Treatments of oscillations of a gravitating Nambu-Goto string

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Contents:

References:

- K.N., A. Ishibashi, H. Ishihara, Phys. Rev. D62 (2000), 101502(R).
- K.N., H. Ishihara, Phys. Rev. D63 (2001), 127501.
- K.N., Class. Quantum Grav. 19 (2002), 783.
- K.N., preprint, (gr-qc/0302057).
- K.N., preprint, (gr-qc/0303090).

§I. Introduction

- **Nambu-Goto membranes** (N-G membrane)
  .... one of the simplest types of the idealization of matter sources

  Nambu-Goto action : $S_{NG} = -\mu \int d^N\sigma \sqrt{-g}$

  eq. of motion : $\frac{1}{\sqrt{-g}} \frac{\partial}{\partial \sigma^i}\left(\sqrt{-g}\, g^{ij} \frac{\partial x^\mu}{\partial \sigma^j}\right) + \Gamma^\mu_{\kappa\sigma} \frac{\partial x^\kappa}{\partial \sigma^i} \frac{\partial x^\sigma}{\partial \sigma^j} g^{ij} = 0$

  - their huge tension comparable to its energy density ("sound speed" = light velocity)

- The gravitational field of N-G membrane have been investigated in many physical contexts.
  - ex. domain walls, cosmic strings, simplest type of brane world, ... etc.
  - In particular, in standard understanding, if cosmic strings (local strings) formed in the eary universe, their energy decayed into gravitational waves by their rapid oscillations.

  $\Rightarrow$ stochastic gravitational wave background. ($h^2\Omega_{GW}$)

  Observation of pulsar timing $\xrightarrow[\text{constraints}]{}$

  $\Downarrow$

  The upper limit of the energy scale of phase transition

  $\left(\frac{M_{break}}{M_{plank}}\right)^2 \sim G\mu/c^2 \lesssim 5.4 \pm 10^{-6}$ (B.Allen 1996)

  - In the estimation of $h^2\Omega_{GW}$, the dynamics of cosmic strings is idealized by the Nambu-Goto action. (self-gravity is ignored.)

- Some reports point out that there is no free oscillations in the dynamics of gravitating N-G membranes
  - Based on the perturbation on exact solutions taking into account of membranes' self-gravity.

- H.Kodama, H.Ishihara, and Y.Fujiwara, PRD**50**,7292,(1994).
  Model : Nambu-Goto wall (codimension 1)
  (4-dim. Minkowski-Minkowski model)
  Method : Gravitational wave scattering
  Resulting Dynamics : Damping oscillations

Almost of these works show the damping oscillation of membranes

- A.Ishibashi and H.Ishihara, PRD**56**,3446,(1997).
  Model : Nambu-Goto wall (codimension 1)
  (4-dim. Minkowski-deSitter model)
  Method : Gravitational wave scattering
  Resulting Dynamics : Damping oscillations
  (**Context** : one bubble open inflation model)

There is an exceptional case

- K.Nakamura, A.Ishibashi, and H.Ishihara, PRD**62**,101502(R),(2000).
  Model : Nambu-Goto string (**codimension 2**)
  (4-dim. Minkowski with deficit angle)
  Method : Gravitational wave scattering
  Resulting Dynamics : No oscillations
- K.Nakamura and H.Ishihara, PRD**63**,127501,(2001).
  Model : Nambu-Goto string (**codimension 2**)
  (4-dim. Minkowski with deficit angle)
  Resulting Dynamics : Can oscillate continuously but ...
  "string oscillations" = "gravitational wave propagation"
  = "pp-wave exact solution"
- K.Nakamura, PRD**66**,084995,(2002).
  Model : Nambu-Goto wall Nambu-Goto wall (codimension 1)
  (4-dim. Cylindrically symmetric spacetime)
  Method : Infinitesimal time evolution
  from mometarily static initial configuration
  Resulting Dynamics : collapse with gravitational emission
- A.Ishibashi and T.Tanaka, (gr-qc/0208006).
  Model : Nambu-Goto wall (codimension 1)
  (n-dim. Maximally symmetric model)
  Method : Gravitational wave scattering
  Resulting Dynamics : Damping oscillations
  (**Context** : brane world)

- Common result :
  - The deformation variable of membranes is completly determined by G.W.
  - There is no other dynamical degree of freedom of their oscillations.

These analyses point out that

oscillatory behaviors of gravitating Nambu-Goto membranes might be quite different from those of test membranes.

(These also imply that

the estimation of $h^2\Omega_{GW}$ from cosmic string might be incorrect. (????))

Anyway ...

■ Ingredents of my talk:

To develop our arguments and to reach to our conclusion, we have also developed some techniques to treat gravity of concentrated matter distribution.

I will show these techniques.

§II. Difficulties.
(concentrated matter source in G.R.)

■ singular behavior of metric (Geroch & Traschen 1987)

This difficulty is related to the codimension of matter concentration.

