2\left(1-\frac{2M}{r}\right)^{-1}\left(K^{RW}+H^{RW}_2\right)+\left(1-\frac{2M}{r}\right)\left[K^{RW}_{,rr}-H^{RW}_{0,rr}\right]+\left(1-\frac{M}{r}\right)\frac{2}{r}K^{RW}_{,r}-2i\omega H^{RW}_{1,r}$$
$$-i\omega\frac{2(r-M)}{r(r-2M)}H^{RW}_1-\frac{1}{r}\left(1-\frac{M}{r}\right)H^{RW}_{2,r}-\frac{1}{r}\left(1+\frac{M}{r}\right)H^{RW}_{0,r}-\frac{(1+\lambda)}{r^2}\left(H^{RW}_2-H^{RW}_0\right)=8\sqrt{2}\pi G^{(s)}$$

$$H^{RW}_0-H^{RW}_2=\frac{16\pi}{\sqrt{2\lambda(1+\lambda)}}r^2F$$

Zerilli's Master variable

F. Zerilli,
Phys. Rev. D. **2** 2141 (1970)

Master Variable $$^{(e)}\chi = \left(\frac{r-2M}{\lambda r+3M}\right)\left[\frac{i\omega r^2}{r-2M}K^{RW} + H_1^{RW}\right]$$

Master equation $$[\partial^2_{r*} + \omega^2 - V(r)]^{(e)}\chi = \mathcal{S}^{(^{(e)}\chi)}$$

Potential $$V(r) = \left(1-\frac{2M}{r}\right)\frac{2\lambda^2(\lambda+1)r^3 + 6\lambda^2 M r^2 + 18\lambda M^2 r + 18M^3}{r^3(r\lambda+3M)^2}$$

Source

$$\begin{aligned} S^{(^{(e)}\chi)} &= 8\pi\left[\frac{(r-2M)^2}{(r\lambda+3M)\sqrt{1+\lambda}}B_r^{(0)} + \frac{(r-2M)(-12M^2+9Mr+r^2\lambda)}{r\sqrt{1+\lambda}(r\lambda+3M)^2}B^{(0)} - \sqrt{2}\lambda\frac{(r-2M)^2}{(r\lambda+3M)^2}A^{(1)}\right. \\ &+ \left.\omega\left[-\frac{r^2}{(r\lambda+3M)}A^{(0)} + \frac{(r-2M)^2}{(r\lambda+3M)}A + \frac{(r+2M)^2}{(r\lambda+3M)\sqrt{1+\lambda}}B - \sqrt{2}\frac{(r-2M)}{\sqrt{\lambda(1+\lambda)}}F\right]\right] \end{aligned}$$

Metric Reconstruction

$$\hat{K}^{RW} = \frac{1}{\omega}\left[-\left(1-\frac{2M}{r}\right)^{(e)}\chi_{,r} + \frac{r^2\lambda + (r\lambda+3M)(r\lambda+2M)}{r^2(r\lambda+3M)}{}^{(e)}\chi\right] - \frac{r(r-2M)}{\omega(r\lambda+3M)}\left(\frac{8\pi}{\sqrt{2}}A^{(1)} + \frac{8\pi}{\sqrt{1+\lambda}}B^{(0)}\right)$$

Improved Master variable

Master Variable $$^{(e)}\zeta = \frac{r(r-2M)}{(\lambda+1)(\lambda r+3M)}\left[H_2^{RW} - rK_{,r}^{RW} + \frac{r\lambda+3M}{r-2M}K^{RW}\right]$$

Master equation $$[\partial_{r^*}^2 + \omega^2 - V(r)]^{(e)}\zeta = \mathcal{S}^{(e)\zeta}$$

Potential $$V(r) = \left(1-\frac{2M}{r}\right)\frac{2\lambda^2(\lambda+1)r^3 + 6\lambda^2Mr^2 + 18\lambda M^2 r + 18M^3}{r^3(r\lambda+3M)^2}$$

Source

$$\mathcal{S}^{(e)\zeta} = 8\pi\frac{r-2M}{(1+\lambda)(r\lambda+3M)}\left[r^2A_{,r}^{(0)} - r\left(\frac{r\lambda+2M}{r-2M} - \frac{r\lambda+9M}{r\lambda+3M}\right)A^{(0)} - \omega\frac{r^2}{\sqrt{2}}A^{(1)}\right.$$
$$\left. +(1+\lambda)(r-2M)A + \sqrt{1+\lambda}(r-2M)B - \sqrt{\frac{2(1+\lambda)}{\lambda}}(r\lambda+3M)F\right]$$

