Resonant Growth by Incident Gravitational Waves

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Introduction

- Excitation of neutron star oscillations in a binary
(e.g. Gualtieri et al(2002), Ruoff et al(2001), ...)
Resonance $\omega = l\Omega_K$ by tidal force

- Condition and amplification for resonant growth by incident gravitational waves

Goals of this paper

Resonance condition? $\omega = \Omega$

Amplitude? $\sim ht$ ('secular')

No exponential growth?

Possible astrophysical periodic sources

$$h, \quad \Omega, \quad t$$

- Excitation of Pulsation (f-mode) in Neutron Stars
Frequency $\nu =$ a few kHz, Damping time $\tau =$ a few $\times 10^{-1}$ s.

GWs from SN explosion, coalescence phase of NS-NS binary
Duration is less than a few ms. $\Rightarrow$ Rapid growth is relevant.

- Excitation of Pulsation (f-mode) in White Dwarfs
Frequency $\nu =$ a few $\times 10^{-1}$ Hz, Damping time $\tau \sim 10^{10}$ s.

GWs from spiral-in of BH-BH binary with $M \sim 10^3 M_\odot$
Duration is 10^3 s. $\Rightarrow$ Slow growth may also be relevant.

- Amplitude $h \propto 1/r$

Resonance in a harmonic oscillator

• Resonance occurs when the frequency of external perturbation matches with the intrinsic frequency, $\Omega = \omega$

$$\ddot{X} + \omega^2 X = f \cos(\Omega t). \tag{1}$$

The amplitude increases linearly with the time, $X \propto (f/2\Omega) t \sin(\Omega t)$.

• Parametric resonance occurs for time-dependent frequency. An example (known as the Mathieu's equation) is

$$\ddot{X} + \omega^2 (1 + h \cos(\Omega t)) X = 0. \tag{2}$$

The exponential growth for $\omega/\Omega = n/2, (n = 1, 2, 3, \cdots)$

• The strongest instability occurs for $n = 1$ $(\omega/\Omega = n/2)$

◇ The growth rate $X \propto \exp(st)$ can be calculated for small h as

$$s = -\frac{1}{2}\left[-(\Omega - 2\omega)^2 + \frac{1}{4}h^2\omega^2\right]^{1/2} \sim \frac{1}{4}h\omega \tag{3}$$

in the unstable region

$$-\frac{1}{2}h\omega < \Omega - 2\omega < \frac{1}{2}h\omega \tag{4}$$

◇ In a system with frictional damping $\exp(-\gamma t)$, resonance is possible only when h exceeds a threshold $4\gamma/\omega$.

• The width of resonance range decreases rapidly with increasing n.

Stellar pulsation

The equation of motion for the pulsation driven by external force can be written as

$$\frac{d\vec{v}}{dt} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\phi + \vec{g} \tag{5}$$

◇ Incident wave propagating along z-axis (+ mode)

$$\vec{g} = \frac{1}{2}\ddot{A}(t-z)\left(x\vec{e}_x - y\vec{e}_y\right) = -\frac{1}{2}h\Omega^2\cos(\Omega(t-z))\left(x\vec{e}_x - y\vec{e}_y\right). \tag{6}$$

◇ Radiation loss by quadrupole formula

$$\vec{g} = -\vec{\nabla}\frac{G}{5c^5}I_{ij}^{(5)}x^i x^j. \tag{7}$$

Ellipsoidal model

The dynamical motion is limited to uniform expansion and compression along the three axes. Incompressible fluid is assumed.

$$\ddot{a}_1 = -2\pi G\rho a_1 A_1 + \frac{K}{a_1} - \frac{1}{2}h\Omega^2 \cos(\Omega t)a_1 - \frac{G}{5c^5}I_{xx}^{(5)}a_1 \tag{8}$$

$$\ddot{a}_2 = -2\pi G\rho a_2 A_2 + \frac{K}{a_2} + \frac{1}{2}h\Omega^2 \cos(\Omega t)a_2 - \frac{G}{5c^5}I_{yy}^{(5)}a_2 \tag{9}$$

$$\ddot{a}_3 = -2\pi G\rho a_3 A_3 + \frac{K}{a_3} - \frac{G}{5c^5}I_{zz}^{(5)}a_3 \tag{10}$$

Nonlinear dynamical system of $a_i(t)$

Linearized system for pulsation

Toroidal motion in x-y plane is assumed.

$$X = \frac{\delta a_1}{a} = -\frac{\delta a_2}{a}, \quad \delta a_3 = 0. \tag{11}$$

The system of the equations is reduced to

$$\ddot{X} + \omega^2 \left(1 + \frac{1}{2} h \left(\frac{\Omega}{\omega}\right)^2 \cos(\Omega t)\right) X = -\frac{1}{2} h \Omega^2 \cos(\Omega t), \tag{12}$$

It is clear that two kinds of resonance are possible in eq. (12).

