Easy way of regularization?

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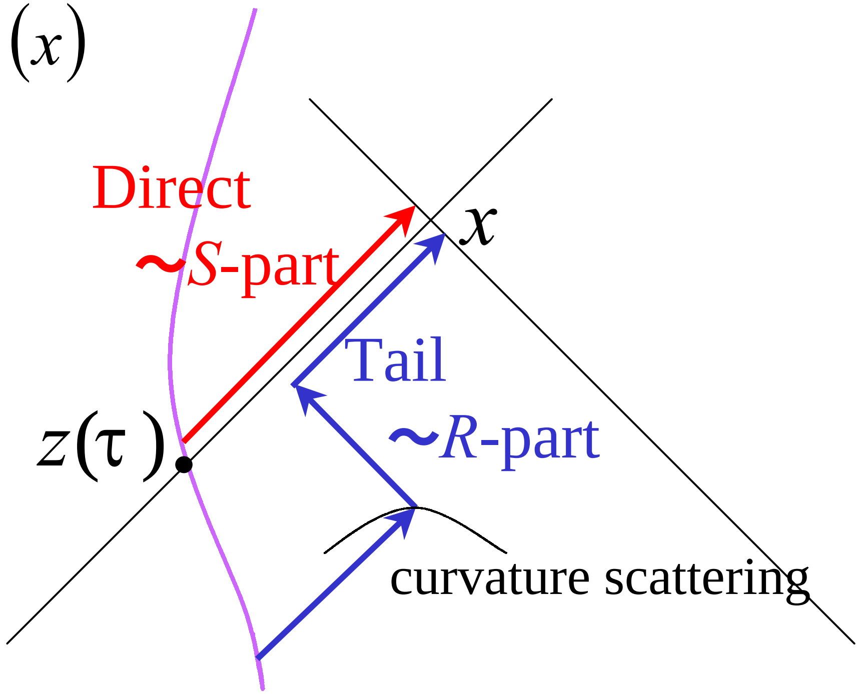
Self force in curved space

DeWitt & Breme (1960) ,Mino et al. (1997),
Quinn & Wald(1999), Detweiler & Whiting (2003)

Regularized self-force is determined by *R*-part

$$\psi^R(x) = \psi^{ret}(x) - \psi^S(x)$$



Since we don't know the way of direct construction of *R*-part, we compute

$$F^{\alpha}[\psi^{R}](\tau)=\lim_{x\to z(\tau)}\left(F^{\alpha}[\psi^{full}(x)]-F^{\alpha}[\psi^{S}(x)]\right)$$

Both terms on the r.h.s. diverge⇒regularization is needed

Mode sum regularization

Decompostion into { mode

$$F^{\alpha}[h^{full}](x)=\sum_{\ell}F^{\alpha}_{\ell}[\psi^{full}](x),\ \ F^{\alpha}[\psi^{S}](x)=\sum_{\ell}F^{\alpha}_{\ell}[\psi^{S}](x)$$

Coincidence limit is taken before summation over {

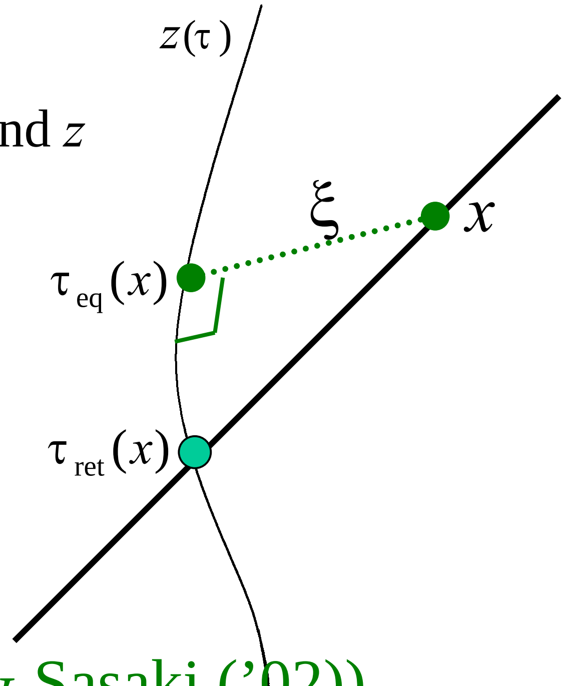
$$F^{\alpha}[h^{R}](\tau)=\sum_{\ell}\lim_{x\to z(\tau)}\left(F^{\alpha}_{\ell}[h^{full}(x)]-F^{\alpha}_{\ell}[h^{S}(x)]\right)$$

