Self-force calculations using the mode-sum method

Amos Ori

Outline

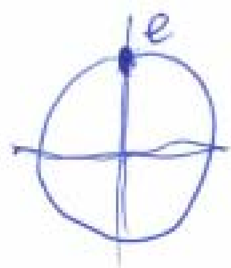
* Introduction: The basic idea
* The need to regularize the mode sum
* Formulation of the mode-sum method:
  Derivation from the tail formulation
* Calculating the regularization parameters
* Previous applications
* Application to EM/gravitational SF in Kerr
* Summary

Introduction

* The basic problem: Divergence of the force field

* The basic idea:

  Although the field diverges, the individual spherical-harmonic modes of the field are regular.



Example -- Static electric charge in flat space:

-- The Coulomb-force field diverges like $(\text{distance})^{-2}$.

– However, the field of each individual mode is *regular* -- even at the particle's location.

Two motivations:

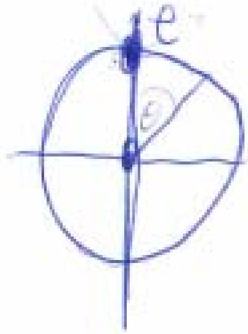
* Regularizing the particle's force field

* Allows practical calculations: Solving ODEs (for each mode) instead of PDEs.

Example: Static electric charge in flat space

Charge e located at the pole ($\theta = 0$), at $r = 1$.

We wish to calculate the r-component, F_r^{self}



The "full-force" field: $f_r = eA^t_{,r}$.

==> Decompose in spherical harmonics:

$$f_r(r,\theta) = \sum_{l=0}^{\infty} f_r^l(r) P_l(\cos\theta)$$

==> Evaluate the force at $\theta = 0$:

$$f_r(r) = \sum_l f_r^l(r)$$

The radial functions:

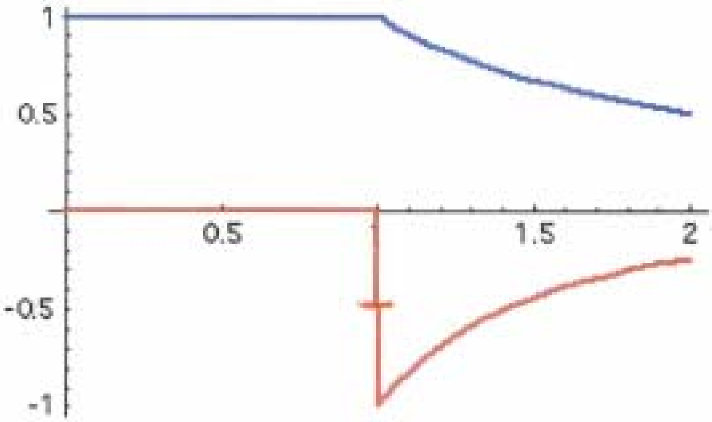
$$f_r^l(r) = \begin{cases} e^2 l r^{l-1} & (r < 1) \\ -e^2(l+1) r^{-l-2} & (r > 1) \end{cases}$$

==> Evaluate at r=1:

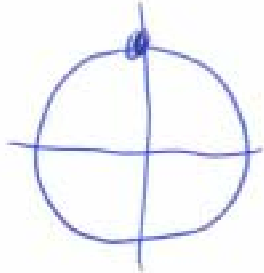
$$f_r^l = \begin{cases} e^2 l & (r \to 1_-) \\ -e^2(l+1) & (r \to 1_+) \end{cases} \Rightarrow \left\langle f_r^l \right\rangle = -(1/2)e^2$$

Example: $l = 0$

$A^{t(l=0)}$



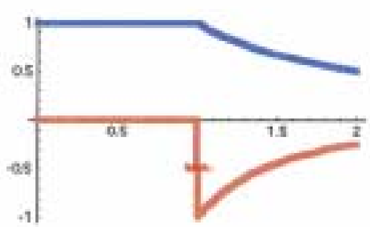
$f_r^{(l=0)}$



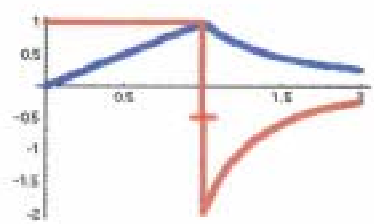
* At r=1, $f_r^{(l=0)}$ has two well-defined values: $(0, -e^2)$.