Numerical calculation

- Numerical calculation depends on $Q = \omega\tau, \omega/\Omega, h$

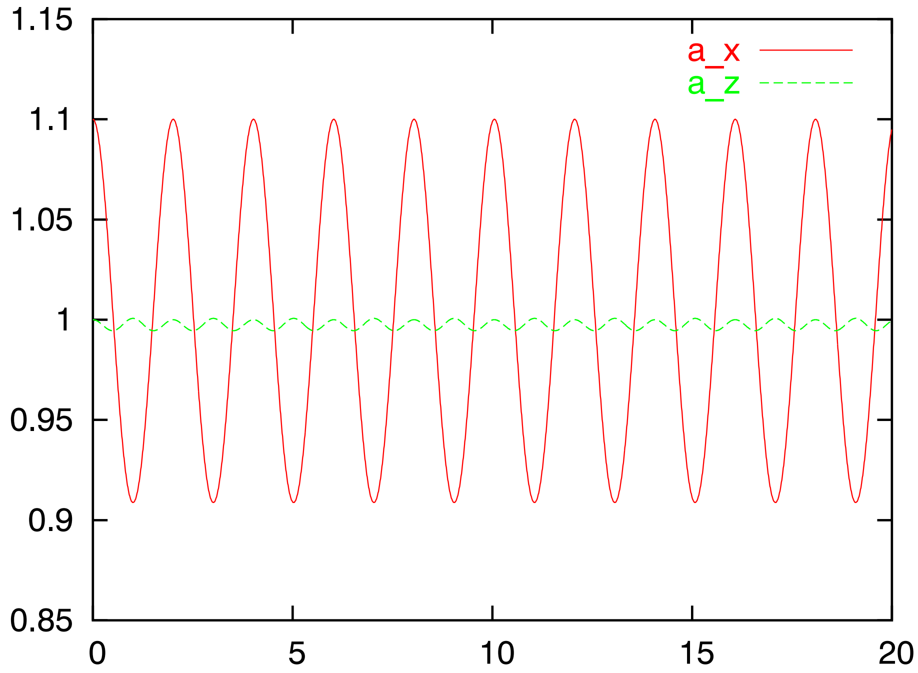
$$\omega = \frac{4GM}{5R^3}, \qquad \tau^{-1} = \frac{2GM\omega^4}{25c^5}.$$

- Initially static state ($a_x = a_y = a_z = 1.0, \dot{a}_x = \dot{a}_y = \dot{a}_z = 0.$)

- Results

◇ Test cases (Free oscillation and Damping oscillation)
◇ Off-resonant cases (Free oscillation and Damping oscillation)
◇ Parametric resonance at $\omega/\Omega = 1/2$ ($n = 1$)
◇ Resonance at $\omega/\Omega = 1$ ($n = 2$)

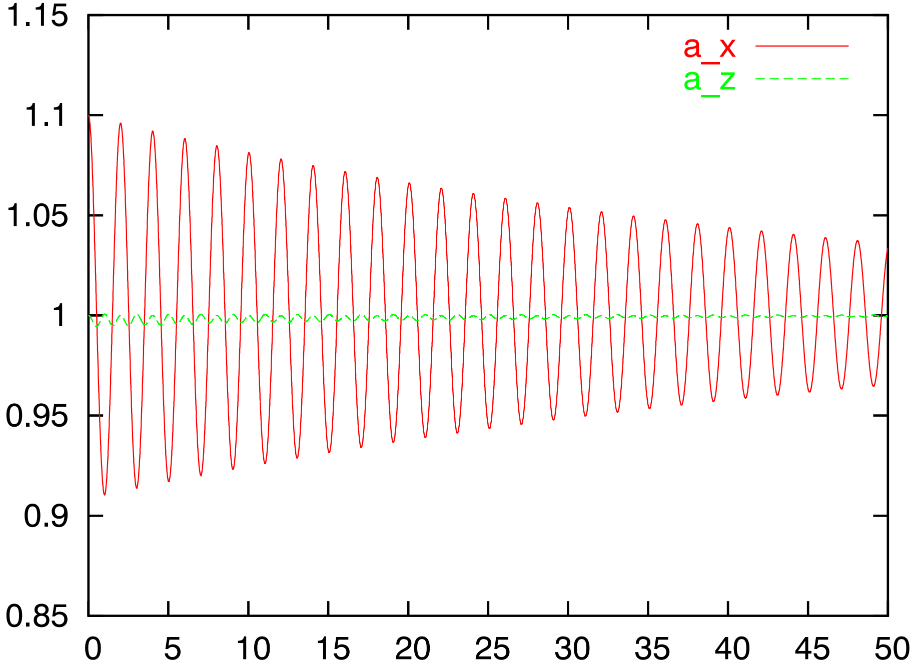
Test - Free oscillation -



Amplitudes for a_x and a_z are shown as a function of time $\Omega t/(2\pi)$. Initial conditions are $a_x = 1.1, a_y = 0.9, a_z = 1.0, \dot{a}_x = \dot{a}_y = \dot{a}_z = 0$.

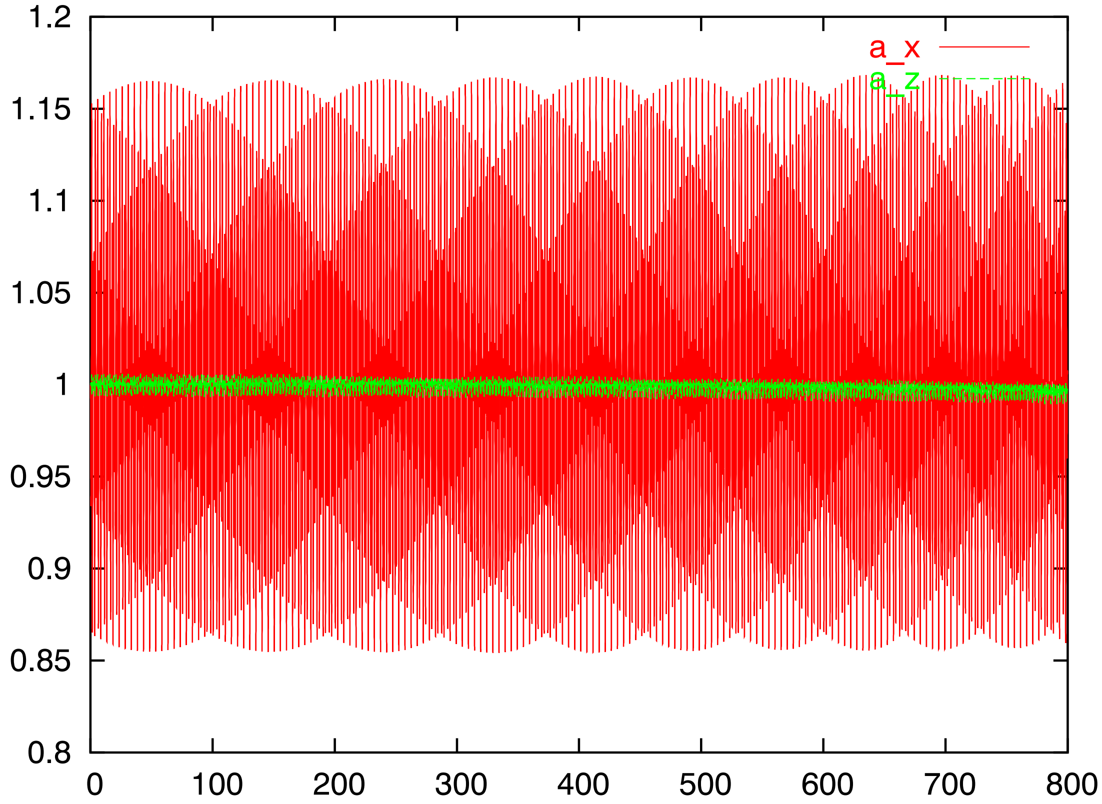
Amplitude of δa_z is not zero exactly, but is of order $(\delta a_x)^2$.

