(t-domain) Numerical Integration of the MP Equations in the Harmonic Gauge

Leor Barack & Carlos Lousto
(UTB)

Organization of talk

- SF via mode-sum: Comments on implementation strategies
- Schwarzschild perturbations in the harmonic gauge: Formalism
- t-domain numerical implementation (circular orbits, Schwarzschild)
  - Scalar field
  - Gravitational field (odd part)
- Comments on generalizations

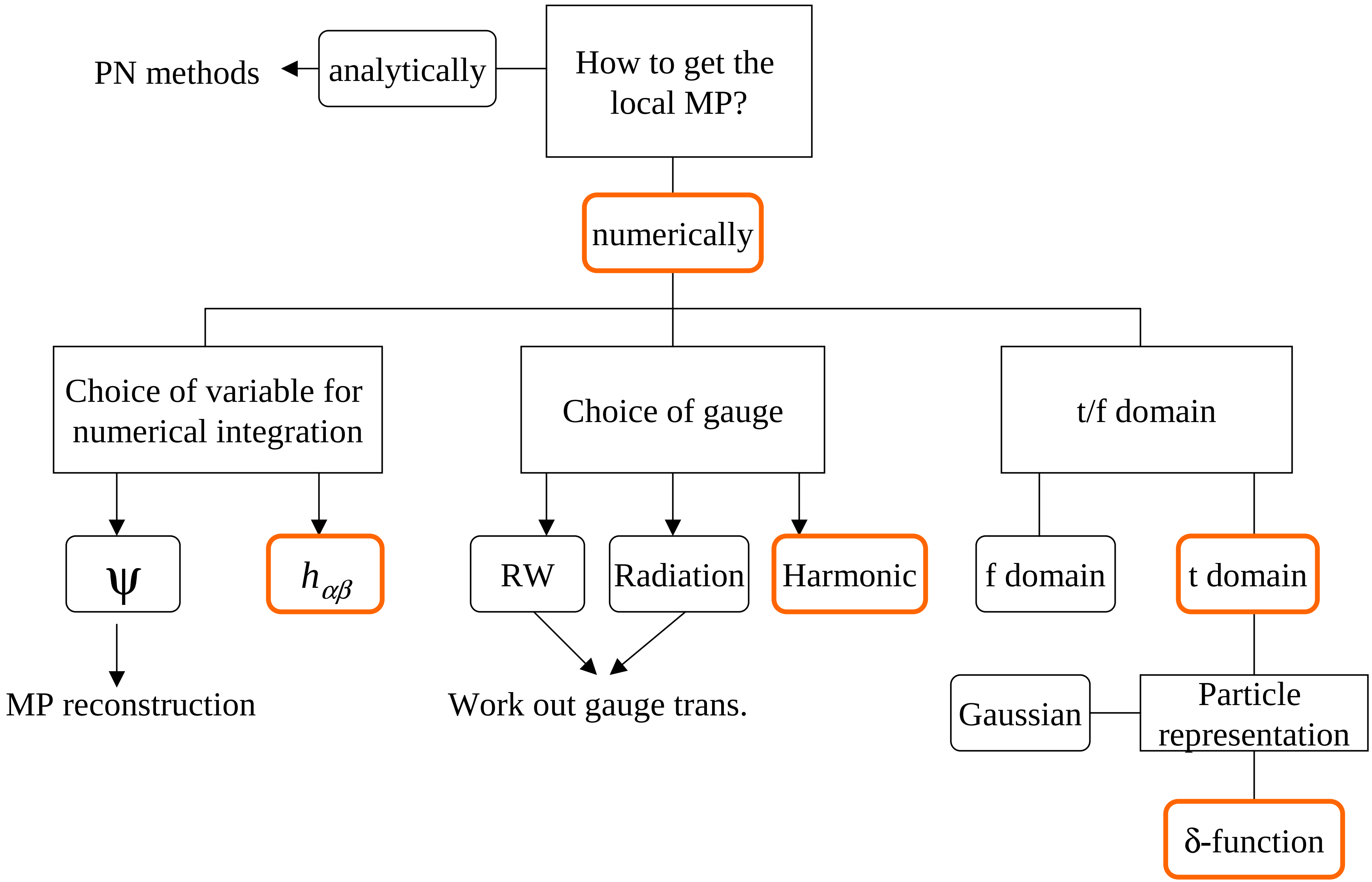
Mode sum formula

$$F_{\rm self}(x_p) = \sum_{l=0}^{\infty} \left[F_{\rm full}(h_{\alpha\beta})|_{x\to x_p} - A(l+1/2) - B - C/(l+1/2) \right] - D$$

Need local MP

- 2 gauge problems/issues:
  - Get $h_{\alpha\beta}$ and $ABCD$ in same gauge
  - For that gauge, make sure $F_{\rm self}$ is well-defined, regular, and can be used for constructing gauge-invariants.

