Gauge Problem in the Gravitational Self-Force

- First Post Newtonian Force under Regge-Wheeler Gauge -

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§1. Introduction

* We want to derive a SELF-FORCE on a particle in a black hole spacetime.

Point particle =====> Singular

Need **Regularization**

Gravitational self-force $O(\mu/M)$: MiSaTaQuWa('97)

μ : Particle's mass

M : Black hole mass

$$\mu \frac{D^2 z^\mu(\tau)}{d\tau^2} = F^\mu(z)$$

$$F^\mu = -\frac{\mu}{2}(g^{\mu\nu} + u^\mu u^\nu)\left(2h^{\mathrm{tail,H}}_{\nu\beta;\alpha} - h^{\mathrm{tail,H}}_{\alpha\beta;\nu}\right) u^\alpha u^\beta$$

$h^{\mathrm{tail,H}}$: Tail part: inside the past null cones

Gives the physical self-force

(identified under a harmonic(H) gauge)

u^μ : Four-velocity of a particle

* We consider to calculate the tail part.

Schwarzschild background

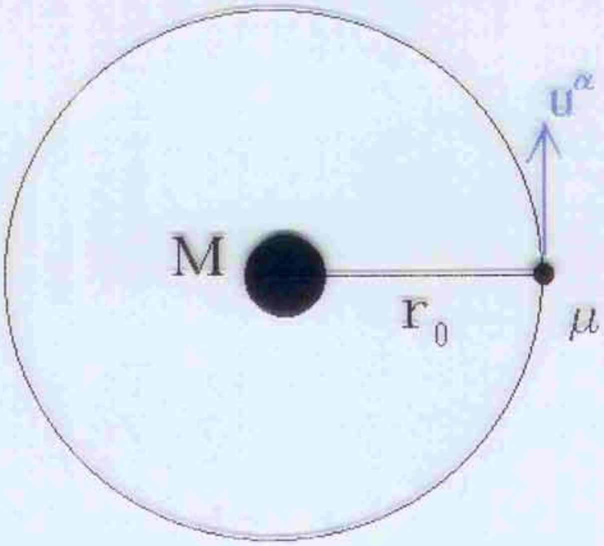
$$g^{(\mathrm{b})}_{\mu\nu} dx^\mu dx^\nu \equiv -f(r)dt^2 + f(r)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$f(r) \equiv 1 - \frac{2M}{r}$$

Particle's motion : Circular orbit

$$\{z_0^\alpha\} \equiv \{t_0,\, r_0,\, \pi/2,\, \phi_0\}$$

$$\{u^\alpha\} \equiv \{1/\sqrt{1-3M/r_0},\, 0,\, 0,\, \sqrt{M/r_0^2(r_0-3M)}\}$$



It is difficult to calculate the tail part **directly**.

$$h_{\alpha\beta}^{\text{tail},\text{H}} = h_{\alpha\beta}^{\text{full},\text{H}} - h_{\alpha\beta}^{\text{dir},\text{H}}$$

Slight modification **Detweiler & Whiting ('02)**

$$h_{\alpha\beta}^{\text{R},\text{H}} = h_{\alpha\beta}^{\text{full},\text{H}} - h_{\alpha\beta}^{\text{S},\text{H}}$$

$h_{\mu\nu}^{\text{S},\text{H}}$: S part : Inhomogeneous solution of the linearized Einstein equation under H gauge

$h_{\mu\nu}^{\text{R},\text{H}}$: R part : Homogeneous solution

Tail force = R force

$$F_\alpha\left[h_{\mu\nu}^{\text{tail},\text{H}}\right] = F_\alpha\left[h_{\mu\nu}^{\text{R},\text{H}}\right]$$

Let's consider a **Finite Gauge Transformation**.

$$x_{\text{RW}}^{\alpha} = x_{\text{H}}^{\alpha} - \xi_{\text{H}\rightarrow\text{RW}}^{\alpha}$$