Test - Radiation damping -



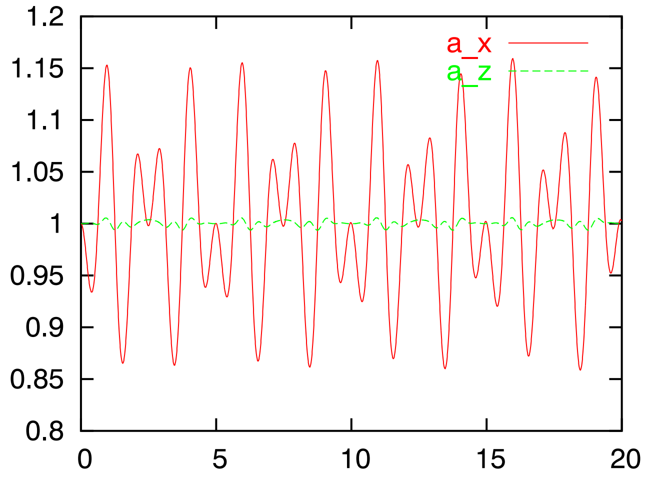
Damping of the oscillation by radiation loss with $Q = 100$.

Off-resonance without damping

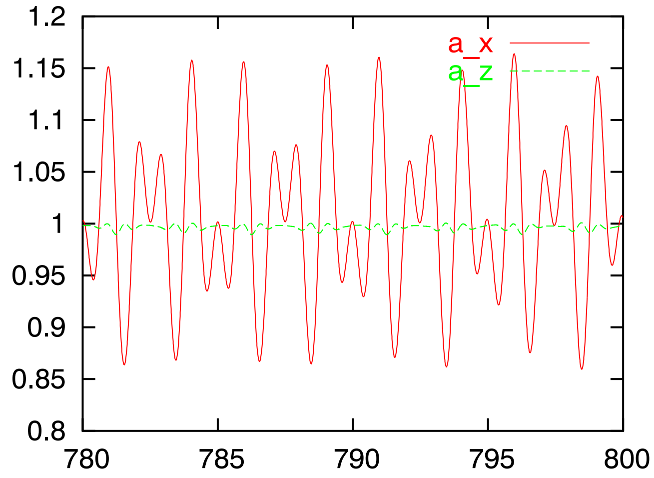


Free oscillation under incident waves with $\omega/\Omega = 0.6$ and $h = 0.1$. The amplitude of the pulsation is always ~ 0.1

Off-resonance without damping 2 - Invariant -



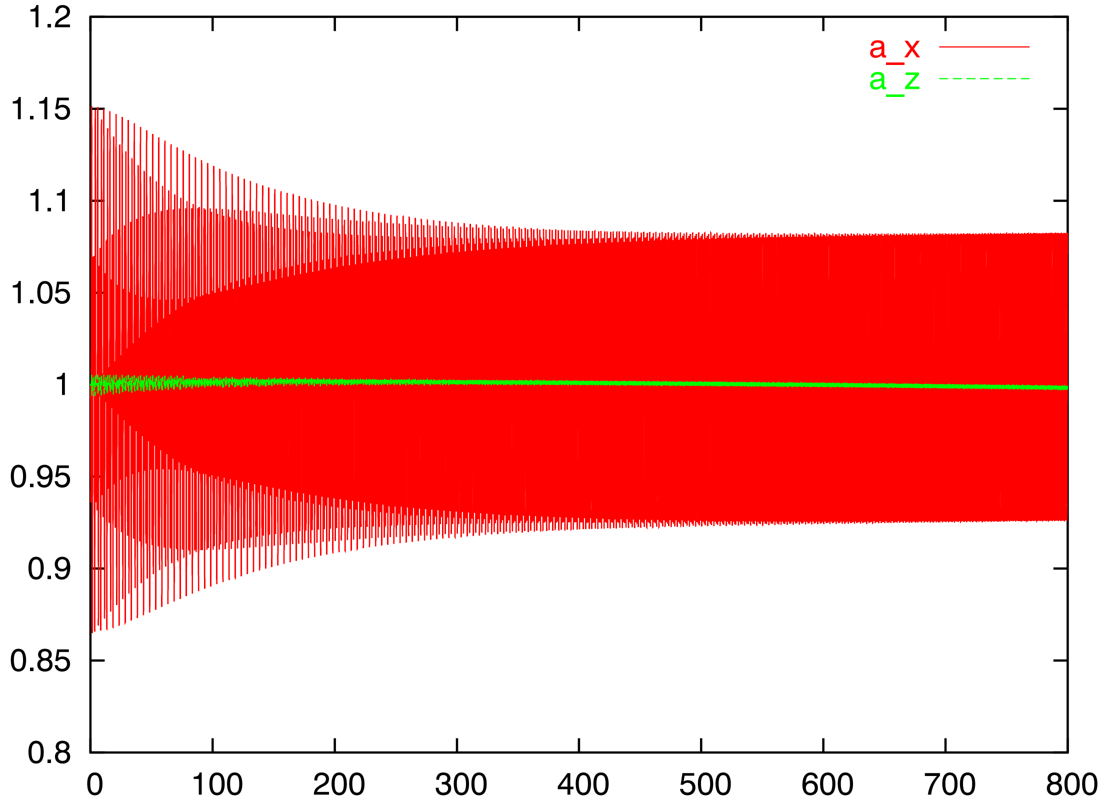
Early stage.