Implementation strategy: many options



Schwarzschild perturbations in the harmonic gauge: formulation

1. Linearized Einstein equations in the harmonic (🕓Lorentz) gauge

Linearize Einstein's equations in perturbation $h_{\alpha\beta}(x)$ about BH background $g_{\alpha\beta}$. Take source to be a point particle moving on a geodesic $x = x_p(\tau)$ of $g_{\alpha\beta}$. Get

$$\Box \bar{h}_{\alpha\beta} + 2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\bar{h}_{\mu\nu} + g_{\alpha\beta}\bar{h}^{\mu\nu}{}_{;\mu\nu} - 2g^{\mu\nu}\bar{h}_{\mu(\alpha;\nu\beta)} = -16\pi\mu \int_{-\infty}^{\infty} (-g)^{-1/2}\,\delta^4[x^\mu - x_p^\mu(\tau)]u_\alpha u_\beta\, d\tau \equiv S_{\alpha\beta},$$

where

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} h.$$

Impose Harmonic gauge condition,

$$g^{\beta\gamma}\bar{h}_{\alpha\beta;\gamma} = 0.$$

Get

$$\Box \bar{h}_{\alpha\beta} + 2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\bar{h}_{\mu\nu} = S_{\alpha\beta}$$

2. Tensor-harmonic decomposition

$$\bar{h}_{\alpha\beta} = \sum_{l,m} \sum_{i=1}^{10} \bar{h}^{(i)lm}(r,t)\, Y^{(i)lm}_{\alpha\beta}(r;\theta,\varphi)$$

$$S_{\alpha\beta} = \sum_{l,m} \sum_{i=1}^{10} S^{(i)lm}(r,t)\, Y^{(i)lm}_{\alpha\beta}(r;\theta,\varphi)$$

3. Make sure variables for numerical evolution ∝ const at $r \to 2M$, ∞

Redefine

$$\bar{h}_{\alpha\beta} = \sum_{l,m} \sum_{i=1}^{10} \frac{R^{(i)}(r)}{r} \, \bar{h}^{(i)lm}(r,t) \, Y^{(i)lm}_{\alpha\beta}(r;\theta,\varphi),$$

with

$$R^{(2,5,9)} = f^{-1}, \quad R^{(3)} = f^{-2}, \quad R^{(i)} = 1 \text{ for rest.}$$

Then all $\bar{h}^{(i)lm}$ are dimensionless and $\propto$ const at both $r \to 2M$ and $r \to \infty$.

[Decide on r-dependent pre-factors with help of

$$(g^{\alpha\alpha} g^{\beta\beta})^{1/2} h_{\alpha\beta}\big|_{r\to\infty} \propto \frac{1}{r} \quad \text{and} \quad h_{\alpha\beta}|_{r\to 2M} \propto \quad \text{const in } (v,r,\theta,\varphi) \text{ coordinates.}]$$

4. Get separated equations for the $h^{(i)}$'s

$$\Box^{2d}_{\rm sc}\bar{h}^{(i)lm} + \mathcal{M}^{(i)}_{(j)}\bar{h}^{(j)lm} = \tilde{S}^{(i)lm},$$

where

$$\Box^{2d}_{\rm sc} \equiv \partial_{uv} + \frac{f}{4}\left[\frac{f'}{r} + \frac{l(l+1)}{r^2}\right],$$

$$\tilde{S}^{(i)lm} = 4\pi r f R^{-1}\int_{\infty}^{\infty} d\tau\, r_p^{-2}\delta(t-t_p)\delta(r-r_p)u_\alpha u_\beta \eta^{\alpha\mu}\eta^{\beta\nu}[Y^{(i)}_{\mu\nu}(\Omega_p)]^*,$$

and, for example,

$$\mathcal{M}^{(8)}_{(j)}\bar{h}^{(j)} = \frac{1}{4}ff'\left(\bar{h}^{(8)}_{,r} - \frac{3}{r}\bar{h}^{(8)} - if^{-1}\bar{h}^{(9)}_{,t}\right)$$

$$\mathcal{M}^{(9)}_{(j)}\bar{h}^{(j)} = \frac{1}{4}ff'\bar{h}^{(9)}_{,r} + \frac{f}{r^2}(1-3.5M/r)\,\bar{h}^{(9)} + \frac{1}{4}if'\bar{h}^{(8)}_{,t} - \frac{f}{r^2}(\lambda/2)^{1/2}\,\bar{h}^{(10)}$$

$$\mathcal{M}^{(10)}_{(j)}\bar{h}^{(j)} = -\frac{f}{2r^2}\left[\bar{h}^{(10)} + (\lambda/2)^{1/2}\,\bar{h}^{(9)}\right]$$