S-part

- S-part is determined by local expansion near the particle.

can be expanded in terms of

$$\begin{cases} \xi : \text{spatial distance between } x \text{ and } z \\ (x-z)^\alpha = (T, R, \Theta, \Phi) \end{cases}$$



$$\boxed{\frac{R^b \Theta^c \Phi^d}{\xi^a} f_{abcd}\left[z\left(\tau_{\mathrm{eq}}(x)\right), u\left(\tau_{\mathrm{eq}}(x)\right)\right]}$$

- Mode decomposition formulae

(Barack and Ori ('02), Mino Nakano & Sasaki ('02))

$$F^{(\mathrm{S})}_{\alpha,\ell} = A_\alpha L + B_\alpha + C_\alpha / L + D_{\alpha\ell} \quad \text{where } L = \ell + \tfrac{1}{2}$$

$$\boxed{C_\alpha = \sum D_{\alpha\ell} = 0}$$

Solving *full* part

Master equation for BH perturbations

$$\boxed{L\psi = 4\pi\sqrt{-g}\,T}$$

Regge-Wheeler eq. or
Teukolsky eq.

Once homogeneous equation is solved,
Green function method is applicable to calculate *full* part.

$$\boxed{L\psi = 0}$$

$$\psi = \sum_{\ell m} R_{\ell m}(r) Y_{\ell m}(\theta, \varphi) e^{-i\omega t}$$

$$\left[\partial_r^2 + \cdots\right] R_{\ell m}(r) = 0$$

Spherical(Spheroidal)
harmonics expansion

Green function method

$$\psi = \int d^4x \, G(x, x') T(x')$$

Green function

$$G(x, x') = \sum_{\ell m} \int d\omega \, e^{-i\omega(t-t')} g_{\ell m\omega}^{full}(r, r') Y_{\ell m}(\Omega) Y_{\ell m}^*(\Omega')$$

Radial part of Green function

$$g_{\ell m\omega}^{full}(r, r') = \frac{1}{W_{\ell m\omega}(R_{\rm in}, R_{\rm up})} \left(R_{\rm in}(r) R_{\rm up}(r') \theta(r' - r) + R_{\rm up}(r) R_{\rm in}(r') \theta(r - r') \right)$$

$$W_{\ell m\omega}(R_{\rm in}, R_{\rm up}) = r^2 \left(1 - \frac{2M}{r}\right) \left(\left(\frac{d}{dr} R_{\rm up}(r)\right) R_{\rm in}(r) - \left(\frac{d}{dr} R_{\rm in}(r)\right) R_{\rm up}(r) \right)$$

Problem

Now, problem is remaining in the *full*-part !

	(ζ, R, Θ, Φ)	Harmonic expansion
Time domain	*S*-part $\dfrac{R^b \Theta^c \Phi^d}{\xi^a}$	$F^S_{\alpha,\ell} = A_\alpha L + B_\alpha + D_{\alpha\ell}$ $\sum D_{\alpha\ell} = 0$ **?**
Frequency domain		*full*-part

As a simple example we discuss scalar charged particle in Schwarzshild spacetime.