The metric for the concentrated matter distribution with codimension two is not "regular".

- essence of their discussion can be seen in the Green functions of the Poisson equation.

@ Singular behavior of the Green functions.

codimension of matter distribution	Singular behavior of G. func.
1 (wall)	$\lvert x \rvert$
2 (string)	$\ln r$
3 (point particle)	$1/r$

⇒ Due to this singular behavior, it is difficult to formulate the general treatment of concentrated matter sources with codimension two.

⇒ Many researchers gave up to formulate.

⇒ The analyses depend on the situations.

Israel's paradox

- This is pointed out by the paper entitled "Cosmic string loops are straight." (Unruh et al. 1989)

- the gravitational field of a straight cosmic string has asymptotic conical structure.

(D. Garfinkle (1985), R. Gregory (1987)
T. Futamase, D. Garfinkle (1988))

⇒ One might think that the dynamics of a cosmic string is also idealized by a conical singularity (???) ⇒ ✗ rejected

The world sheet of conical singularity must be totally geodesic surface in the whole spacetime.

However,

■ These two difficulties are resolved only by taking the string thickness into consideration.

Actually...

It is shown that the Israel's paradox can be resolved only by taking the string thickness into consideration.

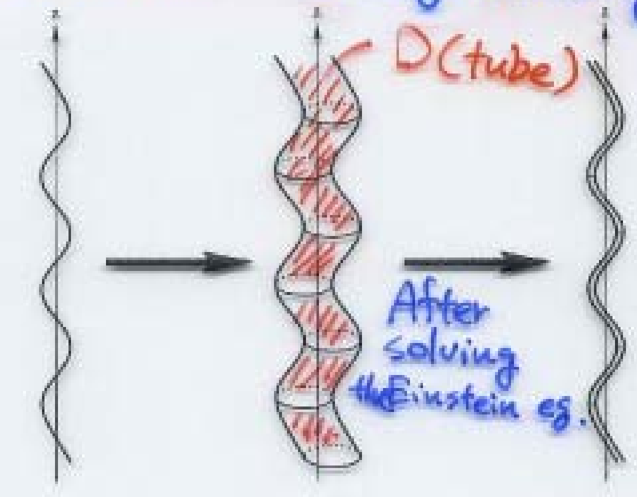
K.N. Class. Quantum Grav. 19 (2002), 783.

§III Regularization by "thick string"

Gravitating N-G thick string

To avoid Israel's paradox, we never consider the zero thickness string.

Instead, we only consider the situation, "thin string situation", in which "string bending scale" ≫ "thickness".



thin string — thick string — thin string situation (finite thickness)

We should not change the dynamics

(if "N-G string" = "cosmic string")

- **Energy momentum tensor:**

$$T_\mu{}^\nu = -\rho q_\mu{}^\nu.$$

ρ: energy density
$q_\mu{}^\nu$: intrinsic metric
support: D(tube)

- $\nabla_\nu T_\mu{}^\nu = 0$:

$$q^{\mu\nu}\nabla_\nu \rho + \rho\gamma^{\nu\rho}\nabla_\nu q^\mu_\rho = 0 \qquad \text{(the equation of continuity)},$$
$$\rho q^{\mu\rho}\nabla_\mu q_\rho{}^\nu =: \rho K^\nu = 0 \qquad \text{(the equation of motion)},$$

where $\gamma^{\mu\nu} := g^{\mu\nu} - q^{\mu\nu}$.

- After solving the Einstein equation, we consider the situation in which the string thickness is much smaller than the wavelength of the string oscillation. ("thin string situation")

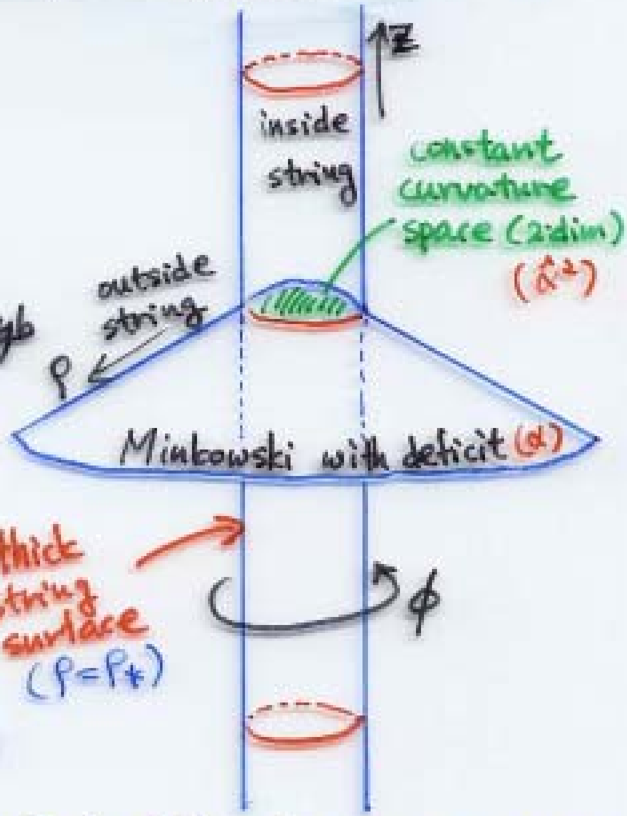
According to this procedure, we have considered the oscillatory behaviors of a gravitating N-G string.