5. Use gauge condition to get rid of "bad" *t*-derivatives

$$\Box^{2d}_{\rm sc}\bar{h}^{(9)} + \frac{1}{4} f f' \bar{h}^{(9)}_{,r} + \frac{f}{r^2}\left(1 - 3.5M/r\right)\bar{h}^{(9)} + \frac{1}{4} i f' \bar{h}^{(8)}_{,t} - \frac{f}{r^2}(\lambda/2)^{1/2}\,\bar{h}^{(10)} = \tilde{S}^{(9)}$$

$$+\left(-\frac{1}{4} i f'\right)\left\{\bar{h}^{(8)}_{,t} - i f\left[\bar{h}^{(9)}_{,r} + \frac{2}{r}\left(\bar{h}^{(9)} - (2\lambda)^{1/2}\,\bar{h}^{(10)}\right)\right]\right\} \quad (= 0,\ \text{by gauge condition})$$

$$\Rightarrow \quad \boxed{\Box^{2d}_{\rm sc}\bar{h}^{(9)} + \frac{f}{r^2}\left(1 - 4.5M/r\right)\bar{h}^{(9)} - \frac{f}{r^2}(\lambda/2)^{1/2}\,\bar{h}^{(10)} = \tilde{S}^{(9)}}$$

Numerical Implementation:

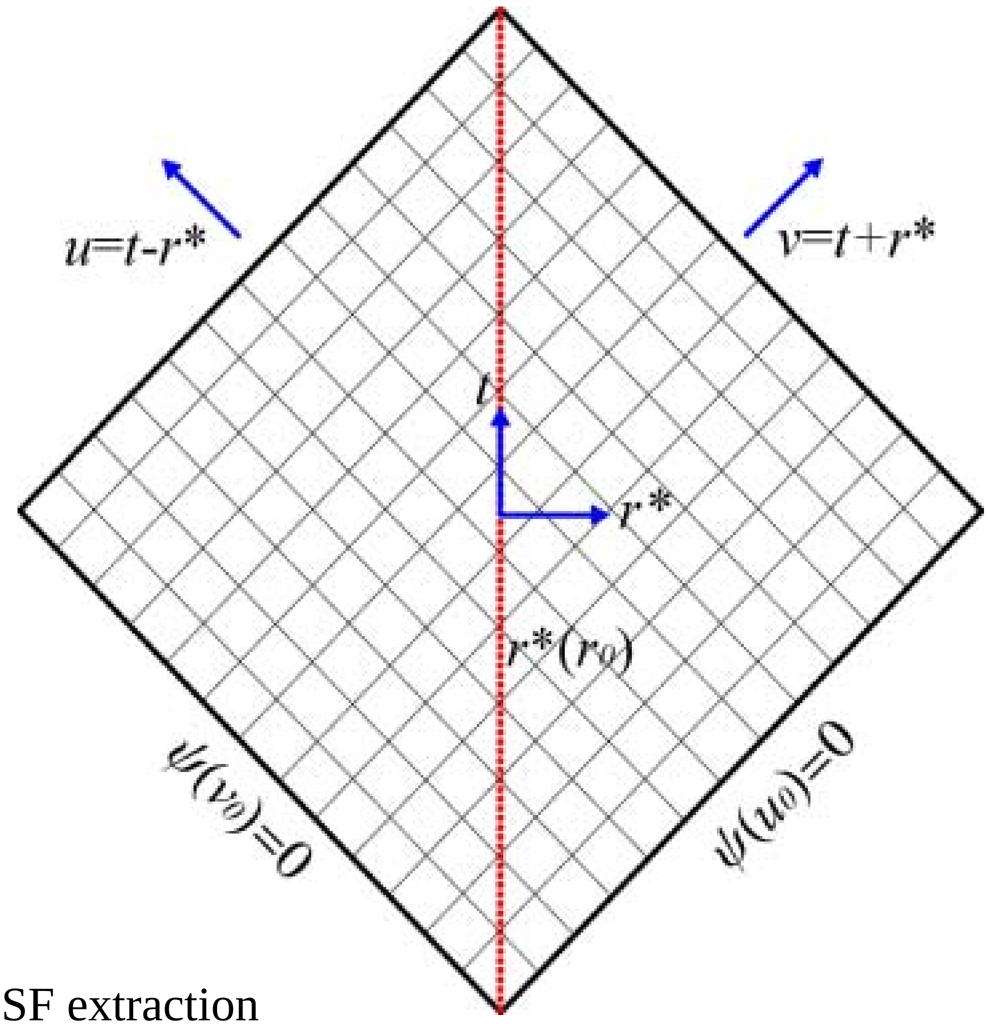
I. Numerical technique demonstrated for scalar field, circular orbit in Schwarzschild