Systematic method of solving the radial functions

(Mano-Takasugi-Suzuki)

Expansion in terms of Coulomb wave fn. $z = \omega r$

$$R_c^\nu(x) \approx \sum_{n=-\infty}^{\infty} a_n^\nu p_{n+\nu}(z),$$

ν: eigenvalue (determined later)

$$p_{n+\nu}(z) \approx z^{n+\nu} e^{-iz} {}_2F_0(*,*;2iz)$$

Problem is reduced to solving 3 terms recursion eq.

$$\alpha_n^\nu a_{n+1}^\nu + \beta_n^\nu a_n^\nu + \gamma_n^\nu a_{n-1}^\nu = 0$$

$\epsilon = 2M\omega$

$$\begin{aligned}
\alpha_n^\nu &= O(\epsilon) \\
\beta_n^\nu &\approx (n+\nu)(n+\nu+1) - l(l+1) \\
\gamma_n^\nu &= O(\epsilon)
\end{aligned}$$

$$\boxed{\frac{a_n^\nu}{a_{n-1}^\nu} \xrightarrow[n\to+\infty]{} 0 \qquad \frac{a_n^\nu}{a_{n+1}^\nu} \xrightarrow[n\to-\infty]{} 0}$$

$\Rightarrow$ determines ν and a_n^ν

New decomposition of *full*-part

Essence of the new method

$$g^{\text{full}}_{\ell m\omega}(r,r') = \frac{1}{W_{\ell m\omega}(R_{\text{in}},R_{\text{up}})}\left(R_{\text{in}}(r)R_{\text{up}}(r')\theta(r'-r) + R_{\text{up}}(r)R_{\text{in}}(r')\theta(r-r')\right)$$

$$W_{\ell m\omega}(R_{\text{in}},R_{\text{up}}) = r^2\left(1-\frac{2M}{r}\right)\left(\left(\frac{d}{dr}R_{\text{up}}(r)\right)R_{\text{in}}(r) - \left(\frac{d}{dr}R_{\text{in}}(r)\right)R_{\text{up}}(r)\right)$$

$$\begin{aligned} R_{\text{in}} &= R^{\nu}_{\text{C}} + \beta_\nu R^{-\nu-1}_{\text{C}} \\ R_{\text{up}} &= \gamma_\nu R^{\nu}_{\text{C}} + R^{-\nu-1}_{\text{C}} \end{aligned} \qquad \nu = \ell + O(\epsilon^2)$$

$$R^{\nu}_{c}(x) \approx z^{\nu}\underline{\left(1 + z^2 + \frac{\epsilon}{z} + \cdots\right)}$$

Post-Newtonian expansion

$$z^2 = (\omega r)^2 = O(v^2)$$

$$\frac{\epsilon}{z} = \frac{2M\omega}{\omega r} = O(v^2)$$

$\boxed{\textit{No log } \omega}$

We divide the Green function into two parts,

$$g^{full}_{\ell m\omega}(r,r') = g^{(\tilde S)}_{\ell m\omega}(r,r') + g^{(\tilde R)}_{\ell m\omega}(r,r')$$

where

$$g^{\tilde S}_{\ell m\omega}(r,r') = \frac{1}{W_{\ell m\omega}(R^{\nu}_{\mathrm{C}}, R^{-\nu-1}_{\mathrm{C}})}\left[\theta(r'-r)R^{\nu}_{\mathrm{C}}(r)R^{-\nu-1}_{\mathrm{C}}(r') + \theta(r-r')R^{-\nu-1}_{\mathrm{C}}(r)R^{\nu}_{\mathrm{C}}(r')\right]$$

$$g^{\tilde R}_{\ell m\omega}(r,r') = \frac{1}{(1-\beta_\nu\gamma_\nu)W_{\ell m\omega}(R^{\nu}_{\mathrm{C}}, R^{-\nu-1}_{\mathrm{C}})}\Big[\beta_\nu\gamma_\nu\left(R^{\nu}_{\mathrm{C}}(r)R^{-\nu-1}_{\mathrm{C}}(r') + R^{-\nu-1}_{\mathrm{C}}(r)R^{\nu}_{\mathrm{C}}(r')\right)$$

$$+\gamma_\nu R^{\nu}_{\mathrm{C}}(r)R^{\nu}_{\mathrm{C}}(r') + \beta_\nu R^{-\nu-1}_{\mathrm{C}}(r)R^{-\nu-1}_{\mathrm{C}}(r')\Big]$$

$O(v^{6\{})$

$O(v^{2\{+1})$ $O(v^{4\{-1})$

$\tilde R$-part
- ○ finite { - sum ⇒ regular
- ○ no step function ⇒ homogeneous solution

All singular behavior is in $\tilde S$-part.