Its average value: $f_{r(\text{avarage})}^{(l=0)} = -(1/2)e^2$

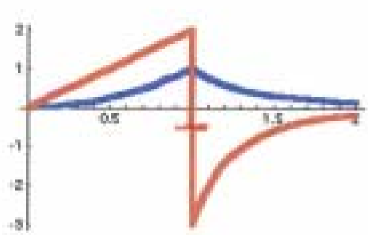
* Same qualitative behavior for all value of l:



$l = 0$



$l = 1$



$l = 2$

* The naive idea: Sum over the average values of all f_r^l .

* The problem: The sum diverges!

For example--for static charge in flat space:

$$f_r^l = \begin{cases} e^2 l & (r \to 1_-) \\ -e^2(l+1) & (r \to 1_+) \end{cases} \qquad \Rightarrow \quad \langle f_r^l \rangle = -(1/2)e^2$$

$\Rightarrow$ Sum over l diverges!

==> The mode sum needs be regularized!

* Important input for the regularization method--
The asymptotic behavior of f_r^l as $l \to \infty$.

In the above example: $f_{\pm r}^l = -e^2(\pm L + \frac{1}{2})$, $\qquad L \equiv l + \frac{1}{2}$

* The most general behavior:

$$f_{\pm\alpha}^l = A_\alpha L + B_\alpha + C_\alpha / L + O(L^{-2}) \quad , \quad L \equiv l + 1/2$$

[In above flat-space case: $A = \mp e^2,\ B = -e^2/2, C = 0$.]

* It turns out that C_α always vanishes---see below.

The tail formulation for the SF

* The *full-force* field:

$$f_\alpha^{full}(x) = \begin{Bmatrix} q(\phi_{,\alpha})_{\text{spatial}} \\ eF_{\alpha\beta}u^\beta \\ -\mu g_{\alpha\beta}(\delta\Gamma^{\beta}_{\gamma\lambda}u^\gamma u^\lambda)_{\text{spatial}} \end{Bmatrix} \equiv \begin{cases} qD(\phi) & (scalar) \\ eD(A_\lambda) & (EM) \\ \mu D(h_{\lambda\nu}) & (grav) \end{cases}$$

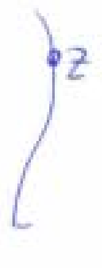
==> Schematically: $f^{full}(x) = QD(\psi^{full})$.

* Separate the full force into *direct part* and *tail part*:

$$f^{full}(x) = f^{dir}(x) + f^{tail}(x) \ .$$

* f^{dir} is known analytically: Mino-Nakano-Sasaki 2001

The regularized SF is then given by



$$F^{self}(z) = f^{tail}(x \to z) \quad (+\ \text{local terms...})$$

For the EM case -- see DeWitt-Brehme 1960; For the grav. SF see Mino-Sasaki-Tanaka 1997, and also Quinn-Wald 1997. For the scalar SF see Quinn 2000. For all cases, see also the alternative formulation by Detweiler-Whiting 2002.