Finite Gauge Transformation

$$\lim_{x\to z(\tau)} F_\alpha\left[h^{\mathrm{R,H}}_{\mu\nu} + 2\,\xi^{\mathrm{H}\to\mathrm{RW}}_{(\mu;\nu)}\left[h^{\mathrm{R,H}}_{\mu\nu}\right]\right](x)$$

$$= \lim_{x\to z(\tau)} F_\alpha[h^{\mathrm{full,H}}_{\mu\nu} - h^{\mathrm{S,H}}_{\mu\nu} + 2\,\xi^{\mathrm{H}\to\mathrm{RW}}_{(\mu;\nu)}\left[h^{\mathrm{full,H}}_{\mu\nu} - h^{\mathrm{S,H}}_{\mu\nu}\right]](x)$$

$$= \lim_{x\to z(\tau)} F_\alpha[h^{\mathrm{full,H}}_{\mu\nu} + 2\,\xi^{\mathrm{H}\to\mathrm{RW}}_{(\mu;\nu)}\left[h^{\mathrm{full,H}}_{\mu\nu}\right] - h^{\mathrm{S,H}}_{\mu\nu} - 2\,\xi^{\mathrm{H}\to\mathrm{RW}}_{(\mu;\nu)}\left[h^{\mathrm{S,H}}_{\mu\nu}\right]](x)$$

$$= \lim_{x\to z(\tau)} F_\alpha\left[h^{\mathrm{full,RW}}_{\mu\nu} - h^{\mathrm{S,H}}_{\mu\nu} - 2\,\xi^{\mathrm{H}\to\mathrm{RW}}_{(\mu;\nu)}\left[h^{\mathrm{S,H}}_{\mu\nu}\right]\right](x)$$

$$= \lim_{x\to z(\tau)} \left(F_\alpha\left[h^{\mathrm{full,RW}}_{\mu\nu}\right](x) - F_\alpha\left[h^{\mathrm{S,H}}_{\mu\nu} + 2\,\xi^{\mathrm{H}\to\mathrm{RW}}_{(\mu;\nu)}\left[h^{\mathrm{S,H}}_{\mu\nu}\right]\right](x)\right)$$

$$= F^{\mathrm{RW}}_\alpha(\tau)$$

This is a **Regge-Wheeler Self-Force**.

§2. Full Force

⋆First, we consider the *Full metric perturbation.*

Linearized Einstein Equation

$$
\begin{aligned}
\boldsymbol{h} = \sum_{\ell m}\Bigg[& f(r)H_{0\ell m}(t,r)\boldsymbol{a}^{(0)}_{\ell m} - i\sqrt{2}H_{1\ell m}(t,r)\boldsymbol{a}^{(1)}_{\ell m} + \frac{1}{f(r)}H_{2\ell m}(t,r)\boldsymbol{a}_{\ell m} \\
& -\frac{i}{r}\sqrt{2\ell(\ell+1)}h^{(e)}_{0\ell m}(t,r)\boldsymbol{b}^{(0)}_{\ell m} + \frac{1}{r}\sqrt{2\ell(\ell+1)}h^{(e)}_{1\ell m}(t,r)\boldsymbol{b}_{\ell m} \\
& +\sqrt{\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)}G_{\ell m}(t,r)\boldsymbol{f}_{\ell m} + \left(\sqrt{2}K_{\ell m}(t,r) - \frac{\ell(\ell+1)}{\sqrt{2}}G_{\ell m}(t,r)\right)\boldsymbol{g}_{\ell m} \\
& -\frac{\sqrt{2\ell(\ell+1)}}{r}h_{0\ell m}(t,r)\boldsymbol{c}^{(0)}_{\ell m} + \frac{i\sqrt{2\ell(\ell+1)}}{r}h_{1\ell m}(t,r)\boldsymbol{c}_{\ell m} \\
& +\frac{\sqrt{2\ell(\ell+1)(\ell-1)(\ell+2)}}{2r^2}h_{2\ell m}(t,r)\boldsymbol{d}_{\ell m}\Bigg]
\end{aligned}
$$

Regge-Wheeler-Zerilli formalism

Regge-Wheeler Equation

$$
\frac{d^2R^{(\mathrm{odd/even})}_{\ell m\omega}}{dr^{*2}} + [\omega^2 - V_\ell(r)]R^{(\mathrm{odd/even})}_{\ell m\omega} = S^{(\mathrm{odd/even})}_{\ell m\omega}
$$

is solved by a Green function method.

[Homogeneous solutions (Mano et al.('96)).]