§IV Treatments of perturbative oscillations.

(☆ the perturbation of an infinite string)

Ref: K. N., A. Ishibashi, H. Ishihara, Phys. Rev. D62 (2000), 101502(R).
K. N., H. Ishihara, Phys. Rev. D63 (2001), 127501.

▨ Background spacetime



■ Background metric

$$g_{\mu\nu}dx^\mu dx^\nu = \underbrace{\eta_{pq}dz^p dz^q}_{\text{2dim}} + \underbrace{\gamma_{ab}dy^a dy^b}_{\text{2-dim}}$$

separated

$$\eta_{pq}dz^p dz^q = -dt^2 + dz^2$$

$\gamma_{ab}dy^a dy^b$

$$= \begin{cases} d\rho^2 + \dfrac{\sin^2(\tilde{\alpha}\rho)}{\tilde{\alpha}^2}d\phi^2, \ (\rho \le \rho_*) \\ \qquad \tilde{\alpha}^2 = 8\pi G \sigma_0 , \quad \sigma_0 \text{ ; string line energy density} \\ d\rho^2 + (1-\alpha)^2\rho^2 d\phi^2, \ (\rho \ge \rho_*) \qquad 0 \le \phi \le 2\pi \end{cases}$$

■ Background energy momentum tensor

$$T_{\mu\nu} = -\sigma\,\eta_{\mu\nu}$$

$\eta_{\mu\nu}$: 4-dim extension of η_{pq}

$$\sigma = \begin{cases} \sigma_0 = \text{"constant"} & (\text{inside string } \rho \le \rho_*) \\ 0 & (\text{outside string } \rho \ge \rho_*) \end{cases}$$

■ Outside deficit angle

$2\pi\alpha = 2\pi\left(1 - \sqrt{1 - 8\pi G r_*^2 \sigma_0}\right)$, r_*: circumference radius
$(r_* = (1-\alpha)\rho_*)$

perturbative variables

metric perturbation $h_{\mu\nu}$

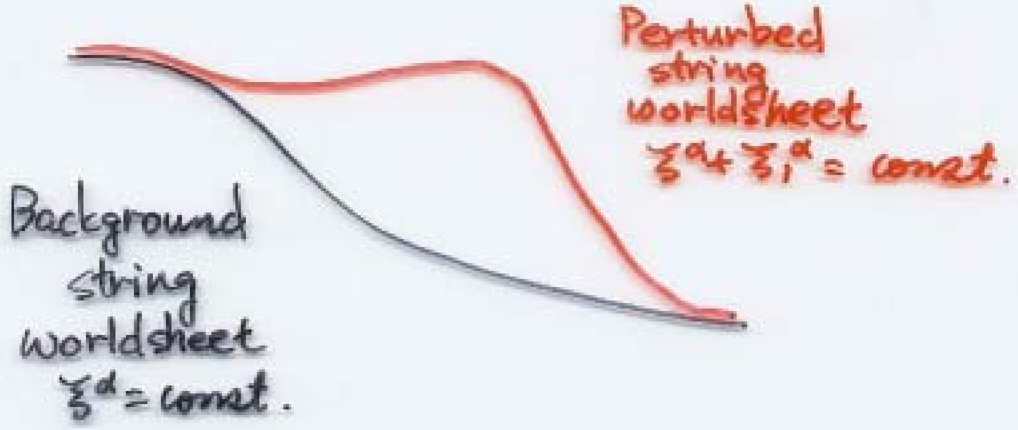
$$g_{\mu\nu} = \overset{(0)}{g}_{\mu\nu} + h_{\mu\nu}$$

String displacement ξ_1^α

$$\xi^\alpha + \xi_1^\alpha$$

$\xi^\alpha = \text{const}$: background string world sheet

$\xi^\alpha + \xi_1^\alpha = \text{const}$: perturbed string worldsheet



energy density perturbation $\delta\rho$

$$\rho = \sigma + \delta\rho$$

The perturbation of the intrinsic metric of the string world sheet is given by the above three perturbations.

$$\underline{\gamma}_\mu{}^\nu = \eta_\mu{}^\nu + \delta\underline{\gamma}_\mu{}^\nu$$

$$\delta\underline{\gamma}_\mu{}^\nu = \gamma_{\mu\rho}\,\gamma_\sigma{}^\alpha h^{\rho\sigma} - (\zeta_\alpha)^\nu \eta_{\mu\rho} \nabla^\rho \xi_1^\alpha - (\zeta_\alpha)_\mu \gamma_\rho{}^\nu \nabla^\rho \xi_1^\alpha$$

$$\left((\zeta_\alpha)_\mu := \nabla_\mu \xi_\alpha \right)$$

Respecting the symmetry of our background spacetime, we apply the Fourier expansion to consider all dynamical mode.