1. Evolution equation

$$\Phi = 2\pi q \sum_{l,m} \frac{\psi^{lm}(r,t)}{r} Y_{lm}(\theta,\varphi)$$

$$\begin{aligned}\Box^{2d}_{\rm sc}\psi^{lm} = S^{lm} &= \frac{f}{2r_0}\int_{-\infty}^{\infty} \delta(r-r_p)\delta(t-t_p)\, Y^*_{lm}(\Omega_p)\, d\tau \\ &= \frac{f^2}{2r_0E}\,\delta(r-r_0)\, a_{lm}\, e^{-im\omega t_p} \quad \text{(circular, equatorial orbit)}\end{aligned}$$

2. Grid for 1+1d numerical evolution



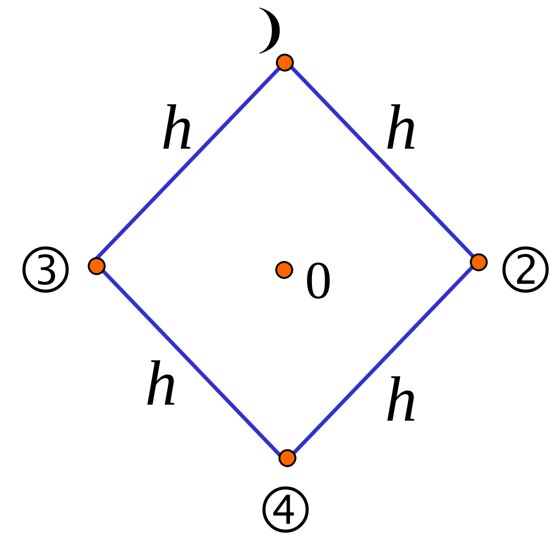
Grid size:

- At least 3-4 T_{orb}
  (>300M for r_0=6M)

Resolution:

- ~1 grid pt/M^2 for fluxes extraction
- ~10^6 grid pts/M^2 (near particle) for SF extraction

3. Finite difference scheme (2nd order)

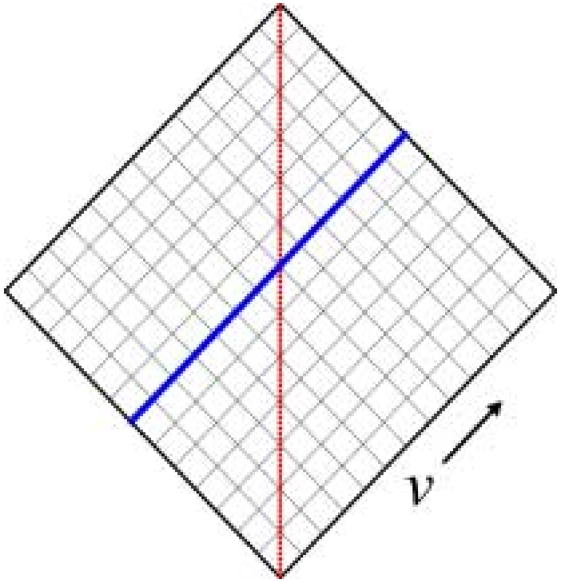


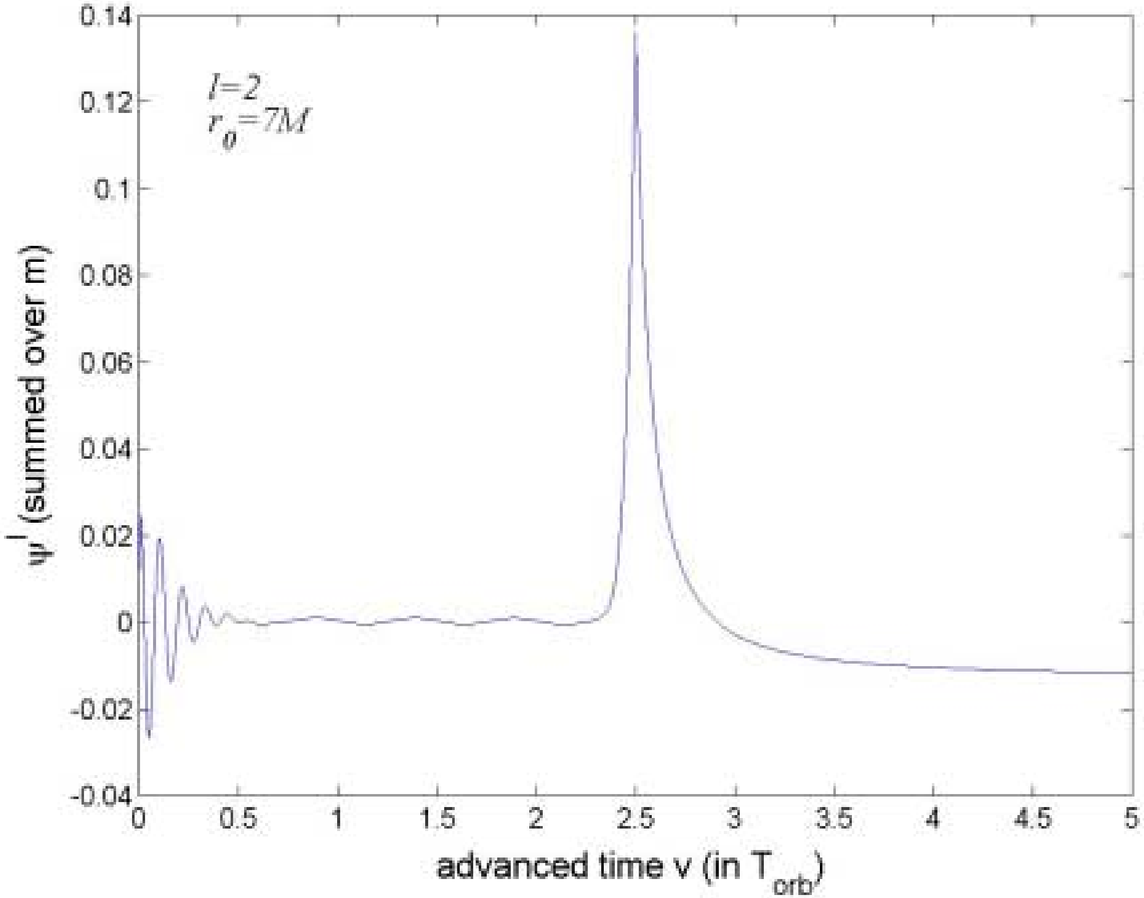
$$\int_{\text{cell}} du\,dv\ \left[\psi^{lm}_{,uv} + V_{\text{sc}}\psi^{lm} = S^{lm}\right]$$