Mode decomposition

* Decompose the full force, also the tail and direct parts:

$$f^{(full,dir,tail)}(x) = \sum_l f_l^{(full,dir,tail)}(x) \ .$$

* Rewrite the SF (ignoring the trivial local terms):

$$F^{self}(z) = f^{tail}(x \to z) = \sum_l f_l^{tail}(x \to z) \ .$$

Now use $f_l^{tail} = f_l^{full} - f_l^{dir}$ to obtain

$$F^{self}(z) = \sum_l [f_l^{full}(x \to z) - f_l^{dir}(x \to z)]$$

$$= \sum_l [f_{l\pm}^{full}(z) - f_{l\pm}^{dir}(z)]$$

The two terms can be summed separately after we subtract the dominant large-l piece:

$$f_{l\pm}^{full} \approx f_{l\pm}^{dir} \approx AL + B + C/L \equiv h_l \qquad (L \equiv l + 1/2)$$

* We obtain:

$$F^{self}(z) = \sum_l [f_{l\pm}^{full}(z) - h_l] - \sum_l [f_{l\pm}^{dir}(z) - h_l] \ .$$

$$F^{self}(z) = \sum_l [f_{l\pm}^{full}(z) - h_l] - \sum_l [f_{l\pm}^{dir}(z) - h_l]$$

Define now

$$D \equiv \sum_l [f_{l\pm}^{dir}(z) - \underbrace{(AL + B + C/L)}_{h_l}] \,.$$

Then,

$$F^{self}(z) = \sum_l [f_{l\pm}^{full}(z) - (AL + B + C/L)] - D \,.$$

* Four *regularization parameters* (RP): $A_\alpha, B_\alpha, C_\alpha, D_\alpha$.

==> The RP A,B,C,D are extracted *analytically* from f^{dir}:

(i) First expand f^{dir} into spherical harmonics $f_{l\pm}^{dir}$,

(ii) Extract A,B,C from the large-l behavior of $f_{l\pm}^{dir}$,

(iii) Then calculate D from the above formula.

* This method yields the SF, provided that

(1) we calculate the *mode contributions* f_l^{full} of the full force (e.g. numerically),

(2) we calculate the RP, A,B,C,D (analytically).

Calculating A,B,C,D

* Analytic expression for f^{dir}: Mino-Nakano-Sasaki 2001

Expanding f^{dir} in powers of $\delta x^{\alpha} \equiv x^{\alpha} - z^{\alpha}$:

$$f^{dir} = \frac{P^{(1)}}{\varepsilon^3} + \frac{P^{(4)}}{\varepsilon^5} + \frac{P^{(7)}}{\varepsilon^7} + (\text{terms vanishing at } x = z)$$

$$\equiv f^{(A)} + f^{(B)} + f^{(C)} + \ldots$$

ε is infinitesimal geodesic distance: $\varepsilon^2 = g_{\alpha\beta}\delta x^{\alpha}\delta x^{\beta}$.

$P^{(N)}$ are polynomials of homogeneous order N in δx^{α}.

[$f^{(A)}$ is the usual flat-space Coulombic term: $\vec{f} \propto \delta\vec{x} / \varepsilon^3$;
$f^{(B,C)}$ are higher-order corrections.]

The magnitude of the various terms:

$$f^{(A)} \propto \delta x^{-2}\,, \quad f^{(B)} \propto \delta x^{-1}\,, \quad f^{(A)} \propto \delta x^{0}\,.$$

To find A,B,C,D we need f_l^{dir} --- therefore we need to decompose $f^{(A,B,C)}$ in spherical harmonics.

Mode decomposition of $f^{(A,B,C)}$:

To find A,B,C,D we need the quantities f_l^{dir}. But

$$f^{dir} = f^{(A)} + f^{(B)} + f^{(C)} + ... \equiv \frac{P^{(1)}}{\varepsilon^3} + \frac{P^{(4)}}{\varepsilon^5} + \frac{P^{(7)}}{\varepsilon^7} + ...$$

so we need to decompose $f^{(A,B,C)}$ in spherical harmonics.