■ Background metric

$$ds^2 = \underbrace{\eta_{pq} dz^p dz^q}_{-dt^2 + dz^2} + \gamma_{ab} dy^a dy^b \quad , \quad \begin{cases} p, q, \cdots = t, z \\ a, b, \cdots = \rho, \phi \end{cases}$$

$$\mathcal{M} = \mathcal{M}_2 \times \mathcal{M}_1$$

■ harmonic expansion of tensors on $\mathcal{M}$

- scalar harmonics : $S = e^{-i\omega t + k_z z}$
- $\boxed{K^2 := \omega^2 - k_z^2}$
- $h_{\mu\nu}$: tensors (perturbation) on $\mathcal{M}$

$$\boxed{h_{**} = \underset{(\omega = k_z = 0)}{\text{"zero mode"}} + \underset{(\omega \neq 0,\ \text{null mode})}{\text{"}K^2 = 0 \text{ mode"}} + \text{"}K^2 \neq 0 \text{ mode"}}$$

- "zero mode" : ∘ cylindrically symmetric static perturbation. ∘ inertia motion of a string

→ irrelevant to the oscillatory motion of the string.

- "$K^2 = 0$ mode" (null mode) : perturbations propagate along the string with the light velocity. (including traveling waves)
- "$K^2 \neq 0$ mode" : perturbations propagate to the codimension of the string worldsheet.

→ Here, we consider these mode.

Further we introduce the following tensor harmonics to expand the tensors on the two dimensional spacetime $(\mathcal{M}_2, \eta)$

$\begin{cases} \eta = -dt^2 + dz^2 \\ \mathcal{M}_2 : \text{background string world sheet.} \end{cases}$

$\mathbf{k}^2 = \omega^2 - \mathbf{k}_z^2 \neq 0$ case :

$$
\begin{aligned}
S &:= e^{-i\omega t + ik_z z}, \\
V^p_{(e1)} &:= \eta^{pq}\hat{D}_q S, \\
V^p_{(o1)} &:= \epsilon^{pq}\hat{D}_q S, \\
T_{(e0)pq} &:= \frac{1}{2}\eta_{pq} S, \\
T_{(e2)pq} &:= \left(\hat{D}_p\hat{D}_q - \frac{1}{2}\eta_{pq}\hat{D}^r\hat{D}_r\right) S, \\
T_{(o2)pq} &:= -\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S,
\end{aligned}
$$

where ϵ_{pq} is the totally antisymmetric tensor on $(\mathcal{M}_2, \eta_{pq})$ and $\hat{D}_p$ is the covariant derivative associated with the metric η_{pq}.

$\mathbf{k}^2 = 0$ case :

$$
\begin{aligned}
S &:= e^{-i\omega(t + i\epsilon z)}, \\
V^p_{(e1)} &:= \eta^{pq}\hat{D}_q S =: ik^p S, \\
V^p_{(l1)} &:= il_p S, \\
T_{(e0)pq} &:= \frac{1}{2}\eta_{pq} S = -\frac{1}{2\omega^2}k_{(p}l_{q)} S, \\
T_{(e2)pq} &:= \left(\hat{D}_p\hat{D}_q - \frac{1}{2}\eta_{pq}\hat{D}^r\hat{D}_r\right) S = -k_p k_q S, \\
T_{(l2)pq}(\omega, \epsilon) &:= -l_p l_q S.
\end{aligned}
$$

where the null vector l_p is defined for each k^p by

$$k_p k^p = 0, \quad l_p l^p = 0, \quad k_p l^p = -2\omega^2.$$

Gauge freedom in the perturbations. (J.M. Stewart (1974))

To discuss the perturbative oscillations of the string, we must exclude gauge freedom in the perturbations and we develop the gauge invariant formalizm.

I think....

- It is natural to consider the gauge freedom in perturbations of manifold is the degree of freedom of point identification of each point on physical manifold to a point on the background manifold. (rather than the degree of freedom of the choice of coordinate labels.)

- In one-parameter perturbation, we are implicitly considering an one-parameter family of spacetimes which connects the background spacetime and physical spacetime.

From this point of view, gauge transformations are the changes of the point identification map.

- gauge transformation

$$\phi_Y^* Q - \phi_X^* Q = \lambda \mathcal{L}_\xi Q_0 + O(\lambda^2)$$ (Tylar expansion of ϕ^* and Q)

coordinate function

$$: \phi_Y^* x^\mu - \phi_X^* x^\mu = -\xi^\mu$$ (passive point of view)

Gauge Invariant variables.

Inspecting the gauge transformation, we define the gauge invariant variables.