Late stage.

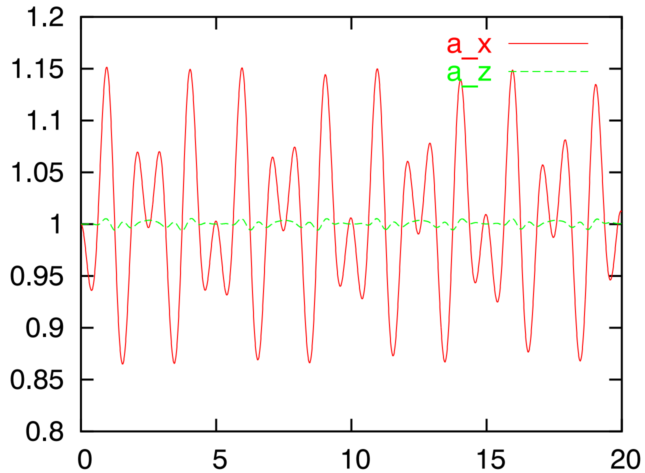
Temporal profiles are the same. They are mixture of two modes, Ω and $\omega = 0.6\Omega$.

Off-resonance with damping

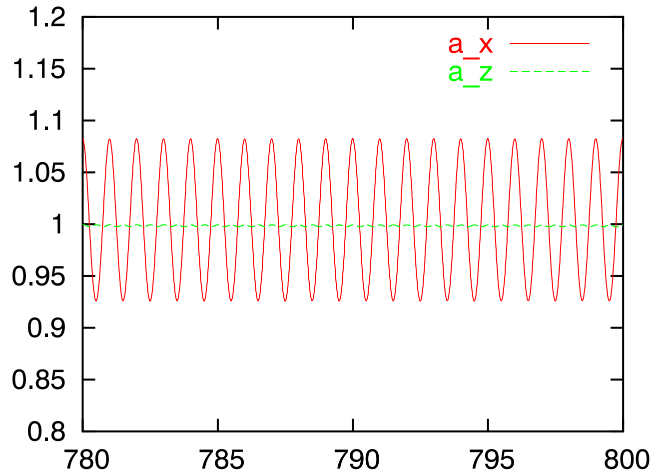


Oscillation with damping $Q = 500$ under incident waves with $\omega/\Omega = 0.6$ and $h = 0.1$. The amplitude of the pulsation tends to 0.1

Off-resonance with damping 2 - Transition -



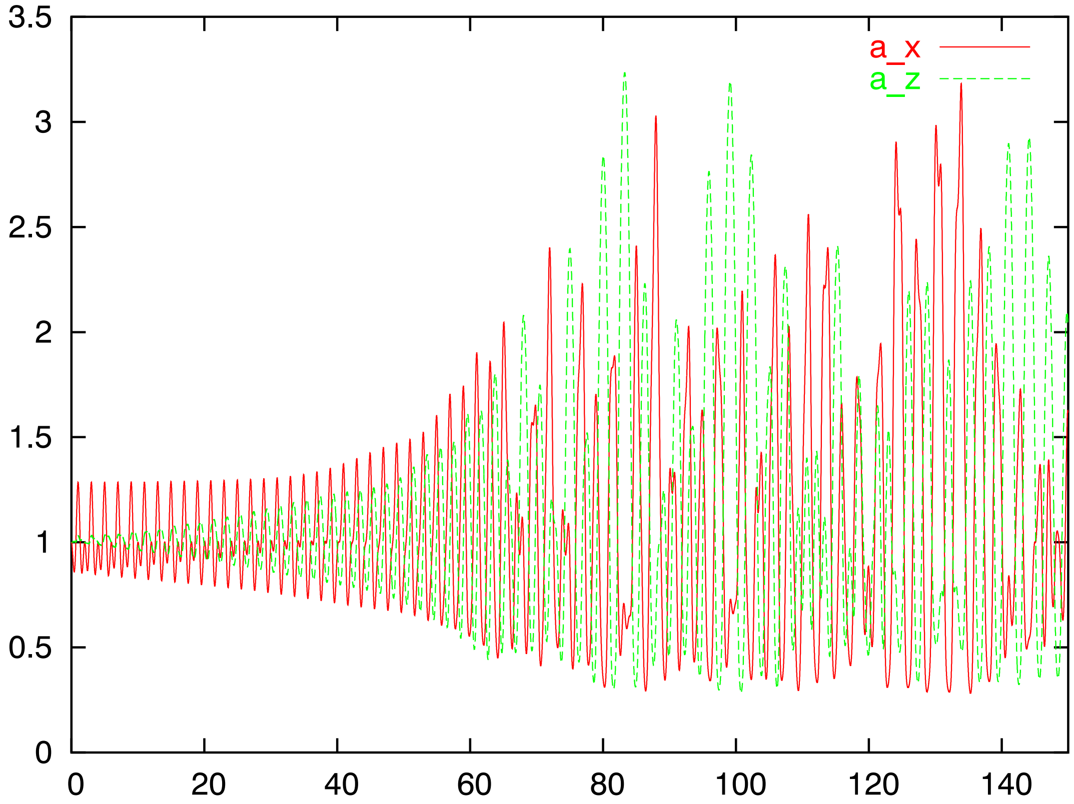
Early stage.