==> One obtains the exact result:

$$f_l^{(A)} = \pm aL\ ,\ \ f_l^{(B)} = b\ ,\ \ f_l^{(C)} = 0 \qquad (L \equiv l + 1/2).$$

Recalling $f_{l\pm}^{dir} \approx AL + B + C/L$, we find

$$A = \pm a \quad , \quad B = b \quad , \quad C = 0\ .$$

The definition of D:

$$D \equiv \sum_l [f_{l\pm}^{dir}(z) \ - (AL + B + C/L)],$$

now yields:

$$D = 0.$$

Calculating A,B,C,D --- general results:

* In all cases (generic geodesic orbit in Kerr, and for scalar, EM or grav. SF), we find

$$C = D = 0.$$

* The two-sided values of A,B are

$$A_{\pm} = \pm A \quad , \quad B_{\pm} = B \, .$$

==> The final expression for the self force:

$$F^{self}(z) = \sum_{l} [f_{l\pm}^{full}(z) - (\pm AL + B)] \, .$$

* F^{self} may be calculated from either the "+" side or the "-" side.

* Alternatively one can take their average,

$$f_{l(ave)}^{full} \equiv [f_{l+}^{full} + f_{l-}^{full}]/2 \, ,$$

in which case A cancels out and the SF is

$$F^{self} = \sum_{l} (f_{l(ave)}^{full} - B) \, .$$

Actual calculations of the RP values

Scalar SF:

* Static and circular orbits in Schwarzschild: A.O. (1999) (unpublished).

* Radial orbits in Schwarzschild: Barack and Ori (2000).

Gravitational SF:

* For radial orbits in Schwarzschild: Barack (2001).

For all three cases (scalar, EM, and gravitational SF):

* For generic (equatorial) geodesic orbits in Schwarzschild: Barack, Mino, Nakano, Ori, Sasaki (2002).

* For generic geodesic orbits in Kerr: Barack and Ori (2003).

Closely related mode-sum methods

* Gravitational SF for radial geodesics in Schwarzschild (RW gauge): Lousto (2000); Barack and Lousto (2002).

* Gravitational SF for circular geodesics in Schwarzschild (RW gauge): Barack and Lousto (in progress).

* Scalar SF for circular geodesics in Schwarzschild: Detweiler and Whiting (2002).

Implementations of the mode-sum method

* Implementation requires two ingredients:

(i) The RP --- already calculated analytically for all relevant cases.

(ii) The mode contributions f_l^{full}: These quantities were calculated, numerically, in several case:

Scalar SF in Schwarzschild:

Static charge: Burko (2000);
Circular orbits: Burko (2000); Detweiler & Whiting (2002)
Radial orbits: Barack and Burko (2000).

Scalar SF for static charge in Kerr: Burko&Liu (2001) #

Gravitational SF in Schwarzschild (RW gauge): #

Radial geodesic: Lousto (2000); Barack & Lousto (2002)
Circular geodesics: Barack and Lousto (in progress)

Using other variants of the mode-sum method

Calculating EM/gravitational SF in Kerr

In Kerr, separability is achieved using the Teukolsky formalism: The equations for the Weyl scalars ψ_0, ψ_4 (grav. case) or the Maxwell scalars ϕ_0, ϕ_2 (EM case) are separable.

However, for calculating the SF, we need the full-force field

$$f_\alpha^{full}(x) = \begin{Bmatrix} eF_{\alpha\beta}u^\beta \\ -\mu g_{\alpha\beta}(\delta\Gamma^\beta_{\gamma\lambda}u^\gamma u^\lambda)_{\text{spatial}} \end{Bmatrix} \equiv \begin{cases} eD(A_\lambda) & (EM) \\ \mu D(h_{\lambda\nu}) & (grav) \end{cases}$$

This requires the basic perturbations $A_\lambda, h_{\lambda\nu}$.

==> Need to reconstruct A,h from ψ_0, ψ_4 or ϕ_0, ϕ_2

Reconstruction formalism (Chrzanowski,Wald):

$$\begin{Bmatrix} \phi_0, \phi_2 \\ \psi_0, \psi_4 \end{Bmatrix} \rightarrow \begin{Bmatrix} \Psi_{EM} \\ \Psi_g \end{Bmatrix} \rightarrow \begin{Bmatrix} A \\ h \end{Bmatrix}$$

(A,h in the radiation gauge)

Construction of Ψ form $\phi_{0,2}$ or $\psi_{0,4}$: A.O. (2003).