⇒ The coefficients A_α and B_α must be common with S-part.

$$\lim_{x\to z_0} F^{\tilde S}_{\alpha\ell} = A_\alpha L + B_\alpha + \tilde D_{\alpha\ell}$$

⇒ The resulting force from $\tilde S$-part after subtracting S-part is

$$\sum_{\ell=0}^{\infty}\lim_{x\to z_0} F^{\tilde S - S}_{\alpha\ell} = \sum_{\ell=0}^{\infty}\tilde D_{\alpha\ell}$$

Our strategy

	(ζ, R, Θ, Φ)	Harmonic expansion
Time domain	*S*-part $\dfrac{R^b \Theta^c \Phi^d}{\xi^a}$	$F^{S}_{\alpha,\ell} = A_\alpha L + B_\alpha + D_{\alpha\ell}$, $\sum D_{\alpha\ell} = 0$ $F^{\tilde{S}}_{\alpha,\ell} = A_\alpha L + B_\alpha + \tilde{D}_{\alpha\ell}$
Frequency domain		*full*-part $\tilde{S}$-part $\tilde{R}$-part

Regularized force up to n -PN

$$\sum_{\ell=0}^{\infty} F^{R}_{\alpha,\ell} = \sum_{\ell=0}^{\infty} \left(\lim_{x \to z_0} F^{\tilde{S}-S}_{\alpha,\ell} + F^{\tilde{R}}_{\alpha,\ell} \right) = \sum_{\ell=0}^{\infty} \tilde{D}_{\alpha\ell} + \sum_{\ell=0}^{n} F^{\tilde{R}}_{\alpha,\ell}$$

$\tilde{S}$-part in time domain

$$g^{(\tilde{S})}_{\ell m\omega}(r,r') = \frac{1}{W_{\ell m\omega}(R^{\nu}_{\mathrm{C}}, R^{-\nu-1}_{\mathrm{C}})}\left[\theta(r'-r)R^{\nu}_{\mathrm{C}}(r)R^{-\nu-1}_{\mathrm{C}}(r') + \theta(r-r')R^{-\nu-1}_{\mathrm{C}}(r)R^{\nu}_{\mathrm{C}}(r')\right]$$

$$R^{\nu}_{c}(x) \approx z^{\nu}\left(1 + z^2 + \frac{\epsilon}{z} + \cdots\right) \qquad \begin{array}{l} z = \omega r \\ \epsilon = 2M\omega \end{array}$$

$$= \sum_{k=0}^{\infty} \omega^k X_{\ell m k}(r,r')$$

Since there is no logω, ω –integral is easy

$$\int d\omega\, \omega^n e^{-i\omega(t-t')} = 2\pi(-i)^n \partial^n_{t'}\delta(t-t')$$

$$F^{\tilde{S}}_{\alpha,\ell} = q^2 \nabla_\alpha \lim_{x\to z(t)} \sum_{m,k} (i\partial_t)^k \frac{d\tau(t)}{dt} X_{\ell m k}(r, z^r(t)) Y_{\ell m}(\theta,\varphi) Y^*_{\ell m}(z^\theta(t), z^\varphi(t))$$

Result for S-$\tilde{S}$ part

Here we assume geodesic motion to eliminate higher derivatives.
After lengthy calculation, we have

$$F_t^{\tilde{S}-S} = q^2 \frac{v^r(t)}{4\pi (z^r)^2} \sum_i^4 K_t^{(i)} \delta_{\mathcal{E}}^i \qquad \delta_{\mathcal{E}} = 1 - \frac{1}{\mathcal{E}^2}$$