Late stage.

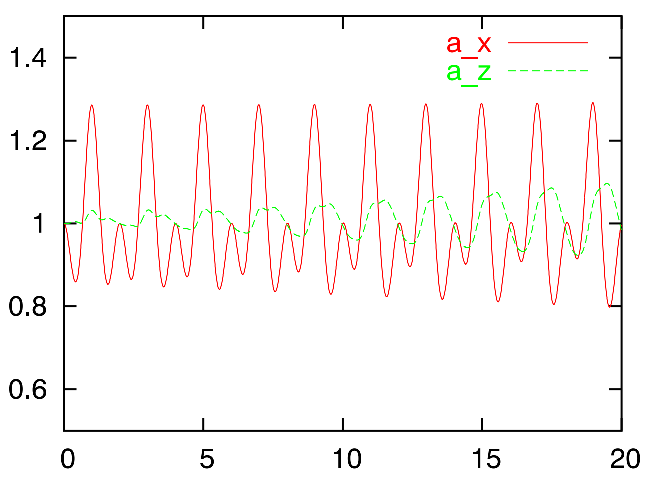
Intrinsic stellar pulsation $\omega(= 0.6\Omega)$ is damped, and oscillation is enforced to external mode with Ω at the late phase.

Parametric Resonance without damping



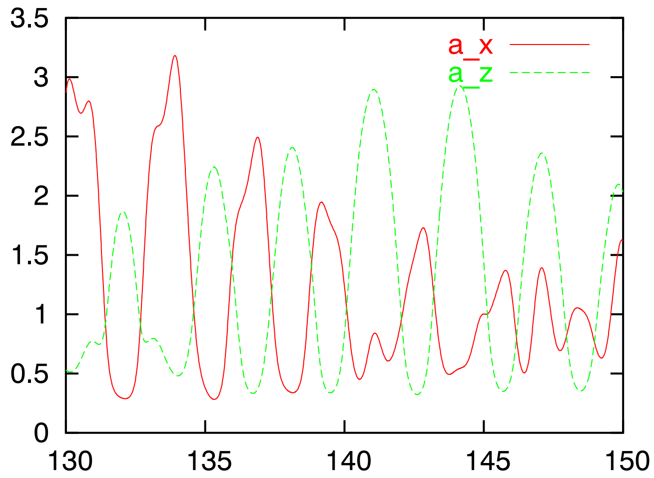
Resonance at $\omega/\Omega = 1/2$. Amplitude of incident wave is $h = 0.1$. Amplified stellar pulsation is realized.

Parametric Resonance without damping 2



Early stage

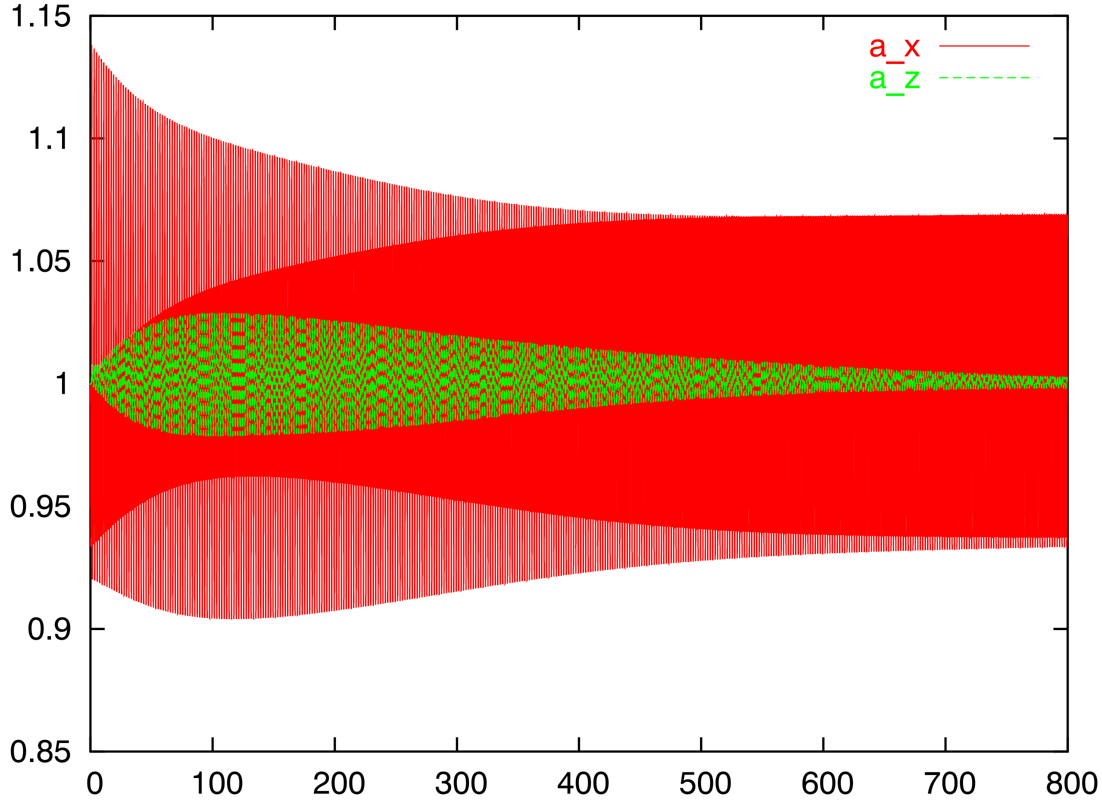
Mixture of two frequencies $\Omega/2$ and Ω can be seen.