Gauge transformation

$$\delta_G x^\mu = \xi^\mu \qquad \delta_G h_{\mu\nu} = -\pounds_\xi \overset{(B)}{g}_{\mu\nu}$$

harmonic expansion of metric perturbations.

• $k^2 \neq 0$ mode

$$h_{ab} = \int d^2k \left(\boxed{F_{ab}} + 2D_{(a}X_{b)} \right) S$$

$$h_{ap} = \int d^2k \left\{ \left(X_a + \frac{1}{2} D_a f_{(e2)} \right) V_p^{(e1)} + \left(\boxed{F_a} + \frac{1}{2} D_a f_{(o2)} \right) V_p^{(o1)} \right\}$$

$$h_{pq} = \int d^2k \left\{ \left(\boxed{F} + k^2 f_{(e2)} \right) T_{pq}^{(e0)} + f_{(e2)} T_{pq}^{(e2)} + f_{(o2)} T_{pq}^{(o2)} \right\}$$

F_{ab}, F_a, F ..... gauge invariant.

• $k^2 = 0$ mode

$$h_{ab} = \sum_\epsilon \int d\omega \left(\boxed{H_{ab}} + 2D_{(a}X_{b)} \right) S$$

$$h_{ap} = \sum_\epsilon \int d\omega \left\{ \left(X_a + \frac{1}{2} D_a f_{(e2)} \right) V_p^{(e1)} + \left(\boxed{H_a} + \frac{1}{4\omega^2} D_a f_{(e0)} \right) V_p^{(e1)} \right\}$$

$$h_{pq} = \sum_\epsilon \int d\omega \left\{ f_{(e0)} T_{pq}^{(e0)} + f_{(e2)} T_{pq}^{(e2)} + \boxed{H} T_{pq}^{(e2)} \right\}$$

H_{ab}, H_a, H ..... gauge invariant.

Using gauge variant metric perturbations, we can also define the gauge invariant variables for the perturbations of the string displacement and energy density.

Linearized Einstein eq. (string interior)

■ $k^2 \neq 0$ mode

- odd mode (Linearized Einstein eq.)

$$D^a F_a = 0 \;, \quad (\Delta + k^2) F_a - D^c D_a F_c = 0$$

- even mode (Linearized Einstein eq.)

$$D^c F_{ac} - \frac{1}{2} D_a F = V_a \;, \quad F_c{}^c = 0$$

$$(\Delta + k^2) F_{ab} = R F_{ab} + 2 D_{(a} V_{b)} - \gamma_{ab} D_c V^c$$

$$(\Delta + k^2) F = 0$$

($\Delta = D_a D^a$, R : curvature of string interior)

- $\delta(\nabla_\mu T^{\mu\nu}) = 0$

$$\boxed{k^2 V^a + \frac{1}{2} R D^a F = 0} \;, \quad D_a V^a + \frac{1}{2} \Sigma = 0$$

■ $k^2 = 0$ mode

- Linearized Einstein eq.

$$\Delta H_{ab} = R H_{ab} + 2 D_{(a} V_{b)} - \gamma_{ab} D_c V^c$$

$$\Delta H_a - D^c D_a H_c - 2\omega^2 D_a H = 0$$

$$\Delta H = 0 \;, \quad D^a H_a + 2\omega^2 H = 0$$

$$H_c{}^c = 0 \;, \quad D^c H_{ac} + 2\omega^2 H_a = V_a$$

- $\delta(\nabla_\mu T^{\mu\nu}) = 0$

$$\boxed{2\omega^2 \sigma H^a = 0} \;, \quad D^a V_a + \frac{1}{2} \Sigma = 0$$

These equations are also derived from the perturbation of N-G action within the first order w.r.t. the string oscillation amplitude.

■ Displacement of a thin string

After constructing global solutions,

We consider the situation where the string thickness is much smaller than the wavelength of perturbations.

Then we obtain the displacement of a thin string.

- $K^2 \neq 0$ mode displacement

$$X_S^a := \int d^2k\, S\left(-\frac{1}{R}V^a\right)\Big|_{\substack{\text{string surface} \\ m=1 \\ \text{long wavelength}}}$$

$$= \int d^2k \frac{\kappa \boxed{B} S}{2(1-\alpha)\Gamma\left(\frac{2-\alpha}{1-\alpha}\right)} \boxed{\left(\frac{\kappa \rho_*}{2}\right)^{\frac{\alpha}{1-\alpha}}} e^{i\phi}(n^a + i\tau^a)$$

B … amplitude of incidental gravitational waves.

ρ_* … string thickness.