where the coefficients $K_t^{(i)}$ are

$$\begin{aligned}
K_t^{(0)} &= -\Big[\frac{9}{19} + \frac{10364}{1659}\frac{M}{(z^r)} + \frac{20728}{1659}\frac{M^2}{(z^r)^2} + \left(\frac{5246140232891518}{35013238792623} - \frac{27\pi^2}{2}\right)\frac{M^3}{(z^r)^3} - \Big(\frac{30357785237878215893339}{1214294134566958263} \\
&\quad + \frac{2007\pi^2}{16}\Big)\frac{M^4}{(z^r)^4}\Big] + \Big[\frac{38844}{10507} + \frac{12572900}{1775683}\frac{M}{(z^r)} + \left(\frac{1443514854479884}{11671079597541} - \frac{585M^2}{64}\right)\frac{M^2}{(z^r)^2} - \Big(\frac{7173\pi^2}{32} \\
&\quad + \frac{2092360235136608219218 04}{425002947098435392 05}\Big)\frac{M^3}{(z^r)^3}\Big]\frac{\mathcal{L}^2}{(z^r)^2} - \Big[\frac{3078617}{253669} + \frac{13691383240}{513172387}\frac{M}{(z^r)} - \Big(\frac{4332202056238584185911}{202382355761159 7105} \\
&\quad - \frac{111825\pi^2}{1024}\Big)\frac{M^2}{(z^r)^2}\Big]\frac{\mathcal{L}^4}{(z^r)^4} + \Big[\frac{16973925730}{513172387} + \frac{1332278699099876}{23654681178765}\frac{M}{(z^r)}\Big]\frac{\mathcal{L}^6}{(z^r)^6} - \frac{94008905915838}{1126413389465}\frac{\mathcal{L}^8}{(z^r)^8}, \\
K_t^{(1)} &= -\Big[\frac{145329}{105070} + \frac{224752726}{26635245}\frac{M}{(z^r)} + \left(\frac{3961114172666372}{58355397987705} - \frac{117\pi^2}{32}\right)\frac{M^2}{(z^r)^2} - \Big(\frac{6096532685157103316489}{607147067283479131 5} \\
&\quad + \frac{1413\pi^2}{16}\Big)\frac{M^3}{(z^r)^3}\Big] + \Big[\frac{81010078}{8878415} + \frac{26029992074}{2565861935}\frac{M}{(z^r)} - \left(\frac{5613396674353868549121 4}{42500294709843539205} + \frac{5985}{64}\right)\frac{M^2}{(z^r)^2}\Big]\frac{\mathcal{L}^2}{(z^r)^2} \\
&\quad - \Big[\frac{163684391287}{5131723870} + \frac{98611138120}{13251922229}\frac{M}{(z^r)}\Big]\frac{\mathcal{L}^4}{(z^r)^4} + \frac{21116821648567}{225282677893}\frac{\mathcal{L}^6}{(z^r)^6}, \\
K_t^{(2)} &= -\Big[\frac{230022021}{99438248} + \frac{5642118696757}{538831006350}\frac{M}{(z^r)} + \left(\frac{936134068131746562715 9}{21250147354921769602 5} - \frac{2061\pi^2}{128}\right)\frac{M^2}{(z^r)^2}\Big] + \Big[\frac{535002897207}{35922067090} \\
&\quad + \frac{130651280359811}{78848937262550}\frac{M}{(z^r)}\Big]\frac{\mathcal{L}^2}{(z^r)^2} - \frac{3424552998566313}{63079149810040}\frac{\mathcal{L}^4}{(z^r)^4} \qquad \cdots\cdots
\end{aligned}$$

Conclusion

- $\tilde{S}$-$\tilde{R}$ decomposition was proposed.

  Regularized force up to n -PN

  $$\sum_{\ell=0}^{\infty} F^{R}_{\alpha,\ell} = \sum_{\ell=0}^{\infty}\left(\lim_{x\to z_0} F^{\tilde{S}-S}_{\alpha,\ell} + F^{\tilde{R}}_{\alpha,\ell}\right) = \sum_{\ell=0}^{\infty}\tilde{D}_{\alpha\ell} + \sum_{\ell=0}^{n} F^{\tilde{R}}_{\alpha,\ell}$$

- Expression for the singular $\tilde{S}$-part force in time domain is calculable for general orbits.
- The remaining $\tilde{R}$-part is truncated at finite {.
- ($\tilde{S}$-S)-part was explicitly computed for a scalar charged particle in Schwarzshild spacetime.