Late stage.

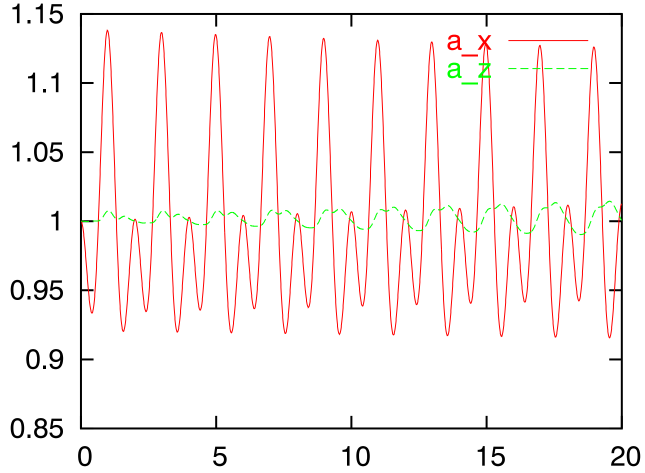
Oscillation of $\Omega/2$ can be seen.

Damped Parametric Resonance

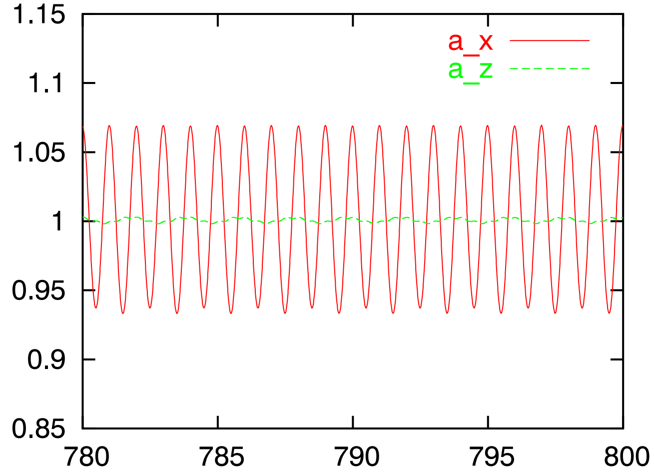


Parametric excitation is damped for small incident amplitude and rapid decay ($h = 0.1, Q = 300$). The final oscillation is the same as the incident one.