Slow motion approximation

$$z = \omega r \sim v$$

$$\epsilon = 2M\omega \sim v^3$$

* $O(v^2)$: Up to the first post-Newtonian order

Regge-Wheeler gauge :

$$h^{(e)\mathrm{RW}}_{0\ell m\omega} = h^{(e)\mathrm{RW}}_{1\ell m\omega} = G^{\mathrm{RW}}_{\ell m\omega} = h^{\mathrm{RW}}_{2\ell m\omega} = 0$$

Regge-Wheeler full metric perturbation :

$$h^{\mathrm{RW}}_{1\ell m\omega} = \frac{r^2}{(r-2M)} R^{(\mathrm{odd})}_{\ell m\omega}$$

We obtain the **Full force** $(\ell \geq 2)$

under Regge-Wheeler gauge.

$$F^{t}_{\mathrm{full,RW}}\Big|_{\ell} = 0$$

$$F^{r(+)}_{\mathrm{full,RW}}\Big|_{\ell} = -\frac{(\ell+1)\,\mu^2}{r_0^2} - \frac{1}{2}\frac{\mu^2\,(12\,\ell^3 + 25\,\ell^2 + 4\,\ell - 21)\,M}{r_0^3\,(2\,\ell+3)\,(2\,\ell-1)}$$

$$F^{r(-)}_{\mathrm{full,RW}}\Big|_{\ell} = \frac{\ell\,\mu^2}{r_0^2} + \frac{1}{2}\frac{\mu^2\,(12\,\ell^3 + 11\,\ell^2 - 10\,\ell + 12)\,M}{(2\,\ell-1)\,(2\,\ell+3)\,r_0^3}$$

$$F^{\theta}_{\mathrm{full,RW}}\Big|_{\ell} = 0$$

$$F^{\phi}_{\mathrm{full,RW}}\Big|_{\ell} = 0$$

$\cdots\big|_{\ell}$: ℓ mode in a coincidence limit

to the particle location

(+) $r > r_0$

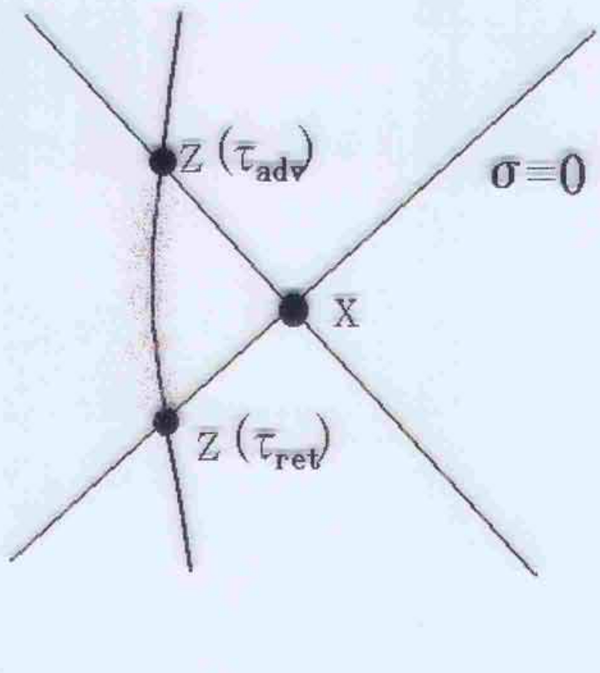
(−) $r < r_0$

§3. S-part Force

S-Part of the Metric Perturbation is derived as

$$\bar{h}^{\rm S,H}_{\mu\nu} = 4\mu\left[\frac{\bar{g}_{\mu\alpha}(x,z(\tau_{\rm ret}))\bar{g}_{\nu\beta}(x,z(\tau_{\rm ret}))u^{\alpha}(\tau_{\rm ret})u^{\beta}(\tau_{\rm ret})}{\sigma_{;\gamma}(x,z(\tau_{\rm ret}))u^{\gamma}(\tau_{\rm ret})}\right]$$
$$+2\mu\int_{\tau_{\rm ret}}^{\tau_{\rm adv}} \bar{g}_{\mu}{}^{\alpha}(x,z(\tau))\bar{g}_{\nu}{}^{\beta}(x,z(\tau))R_{\gamma\alpha\delta\beta}(z(\tau))u^{\gamma}(\tau)u^{\delta}(\tau)\,d\tau + O(y^2)$$