- $K^2 = 0$ mode displacement

$$X_S^a = -\int \frac{\boxed{B_1}}{2(1-\alpha)} \boxed{\rho_*^{\frac{\alpha}{1-\alpha}}} S e^{i\phi}(n^a + i\tau^a)$$

B_1 … amplitude of "cosmic string traveling wave"
(can be regarded as gravitational wave)

Two remarkable points

- A thin string cannot bend nor oscillate without gravitational waves.
  (Oscillations of a thin string are gravitational wave propagation itself.)
- A zero thickness string does not bend.
  (c.f. Unruh et.al (1989))

Solutions and Displacement of a thin string

Solutions to linearized Einstein eq. (interior and exterior)

- $K^2 \neq 0$ mode

$$F_a = \epsilon_{ab} D^b \Phi_{(o)}$$
$$F_{ab} = (D_a D_b - \tfrac{1}{2}\gamma_{ab}\Delta)\Phi_{(e)}, \quad F = \Delta \Phi_{(e)}$$
$$V_a = \tfrac{1}{2} R D_a \Phi_{(e)}, \quad \Sigma = -R\Delta\Phi_{(e)}$$

- exterior of string

$$\Phi^{(ex)}_{(o),(e)} = \sum_{m=0}^{\infty} e^{im\phi}\left\{ A H^{(1)}_{\mu}(k\rho) + B H^{(2)}_{\mu}(k\rho) \right\}$$

- interior of string

$$\Phi^{(in)}_{(o),(e)} = \sum_{m=0}^{\infty} e^{im\phi}\, C P^{\mu}_{\nu}(x)$$

$$x := \sqrt{1-\bar{\alpha}^2\rho^2}, \quad \mu := \frac{m}{1-\alpha}, \quad \nu(\nu+1) := \frac{K^2}{\bar{\alpha}^2}$$

- $K^2 = 0$ mode

$$H = 0 = H_a$$

- exterior of string

$$H_{ab} = D_a D_b \Phi^{(ex)}, \quad \Phi^{(ex)} = A_0 \ln\rho + \sum_{m=1}^{\infty}\left(A_m \rho^{-\frac{m}{1-\alpha}} + B_m \rho^{\frac{m}{1-\alpha}}\right) e^{im\phi}$$

- interior of string

$$H_{ab} = (D_a D_b - \tfrac{1}{2}\gamma_{ab}\Delta)\Phi^{(in)} - \epsilon_{c(a}D_{b)}D^c \Psi^{(in)}$$
$$V_a = \tfrac{1}{2} D_a(\Delta + R)\Phi^{(in)} + \tfrac{1}{2}\epsilon_{ac}D^c(\Delta+R)\Psi^{(in)}$$
$$\Sigma = -\Delta(\Delta+R)\Phi^{(in)}$$

$\Phi^{(in)}, \Psi^{(in)}$: arbitrary functions on $\mathcal{M}$

Global solution

Global solution should be obtained by matching these interior and exterior solutions at the string surface using Israel's junction condition.

§V Consistency with the dynamics of a test string

I have also checked the consistency of our result with the dynamics of a test string (K.N. preprint gr-gc/0302057)

To do this, we change the background spacetime from an exact solution to the Einstein equation to the Minkowski spacetime.

By this change, we have two infinitesimal parameters for the perturbations.

Two-infinitesimal parameters:

$$\begin{cases} \epsilon: & \text{string oscillation amplitude} \\ \lambda: & \text{string energy density (additional)} \end{cases}$$

I developed two-parameter gauge invariant perturbation theory.

The analyses are completely parallel to those shown before but gauge transformations are slightly different from the linear order perturbations.

IV. GAUGE TRANSFORMATION

We must care the gauge freedom arise by the perturbation theory (degree of freedom of point identification).

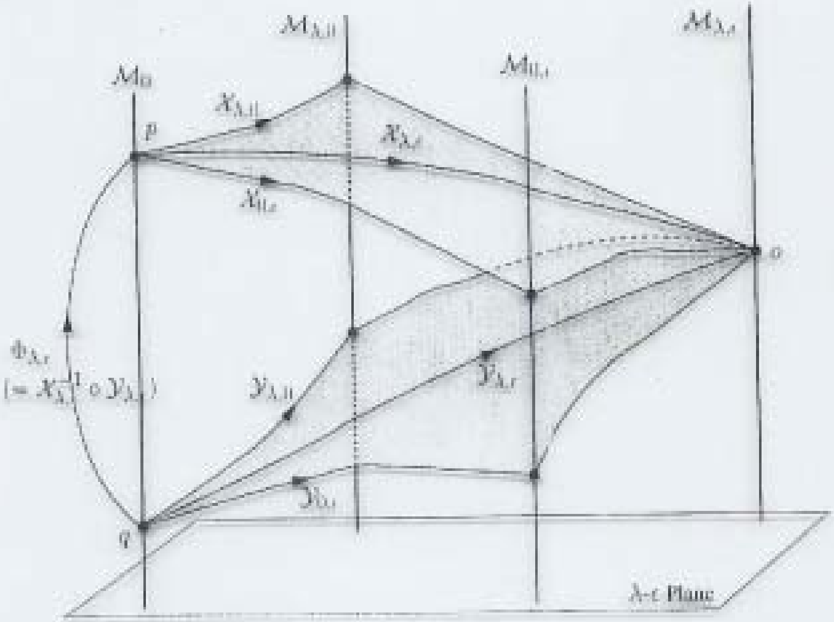


Figure 6: Point identification in two parameter perturbation

In two parameter perturbation of $O(\epsilon\lambda)$, we must consider three kind of gauge transformation. (Q : physical variables on the full spacetime)