Damped Parametric Resonance 2



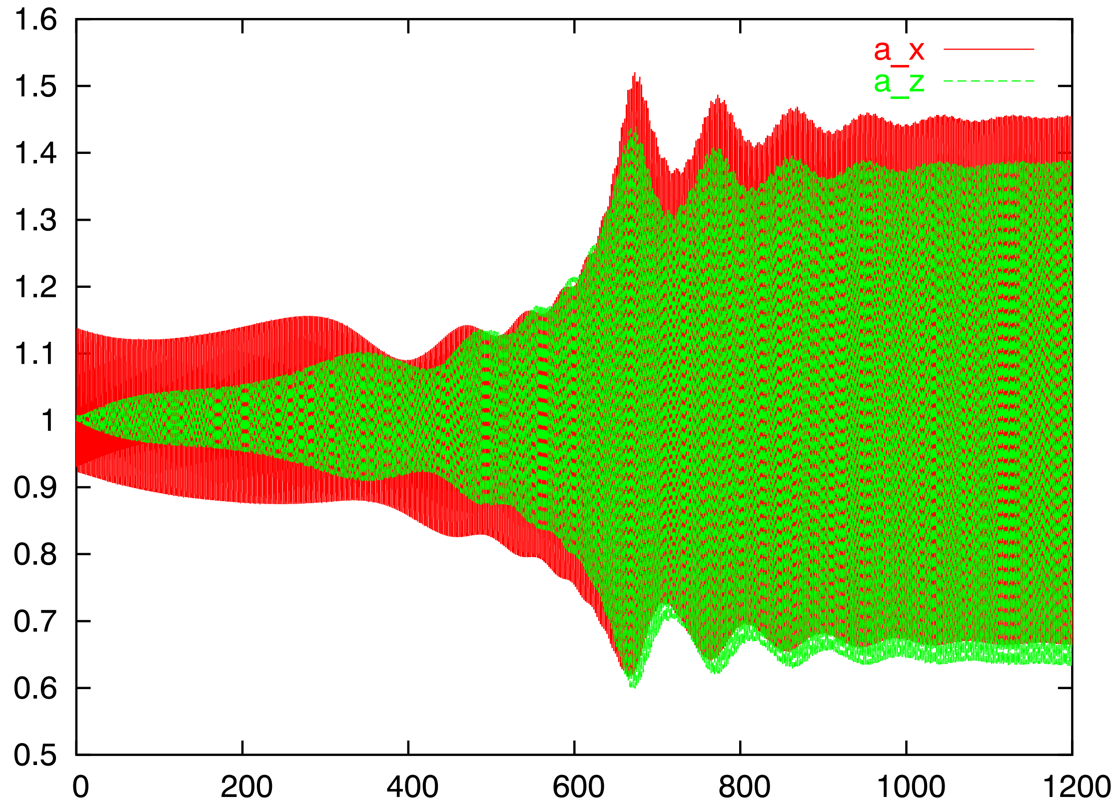
Early stage



Late stage

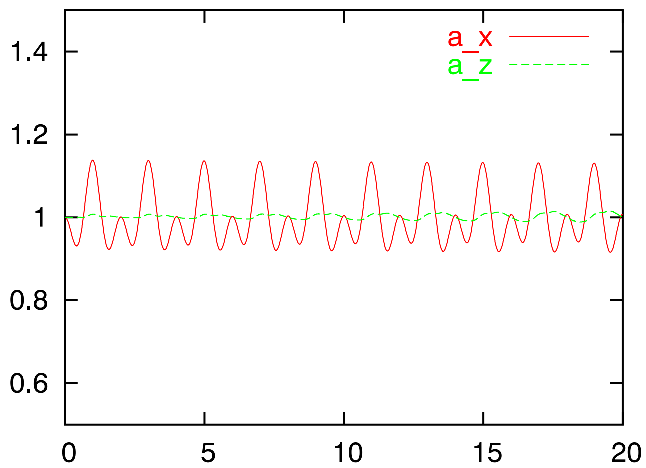
Intrinsic stellar pulsation $\omega(=\Omega/2)$ is damped, and oscillation is enforced to external mode with Ω at the late phase. This is the same as in off-resonant case.

Excited Parametric Resonance

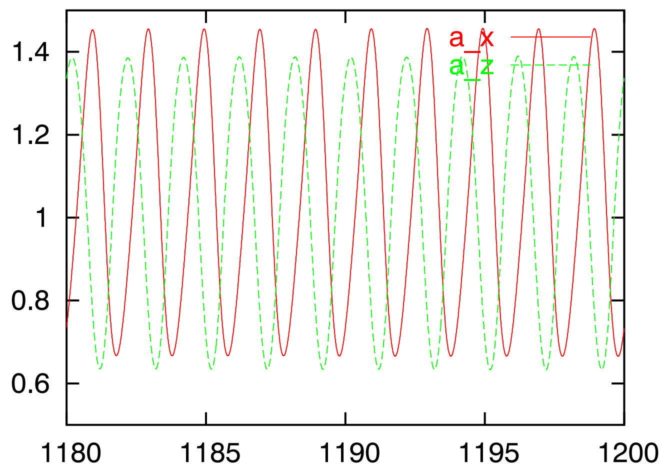


Parametric resonance occurs even with radiation loss for large incident amplitude and slow decay ($h = 0.1, Q = 500$).