- $\sigma(x,z)$: bi-scalar of half the squared geodesic distance
- $\bar{g}_{\alpha\beta}(x,z)$: parallel displacement bi-vector
- $\tau_{\rm ret}(x)$: Retarded time for x
- y : Coordinate difference between x and z_0



⋆ Next, we write down by the general coordinates.

$$
\begin{aligned}
h_{tt}^{\rm S,H} = \mu\Big[& 2\,\frac{1}{\epsilon} + \Big(+2\,\frac{\Phi\,T\,{r_0}^2}{\epsilon^3} + \frac{\Phi\,T\,R^2}{\epsilon^3} - \frac{\Phi\,T}{\epsilon} + \frac{2}{3}\,\frac{\Phi^3\,T\,{r_0}^2}{\epsilon^3} + 2\,\frac{\Phi\,T\,r_0\,R}{\epsilon^3}\Big)\,u^{\phi} \\
&+\Big(-\frac{1}{2}\,\frac{\epsilon}{{r_0}^3} - \frac{R\,T^2}{\epsilon^3\,{r_0}^2} - 4\,\frac{1}{\epsilon\,r_0} - 2\,\frac{R^2}{\epsilon^3\,r_0} + \frac{R^3}{\epsilon^3\,{r_0}^2} - \frac{R^4}{\epsilon^3\,{r_0}^3} - \frac{1}{2}\,\frac{R^2\,T^2}{\epsilon^3\,{r_0}^3} + \frac{1}{2}\,\frac{T^2}{\epsilon\,{r_0}^3} \\
&\quad -\frac{5}{2}\,\frac{R^2}{\epsilon\,{r_0}^3} + 4\,\frac{R}{\epsilon\,{r_0}^2}\Big)M \\
&+\Big(-2\,\frac{{r_0}^4\,\Phi^4\,T^2}{\epsilon^5} + \frac{{r_0}^2\,T^2}{\epsilon^3} - \frac{{r_0}^2\,\Phi^2}{\epsilon} - 3\,\frac{{r_0}^4\,\Phi^2\,T^2}{\epsilon^5} - 6\,\frac{{r_0}^2\,\Phi^2\,R^2\,T^2}{\epsilon^5} \\
&\quad +\frac{2}{3}\,\frac{{r_0}^4\,\Phi^4}{\epsilon^3} - 4\,\frac{{r_0}^2}{\epsilon} - 6\,\frac{{r_0}^3\,\Phi^2\,T^2\,R}{\epsilon^5} + 3\,\frac{{r_0}^2\,\Phi^2\,T^2}{\epsilon^3} + 2\,\frac{{r_0}^3\,\Phi^2\,R}{\epsilon^3} \\
&\quad +2\,\frac{{r_0}^2\,\Phi^2\,R^2}{\epsilon^3} + \frac{{r_0}^4\,\Phi^2}{\epsilon^3}\Big)(u^{\phi})^2\Big] + O(y^2)
\end{aligned}
$$

$$
\epsilon := \left(r_0^2 + r^2 - 2\,r_0\,r\,\cos\Theta\cos\Phi\right)^{1/2},
$$

$$
T := t - t_0\,, \quad R := r - r_0\,, \quad \Theta := \theta - \frac{\pi}{2}\,, \quad \Phi := \phi - \phi_0\,.
$$

Local

↓

Tensor Harmonic Expansion

Global

↓

Gauge transformation to RW gauge

S-Part of Metric Perturbation under RW Gauge

S-Force (Mode decomposition regularization)

$$F^{t}_{\mathrm{S,RW}}\Big|_{\ell} = 0$$

$$F^{r(\pm)}_{\mathrm{S,RW}}\Big|_{\ell} = \mp\frac{1}{2}\,\frac{\mu^2\,(3\,M + 2\,r_0)}{r_0^3}\,L - \frac{1}{8}\,\frac{\mu^2\,(4\,r_0 - 7\,M)}{r_0^3} + O\!\left(\frac{1}{L^2}\right)$$

$$F^{\theta}_{\mathrm{S,RW}}\Big|_{\ell} = 0$$

$$F^{\phi}_{\mathrm{S,RW}}\Big|_{\ell} = 0\,, \qquad L = \ell + 1/2$$

* $O(\frac{1}{L^2})$ terms vanish after summing over ℓ.