Gravitational SF: Gauge transformations

* The gravitational SF is gauge-dependent:

$$\delta F^{\alpha}_{self} = -\mu[(g^{\alpha\lambda} + u^{\alpha}u^{\lambda})\dot{\dot{\xi}}_{\lambda} + R^{\alpha}_{\mu\lambda\nu}u^{\mu}\xi^{\lambda}u^{\nu}] \equiv \Delta(\xi)$$

* f^{α}_{full} undergoes the same transformation:

$$\delta f^{\alpha}_{full} = \delta F^{\alpha}_{self} = \Delta(\xi)$$

$\Rightarrow f^{\alpha}_{dir}$ is gauge-invariant.

Implication to the mode-sum method:

$$F^{self\,(G)}(z) = \sum_{l} [f^{full(G)}_{l\pm}(z) - (\pm AL + B + \underset{\substack{\parallel \\ 0}}{C}/L)] - \underset{\substack{\parallel \\ 0}}{D}$$

* The RP are gauge-invariant!

In particular: $C = D = 0$ in all gauges.

==> Calculate the RP once and forever, e.g. in the harmonic gauge.

The gauge-regularization problem

The radiation gauge is singular: $\xi^{(\text{harmonic}\to\text{rad})}$ is ill-defined at the particle's worldline.

==> F^{self} is ill-defined in the radiation gauge!

==> Need to transform to a regular gauge, $F^{self\,(reg)}$.

Implications to the mode-sum method:

In principle: $F^{self\,(reg)}(z) = \sum_l [f_{l\pm}^{full(reg)}(z) - (\pm AL + B)]$

But more practical:

$$F^{self\,(reg)}(z) = \sum_l [f_{l\pm}^{full(rad)}(z) - (\pm AL + B)] - \delta D^{(rad\to reg)}$$

* Calculation of $\delta D^{(rad\to reg)}$ is done by decomposing $\delta f_{full}^{\alpha(rad\to reg)} \equiv \hat{D}(\xi)$ into spherical harmonics.

==> Results for $\delta D^{(rad\to reg)}$ (A.O., unpublished):

* General formula (for generic geodesic in Kerr);
* Fully-explicit (but preliminary) result for a specific case---Circular orbit in weak-field Schwarzschild:

$$\delta D^{\varphi} = \pm(3/2)m\Omega / r^3 .$$

Summary

* Mode-sum formula:

$$F^{self}(z) = \sum_l [f_{l\pm}^{full}(z) - (\pm AL + B)].$$

* In the Gravitational case in Kerr, where one uses the (singular) radiation gauge:

$$F^{self\,(reg)}(z) = \sum_l [f_{l\pm}^{full(rad)}(z) - (\pm AL + B)] - \delta D$$

* All required RP were calculated analytically, for a generic geodesic orbit in Kerr.

==> To calculate the SF one needs to calculate f_l^{full} (e.g. numerically) in all cases of interest.

* In the EM/grav case, one may calculate $f_{l\pm}^{full(rad)}$ through

$$\begin{Bmatrix} \phi_0, \phi_2 \\ \psi_0, \psi_4 \end{Bmatrix} \rightarrow \begin{Bmatrix} \Psi_{EM} \\ \Psi_g \end{Bmatrix} \rightarrow \begin{Bmatrix} A \\ h \end{Bmatrix} \rightarrow f^{full(rad)}$$

* Need to decompose $f^{full(rad)}$ in spherical harmonics.

The remaining technical difficulty:

* In Kerr, the construction of f^{full} from (ϕ, A, h) is carried out in (spin-weighted) spheroidal harmonics.

* However, to apply the mode-sum method we need to decompose f^{full} in spherical-harmonics modes f_l^{full}.

==> Needs to decompose the spheroidal harmonics in spherical harmonics.

* This decomposition is known *in principle*, but one needs to explicitly apply it to our problem.