$$\psi(1)+\psi(4)-\psi(3)-\psi(2)+\frac{1}{2}h^2 V_{\text{sc}}(0)[\psi(3)+\psi(2)]+O(h^3)=\int_{\text{cell}} S\,du\,dv \equiv h\cdot Z$$

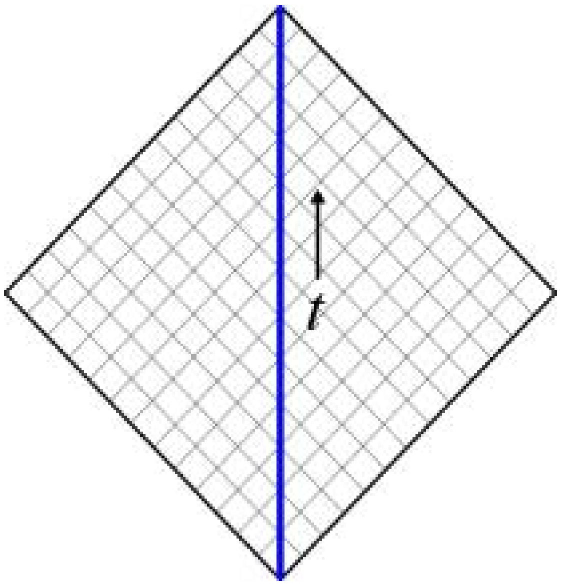
$$Z=\frac{f\alpha_{lm}}{r_0 E}\times\begin{cases} 0, & \text{no particle in cell} \\ 1, & \text{particle in cell, } m=0 \\ \frac{\sin(m\omega h/2)}{m\omega h/2}\exp[-im\omega t(0)], & \text{particle in cell, } m\neq 0 \end{cases}$$