(Barack, Mino, Nakano, Ori and Sasaki ('01))

(+) $r > r_0$

(−) $r < r_0$

§4. Regularization

We consider only r-component. ($\ell \geqq 2$)

[The other components are *zero*.]

$$F^r_{\rm RW}|_\ell = F^r_{\rm full,RW}|_\ell - F^r_{\rm S,RW}|_\ell$$
$$= -\mu^2 \frac{45\,M}{8(2\ell-1)(2\ell+3)\,r_0^3}$$

After summing over ℓ mode,

$$F^r_{\rm RW}(\ell \geqq 2) = -\frac{3\mu^2 M}{4r_0^3}$$

* 1PN order self-force (Reaction Force).

* This is NOT a radiation reaction force.

→ **Correction to the radius of the orbit** that deviates from the geodesic in the unperturbed background.

* How do we consider $\ell = 0$ and 1 modes?

§5. Discussion

* How do we consider $\ell = 0$ and 1 modes?

Detweiler and Whiting ('02) under H gauge.

R part : Homogeneous solution
of the Einstein equation.

$\ell = 0, 1$: DO NOT appear,
DO NOT contribute to the self-force.

——> Regularization for $\ell \geq 2$.

$$F^{H,R} = \sum_{\ell \geq 2} \left(F_\ell^{H,F} - F_\ell^{H,S} \right)$$
$$= \sum_{\ell \geq 0} \left(F_\ell^{H,F} - F_\ell^{H,S} \right)$$

Some ambiguity in the S part.

We calculate only an *approximate* S part $F_\ell^{H,S,Ap}$.

$$
\begin{aligned}
F^{H,R} &= \sum_{\ell\geq 0}\left(F_\ell^{H,F} - F_\ell^{H,S}\right) \\
&= \sum_{\ell\geq 0}\left(F_\ell^{H,F} - F_\ell^{H,S,Ap}\right) - D_{\ell\geq 0} \\
&= \sum_{\ell\geq 2}\left(F_\ell^{H,F} - F_\ell^{H,S,Ap}\right) \\
&\quad + \sum_{\ell=0,1}\left(F_\ell^{H,F} - F_\ell^{H,S,Ap}\right) - D_{\ell\geq 0}\,,
\end{aligned}
$$

Remaining term $D_{\ell\geq 0}$: $D_{\ell\geq 0} = O(y)$

[Regularization must be considered for all $\ell \geq 0$.]

Using an *ambiguity* of the mode decomposition,

$$
\begin{aligned}
\delta F_{\ell\geq 2} &= \delta F_{\ell=0,1} + O(y) \\
&= \sum_{\ell=0,1}\left(F_\ell^{H,F} - F_\ell^{H,S,Ap}\right).
\end{aligned}
$$

The $\ell = 0$, 1 modes are converted to the $\ell \geq 2$ modes.

* We can transform to the Regge-Wheeler gauge.

The $\ell = 0, 1$ odd modes : O.K!

The $\ell = 1$ even mode : Impossible to obtain (?).

$\rightarrow$ Regularization at the level of

the Weyl scalar ψ_4 or the Hertz potential Ψ.

Free from the $\ell = 0, 1$ modes.

— Future works —

* Easy way to Regularization.

* Higher Order Calculation
  in the slow motion approximation.

* Regularization of the self-force
  in the Kerr background.

$l = 1$ even gauge Eq.

$$x^{\mu}_{Z} = x^{\mu}_{RW} + \xi^{\mu}$$

$$\xi_{\mu} = \{ M_0 Y_{lm}, M_1 Y_{lm}, M_2 \nabla Y_{lm} \}$$

RW : $h_0^{(e)} = h_1^{(e)} = G = 0$

Zerilli : $h_0^{(e)} = h_1^{(e)} = K = 0$

RW ↓ Z

$$\begin{cases} h_0^{(e),RW} = -M_0 - \partial_t M_2 \\ h_1^{(e),RW} = -M_1 - r^2 \partial_r (M_2 / r^2) \\ K^{RW} = \frac{1}{r^2}(2(r-2M) M_1 - 2M_2) \end{cases}$$

harmonic → RW → Zerilli

Complete gauge fixing

※ $G^{H} = -\frac{2}{r^2} M_2$ (H → RW)