Excited Parametric Resonance 2



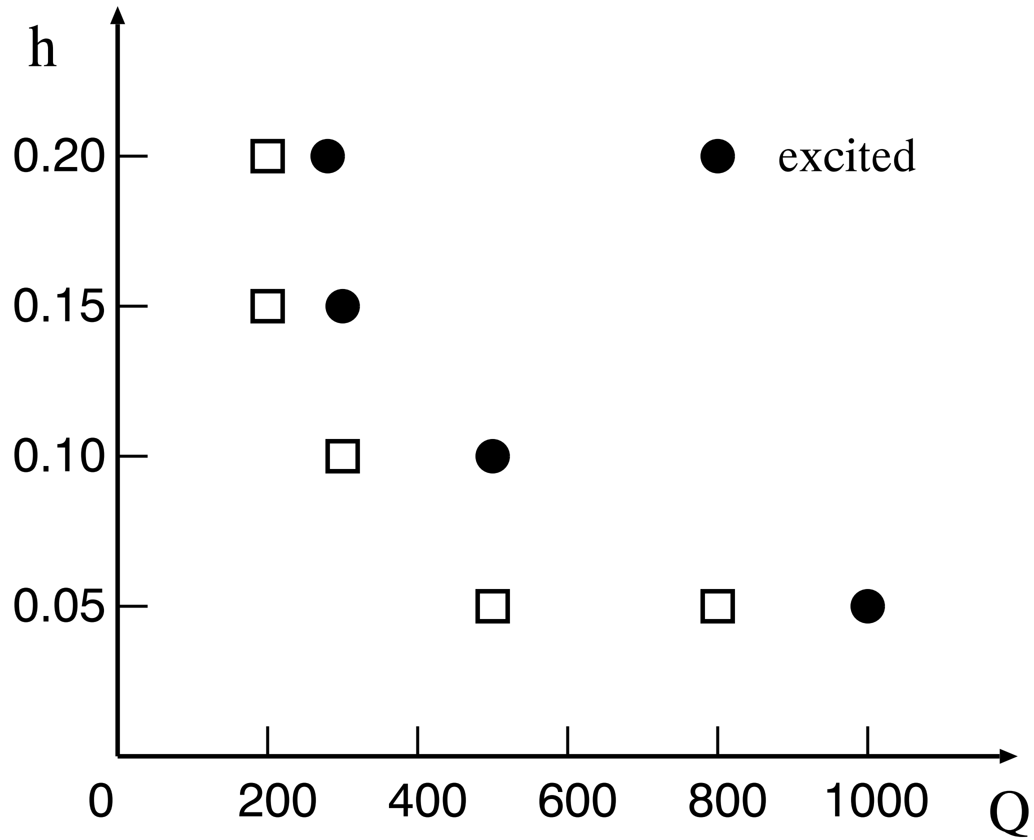
Early stage



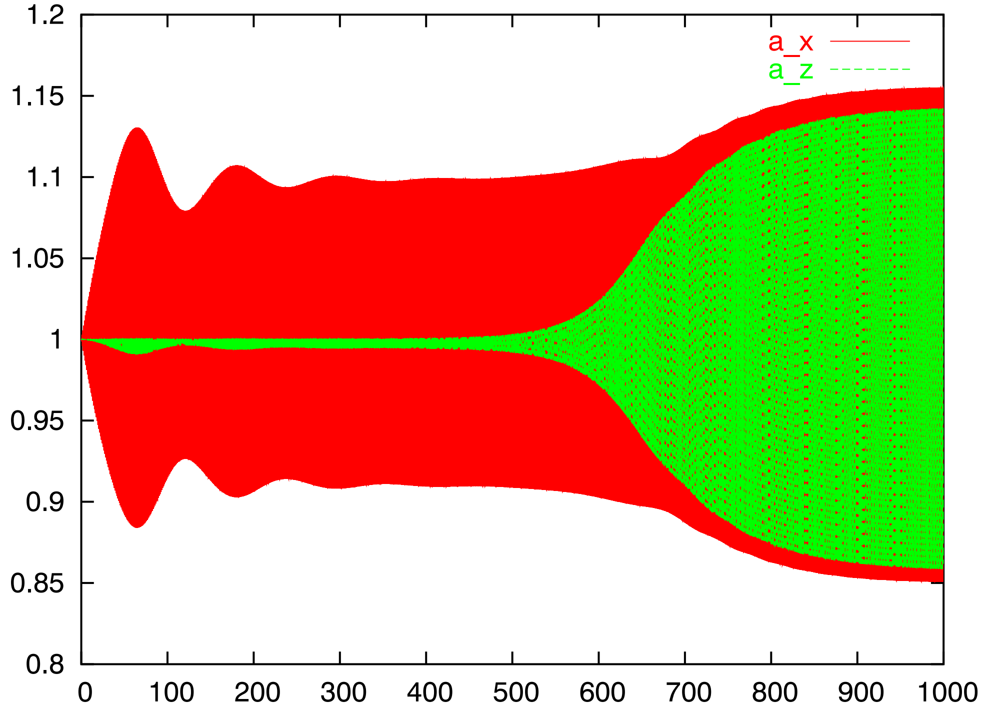
Late stage

Stellar pulsation with large amplitude is realized at the late steady state.

The amplitude is about 4 times larger than that of off-resonant oscillation.

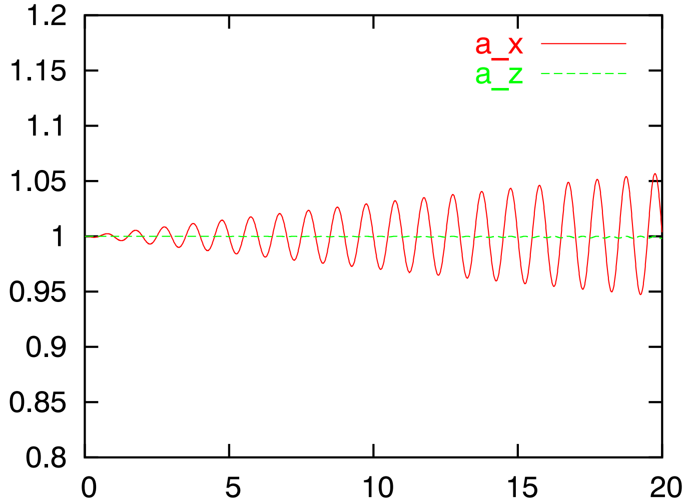

h
0.20
0.15
0.10
0.05
excited
0
200
400
600
800
1000
Q

Resonance at $\omega = \Omega$

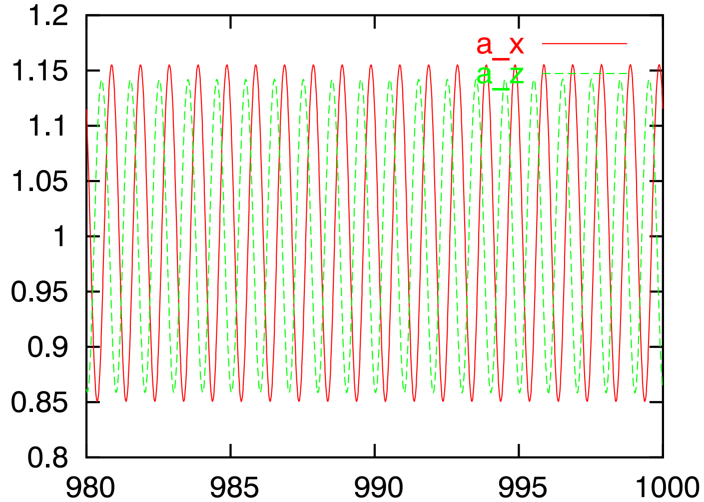


Resonance with $Q = 500$ and $h = 2 \times 10^{-3}$. Oscillation amplitude for a_x increases linearly with time at the initial stage. That for a_z increases with a longer time scale. The amplitudes are saturated by gain and loss.

Resonance 2



Early stage.



Late stage.

Summary and Discussion 1
Resonance at $\Omega = 2\omega$

- The amplitude of the longitudinal motion along the wave propagation gradually increases in the parametric instability.

- The parametric resonance-growth is possible, in the slowly damping system (i.e, $Q >$ a few hundreds) with incident waves of the large amplitude ($h >$ a few $\times 10^{-2}$)

- The amplitude is saturated around $\sqrt{hQ}/10$.

Summary and Discussion 2
Resonance at $\Omega = \omega$

- The amplitude of the motion perpendicular to the wave propagation increases linearly with the time, and can get to large value $\gg h$.

- The amplitude of the motion along the wave propagation subsequently increases.

- The amplitude is saturated around $\sqrt{hQ}/40$.

Concluding Remarks

• Small growth rate of the resonance $\omega/\Omega = 1/2$.
(growth rate $\propto h^2$)

Compare with the Mathieu's equation, in which perturbation $\sim \exp(st)$, $\quad s \propto h$.

◇ Oscillations in the x-y plane are coupled with those of z-axis through self-gravity and pressure in ellipsoidal model.
$\delta a_z \sim (\delta a_x)^2 \sim h^2$
After δa_z gets to a significant value, $\delta a_z \sim \delta a_x$, system of a_x, a_y, a_z blows up.

Number of degrees of freedom relevant to dynamics is important for 'secular' growth (i.e, long term evolution).
Possible in realistic fluid system?