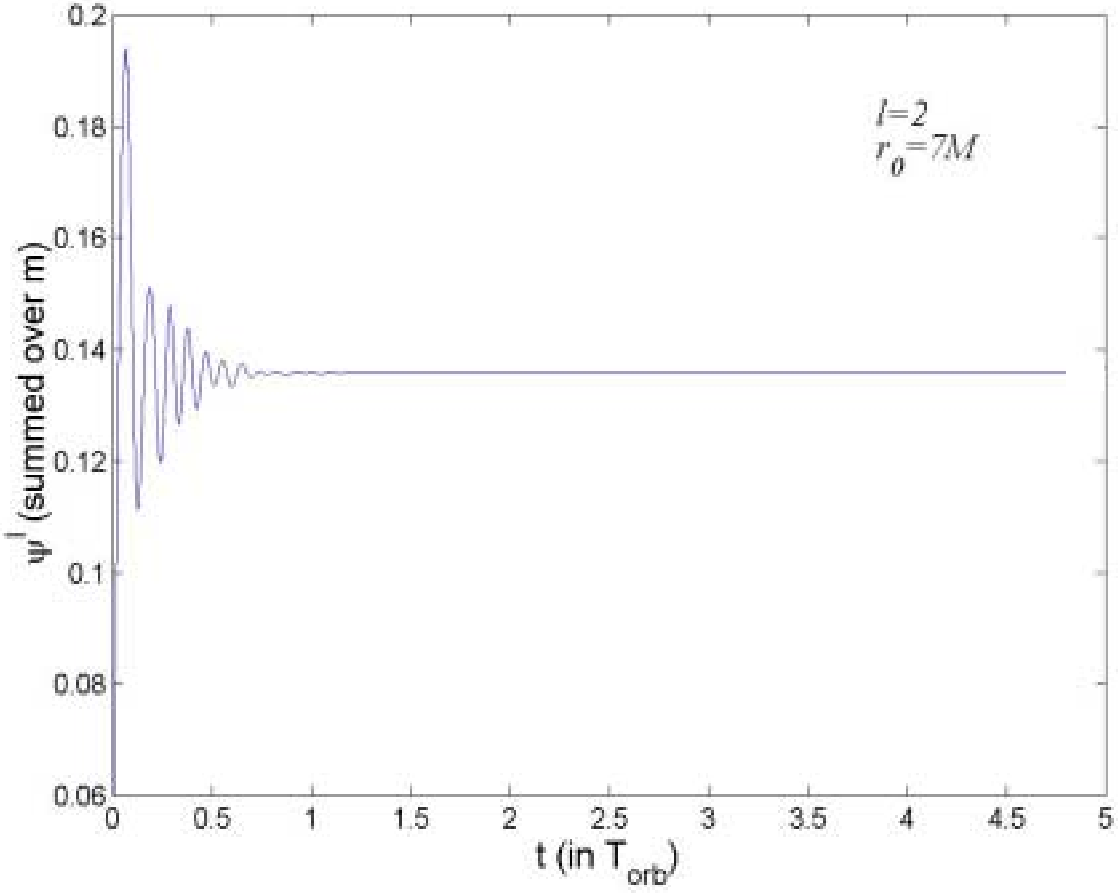
4. Results: Scalar field



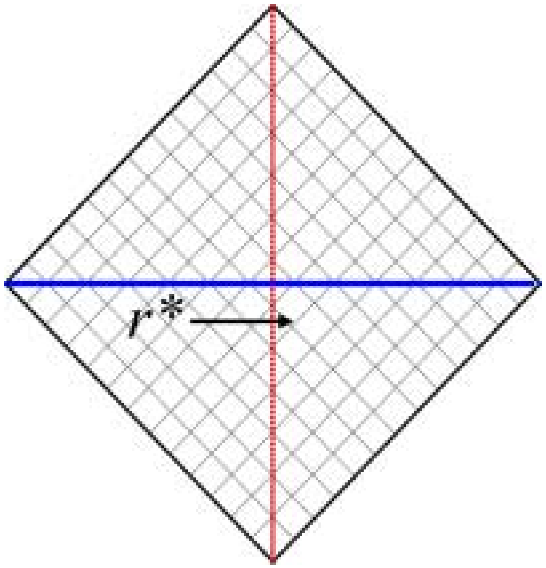