No **ω** in the denominator, source is localized

Metric Reconstruction

$$\hat{K}^{RW} = \frac{\lambda(\lambda+1)r^2 + 3\lambda M r + 6M^2}{r^2(r\lambda+3M)} {}^{(e)}\zeta + \left(1 - \frac{2M}{r}\right) {}^{(e)}\zeta_{,r} - \frac{8\pi r^3}{(\lambda+1)(r\lambda+3M)} A^{(0)}$$

$$\hat{H}_1^{RW} = -i\omega \frac{\lambda r(r-2M) - M(r\lambda+3M)}{(r-2M)(r\lambda+3M)} {}^{(e)}\zeta - i\omega r {}^{(e)}\zeta_{,r} + i\omega \frac{8\pi r^5}{(1+\lambda)(r\lambda+3M)(r-2M)} A^{(0)} + i \frac{8\pi r^2}{\sqrt{2}(1+\lambda)} A^{(1)}$$

$$\begin{aligned}\hat{H}_2^{RW} = \frac{1}{r\lambda+3M} \Bigg[& \left(-\omega^2 r^2 \frac{(r\lambda+3M)}{r-2M} + \lambda^2 + \frac{3M^2}{r^2} + \frac{\lambda(r^2\lambda+6M^2)}{r(r\lambda+3M)}\right) {}^{(e)}\zeta - \left(\frac{M}{r}(r\lambda+3M) - \lambda(2M-r)\right) {}^{(e)}\zeta_{,r} \\ & - \frac{8\pi}{(1+\lambda)} \Bigg(\left(\frac{r\lambda}{r\lambda+3M} - \frac{M}{r-2M}\right) r^3 A^{(0)} + \frac{1}{\sqrt{2}} \omega r^4 A^{(1)} - (r-2M) r^2 [B + (1+\lambda)A] \\ & + \sqrt{\frac{2}{\lambda}} r^2 (r\lambda+3M) F \Bigg) \Bigg]\end{aligned}$$

$$\hat{H}_0^{RW} = \hat{H}_2^{RW} + \frac{16\pi}{\sqrt{2\lambda(1+\lambda)}} r^2 F$$

No **ω** in the denominator; reconstructions are local and do not require time integration

Master variable via Chandrasekhar Transformation

S. Chandrasekhar and S. Detweiler,
Proc. Roy. Soc. London, **A344**, 441 (1975)

Weyl scalar (ψ) contracted with null tetrad satisfies same homogeneous equation irrespective of parity.

S. A. Teukolsky,
Ap. J. **185**, 635(1973)

$$\boxed{\psi \equiv -C_{abcd} l^a m^b l^c m^d}$$

$$(\ell^a) = (r-2M)^{-1}(r, r-2M, 0, 0), \quad (m^a) = (\sqrt{2} r \sin\theta)^{-1}(0, 0, \sin\theta, i)$$

In explicit metric form we have

$$^{(o)}\psi = \frac{i}{2r^2(r-2M)}\left[(r-2M)h^{RW}_{1,r} + r h^{RW}_{0,r} - i\omega r h^{RW}_1 + \frac{(2M + i\omega r^2)}{r-2M} h^{RW}_0\right]$$

$$^{(e)}\psi = -\frac{(H^{RW}_1 + H^{RW}_2)}{r(r-2M)}$$

Using reconstruction formulas in vacuum we can rewrite $^{(o)}\psi$

$$\boxed{\begin{aligned} ^{(o)}\psi = \frac{2}{r^3(r-2M)^2}\Big[&\{\omega^2 r^4 + i\omega r^2(r-3M) + (3M-(\lambda+1)r)(r-2M)\}\,^{(o)}\zeta(r) \\ &+ r(i\omega r^2 + 3M - r)(r-2M)\,^{(o)}\zeta_{,r}(r)\Big]\end{aligned}}$$

$$ {}^{(o)}\zeta(r) = \zeta[{}^{(o)}\psi] = \frac{r^2}{2(3i\omega M + \lambda(\lambda+1))} \left[\left\{ \omega^2 r^3 - 5i\omega M r - 4M + r + \lambda(2M - r) \right\} {}^{(o)}\psi(r) \right. $$
$$ \left. -(r - 2M)(i\omega r^2 + 3M - r)\, {}^{(o)}\psi_{,r}(r) \right] $$

Then, with an arbitrary constant $\mathcal{C}$,

$$ {}^{(e)}\tilde{\zeta} = \mathcal{C}\zeta[{}^{(e)}\psi] $$

satisfies the homogeneous R-W equation.