- $O(\epsilon)$ variable $\delta^\epsilon Q$, (generator : ${}^\epsilon\xi$) :

$$\delta_G\left(\delta^\epsilon Q\right) = \pounds_{{}^\epsilon\xi} Q_0.$$

- $O(\lambda)$ variable $\delta^\lambda Q$, (generator : ${}^\lambda\xi$) :

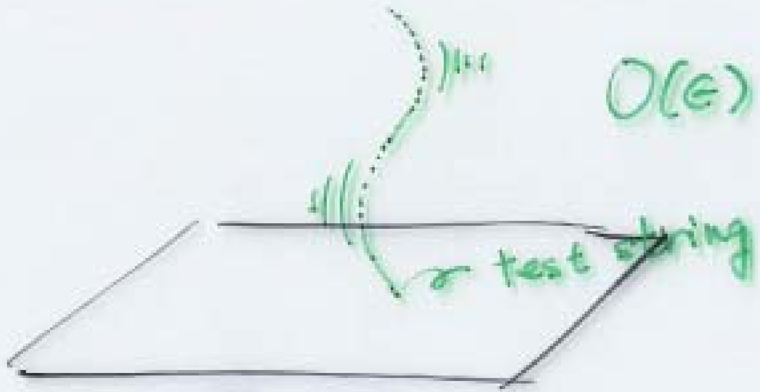
$$\delta_G\left(\delta^\lambda Q\right) = \pounds_{{}^\lambda\xi} Q_0.$$

- $O(\epsilon\lambda)$ variable $\delta^{\epsilon\lambda} Q$, (generator : ${}^{\epsilon\lambda}\xi$) :

$$\delta_G\left(\delta^\epsilon Q\right) = \pounds_{{}^\epsilon\xi} Q_0 + \pounds_{{}^\lambda\xi} Q_0 + \left\{\pounds_{{}^{\epsilon\lambda}\xi} + \frac{1}{2}\pounds_{{}^\epsilon\xi}\pounds_{{}^\lambda\xi} + \frac{1}{2}\pounds_{{}^\lambda\xi}\pounds_{{}^\epsilon\xi}\right\} Q_0.$$

See gr-qc/0303090

Physical meaning of two-parameter perturbations.

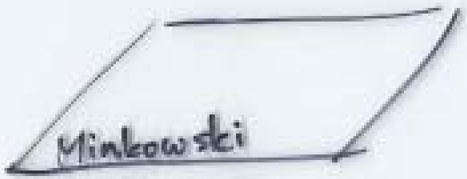


$O(\epsilon)$

$T_{\mu\nu} = 0$

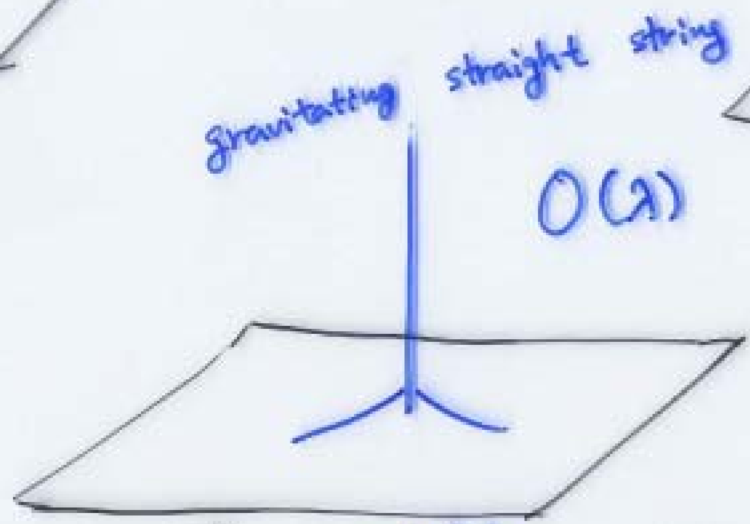
$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}$

$0 = \lambda$
(background)