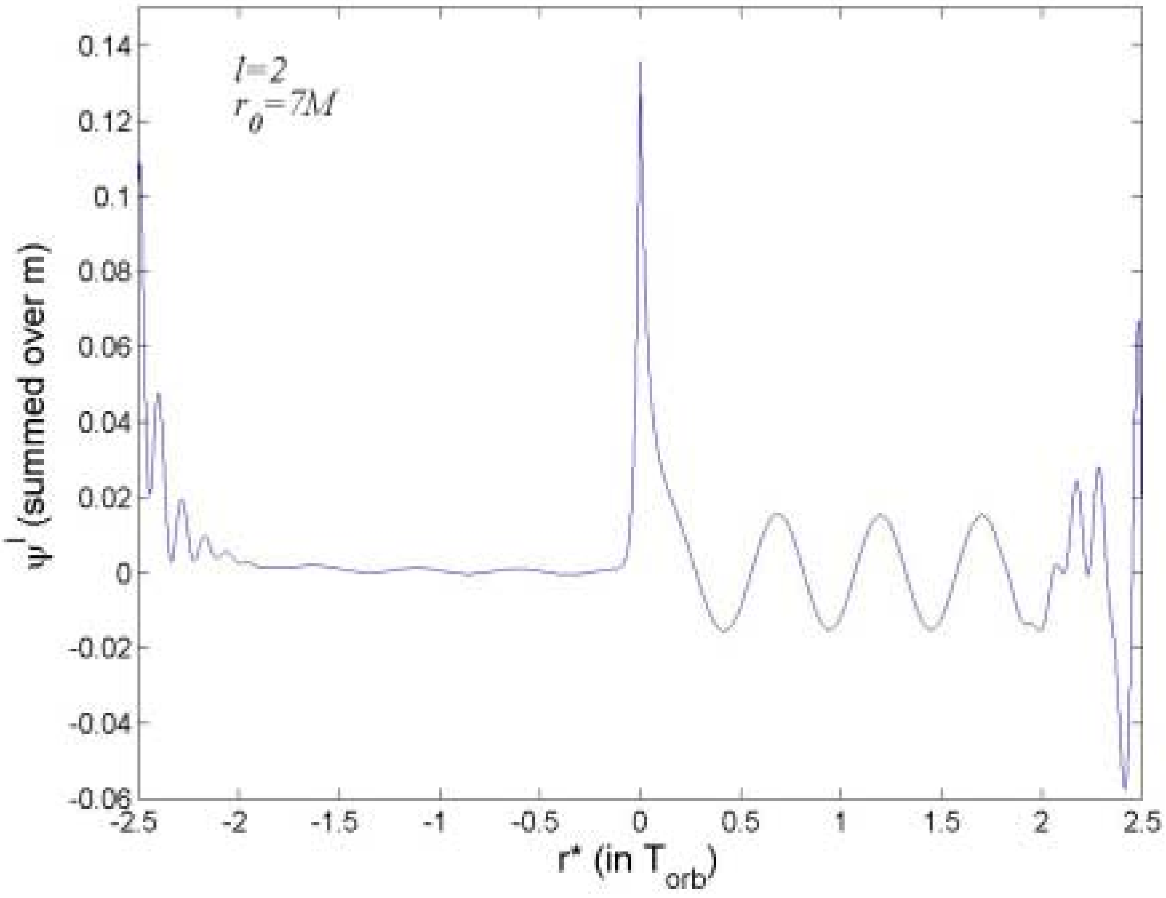
Results: Scalar field



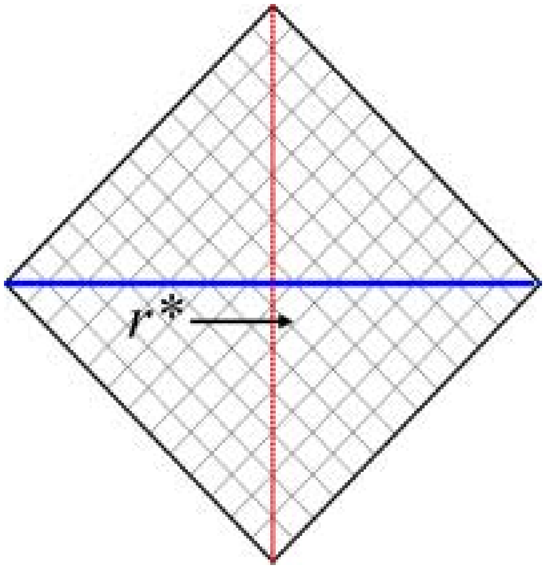


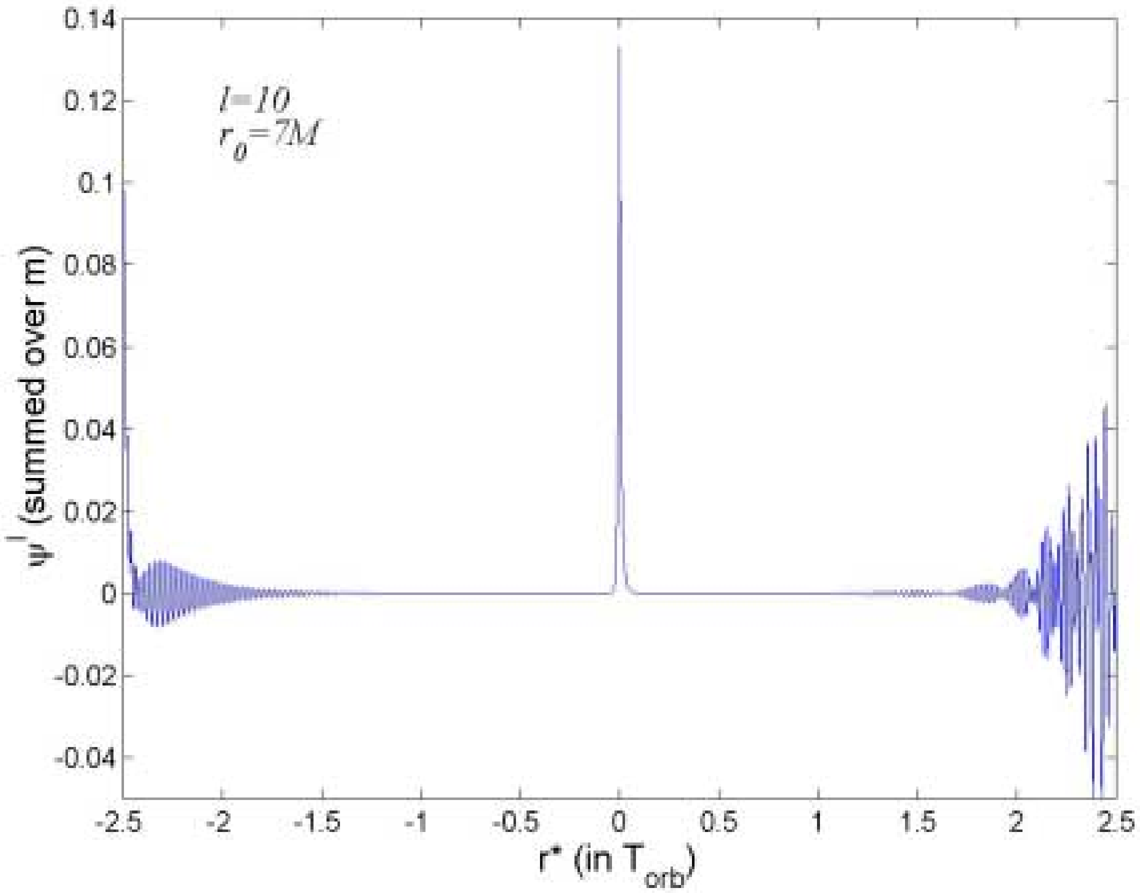