$$ [\partial^2_{r^*} + \omega^2 - V^{RW}(r)]\, {}^{(e)}\tilde{\zeta} = \mathcal{S}^{(e)\tilde{\zeta}} $$

$$ {}^{(e)}\tilde{\zeta}(r) = 2(r - 2M)\left(H_2^{RW} - rK_{,r}^{RW} + \frac{r\lambda}{(r-2M)} K^{RW}\right) $$

with

Starobinsky constant $\quad \mathcal{C} = 4(3i\omega M + \lambda(\lambda+1))$

Source

$$ \mathcal{S}^{(e)\tilde{\zeta}} = 8\pi(r - 2M)\left[-2rA^{(0)}_{,r} + 2\frac{M - r(1-\lambda)}{r - 2M} A^{(0)} + \sqrt{2}\omega r A^{(1)} + 2\frac{[3M - r(1+\lambda)](r-2M)}{r^2} A \right. $$
$$ \left. + \quad 2\frac{[6M - r(1+\lambda)](r-2M)}{r^2\sqrt{1+\lambda}} B - 2\sqrt{2}\frac{[6M^2 - \lambda r^2(1+\lambda)]}{r^2\sqrt{(1+\lambda)\lambda}} F - 6\sqrt{2}\frac{M(r-2M)}{r\sqrt{(1+\lambda)\lambda}} F_{,r} \right] $$

Metric Reconstruction

$$
\begin{aligned}
\bar{K} &= \frac{128\pi}{|\mathcal{C}|^2}\Big[-r(1+\lambda)(r\lambda+3M)A^{(0)} + \frac{3}{r}\sqrt{1+\lambda}M(r-2M)^2\{B+\sqrt{1+\lambda}A\} - \frac{3}{\sqrt{2}}\omega r M(r-2M)A^{(1)} \\
&\quad -3\frac{\sqrt{2(1+\lambda)}M(r-2M)(r\lambda+3M)}{r\sqrt{\lambda}}F + \frac{[(1+\lambda)\{3M(r-2M)+r\lambda(r\lambda+3M)\}-r\mathcal{O}]}{16\pi r^3}{}^{(e)}\bar{\zeta} \\
&\quad + \frac{(r\lambda+3M)(r-2M)(1+\lambda)}{16\pi r^2}{}^{(e)}\bar{\zeta}_{,r}\Big] \\
\bar{H}_1 &= \frac{128\pi}{|\mathcal{C}|^2}\Big[-i\omega\frac{\mathcal{P}r^2(r\lambda+3M)}{(r-2M)}A^{(0)} + i\frac{[3M\omega^2 r+\lambda^2(\lambda+1)]r^2}{\sqrt{2}}A^{(1)} + 3i\omega M\mathcal{P}(r-2M)\{A+\frac{B}{\sqrt{1+\lambda}}\} \\
&\quad -3\sqrt{2}i\omega\frac{M\mathcal{P}(r\lambda+3M)}{\sqrt{(1+\lambda)\lambda}}F + i\omega\frac{[r^2\mathcal{O}+3M\mathcal{P}(r-2M)]}{16\pi r^2(r-2M)}{}^{(e)}\bar{\zeta} + i\omega\frac{(r\lambda+3M)\mathcal{P}}{16\pi r}{}^{(e)}\bar{\zeta}_{,r}\Big] \\
\bar{H}_2 &= \frac{128\pi}{|\mathcal{C}|^2}\Big[\frac{r^2\mathcal{O}}{(r-2M)}A^{(0)} + \omega\frac{\lambda r^2\mathcal{P}}{\sqrt{2}}A^{(1)} - \lambda(r-2M)\mathcal{P}\{(1+\lambda)A+\sqrt{1+\lambda}B\} + \sqrt{2(1+\lambda)\lambda}\mathcal{P}(r\lambda+3M)F \\
&\quad + \frac{[\mathcal{O}-\lambda(\omega^2 r^3+M(1+\lambda))]\mathcal{P}}{16\pi(r-2M)r^2}{}^{(e)}\bar{\zeta} - \frac{\mathcal{O}}{16\pi r}{}^{(e)}\bar{\zeta}_{,r}\Big], \\
\bar{H}_0 &= \bar{H}_2 + \frac{16\pi}{\sqrt{2\lambda(1+\lambda)}}r^2 F
\end{aligned}
$$

$\mathcal{P}=3M-r(1+\lambda)$ $\qquad$ $\mathcal{O}=3M\omega^2r^2+\lambda(\lambda+1)(3M-r)$

In time domain we will need time integrations for the metric reconstruction due to the factor $|\mathcal{C}|^{-2}=[16\lambda^2(1+\lambda)^2-144\omega^2M^2]^{-1}$.

Conclusion/Discussion

- We propose alternative variables which are more suitable for the purpose of metric reconstruction as the formulae are completely local.

- Beyond Chandrasekhar transformation

- As new proposal we introduce gauge invariant regularization.