$T_{\mu\nu} = 0$

$g_{\mu\nu} = \eta_{\mu\nu}$

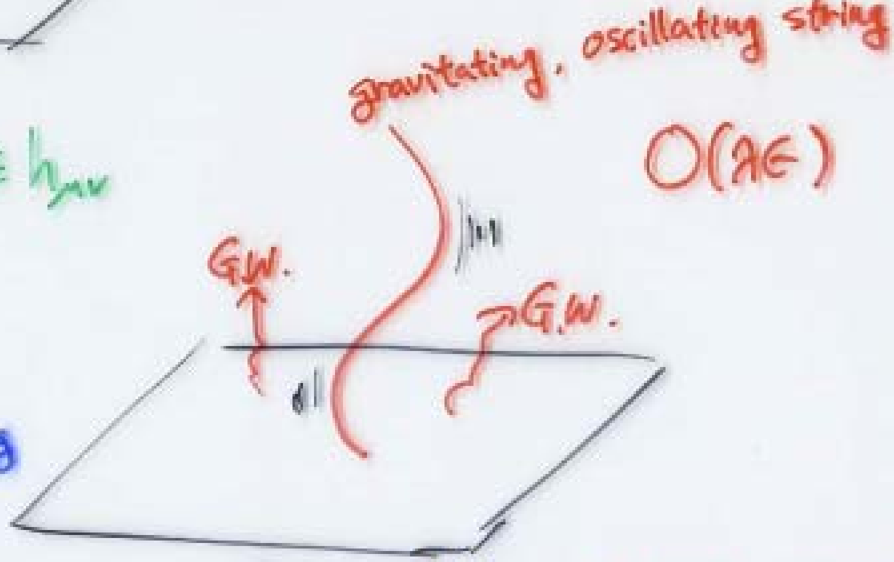


$O(\lambda)$

$T_{\mu\nu} = -\lambda \rho g_{\mu\nu}$

$g_{\mu\nu} = \eta_{\mu\nu} + \lambda l_{\mu\nu}$

static solution.



$O(\lambda\epsilon)$

$T_{\mu\nu} = -\lambda \rho g_{\mu\nu} + \lambda\epsilon(\delta\rho\, g_{\mu\nu} + \rho\, \delta g_{\mu\nu})$

$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} + \lambda l_{\mu\nu} + \epsilon\lambda k_{\mu\nu}$

I only show $\kappa=0$ mode Einstein equation.

- $O(\lambda\epsilon)$ order $\kappa = 0$ mode Einstein eq. (Minkowski Background)

$$
\begin{aligned}
&\Delta K_{ab} = 2\left(D_{(a}V_{b)} - \frac{1}{2}\gamma_{ab}D_cV^c\right),\\
&D^cK_c + 2\omega^2 K = 0,\\
&D^cK_c{}^a + 2\omega^2 K^a = V^a,\\
&\Delta K^a = 0,\\
&K_c{}^c = 0,\\
&\Delta K = 0,\\
&D_aV^a + \frac{1}{2}\Sigma = 0.
\end{aligned}
$$

D_a : covariant derivative associated with the metric $\gamma_{ab}dx^a dx^b = dr^2 + r^2 d\phi^2$ (flat)

$\Updownarrow$ almost same

cf. $\kappa = 0$ mode Einstein eq. in the previous work
(on the background spacetime including gravitating straight string)

$$
\begin{aligned}
&\Delta H_{ab} = \boxed{\mathcal{R}H_{ab}} + 2\left(D_{(a}V_{b)} - \frac{1}{2}\gamma_{ab}D_cV^c\right),\\
&D^cH_c + 2\omega^2 H = 0,\\
&D^cH_c{}^a + 2\omega^2 H^a = V^a,\\
&\Delta H^a = 0,\\
&H_c{}^c = 0,\\
&\Delta H = 0,\\
&D_aV^a + \frac{1}{2}\Sigma = 0.
\end{aligned}
$$

proportional to background curvature

D_a : covariant derivative associated with the metric

$$
\gamma_{ab}\,dx^a dx^b = \begin{cases} dr^2 + (1-\alpha)^2 r^2 d\phi^2 \\ d\rho^2 + \dfrac{\sin^2 \lambda\rho}{\lambda^2} d\phi^2 \end{cases}
$$

(non-flat)

- Due to the diffecence, we obtain the string oscillations without gravitational waves but these corresponds to previous solutions with gravitational waves.
- There is no other solution describing string oscillations.

§VI Summary

- Our treatments of oscillations of a gravitating Nambu-Goto string.
  - Difficulties which should be resolved:
    - singular metric
    - Israel's paradox

⇒ (resolution)
Thickness introduction without changing the dynamics.

⇒ (conclusion)
The perturbative oscillations of a gravitating Nambu-Goto string is nothing but the propagation of gravitational wave.

(There is no other dynamical degree of freedom of their oscillations in the case of an infinite string.)

- Consistency check by two-parameter perturbations.

The above conclusion is consistent with the dynamics of a test string.

(by developing the two-parameter perturbation theory on the Minkowski background.)

- These analyses are for the exceptional case in which traveling waves exist.
  - Traveling waves are PP-wave exact solutions.

The existence of pp-wave $\xLeftrightarrow{\text{closely related}}$ spacetime symmetry

Such solutions will not exist in generic situations.
(This is my speculation. Some proofs are necessary.)

Actually...

There are some exact soluble models in which the existence of traveling waves is not confirmed.

$\Downarrow$

It is interesting to consider the consistency with the dynamics of a test membrane using such a model. ($\Rightarrow$ future work)

Anyway, ...

I hope that our attempts shown here are instructive to understand the radiation reaction problem of gravitational waves.