Results: Scalar field



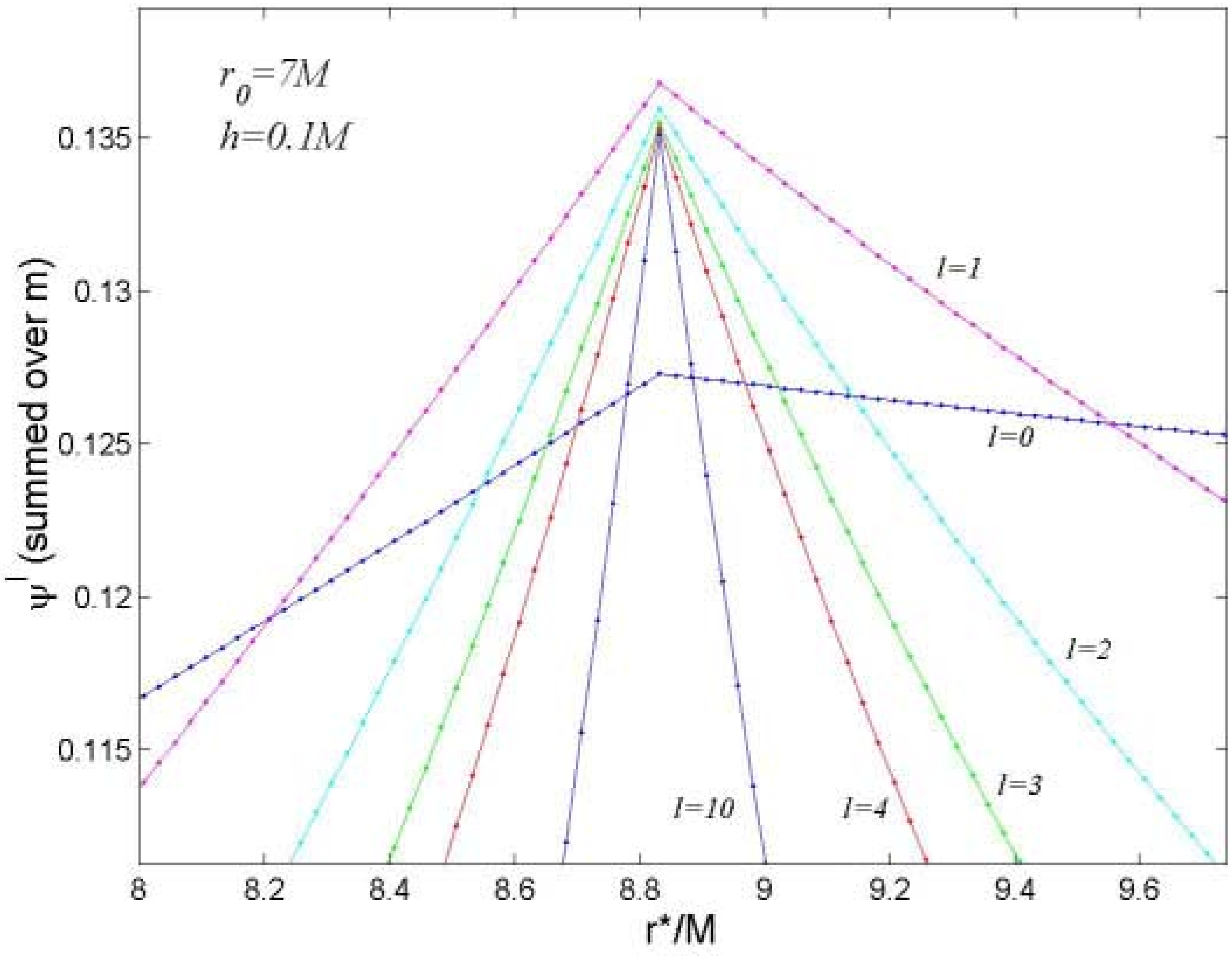


Results: Scalar field





Results: Scalar field

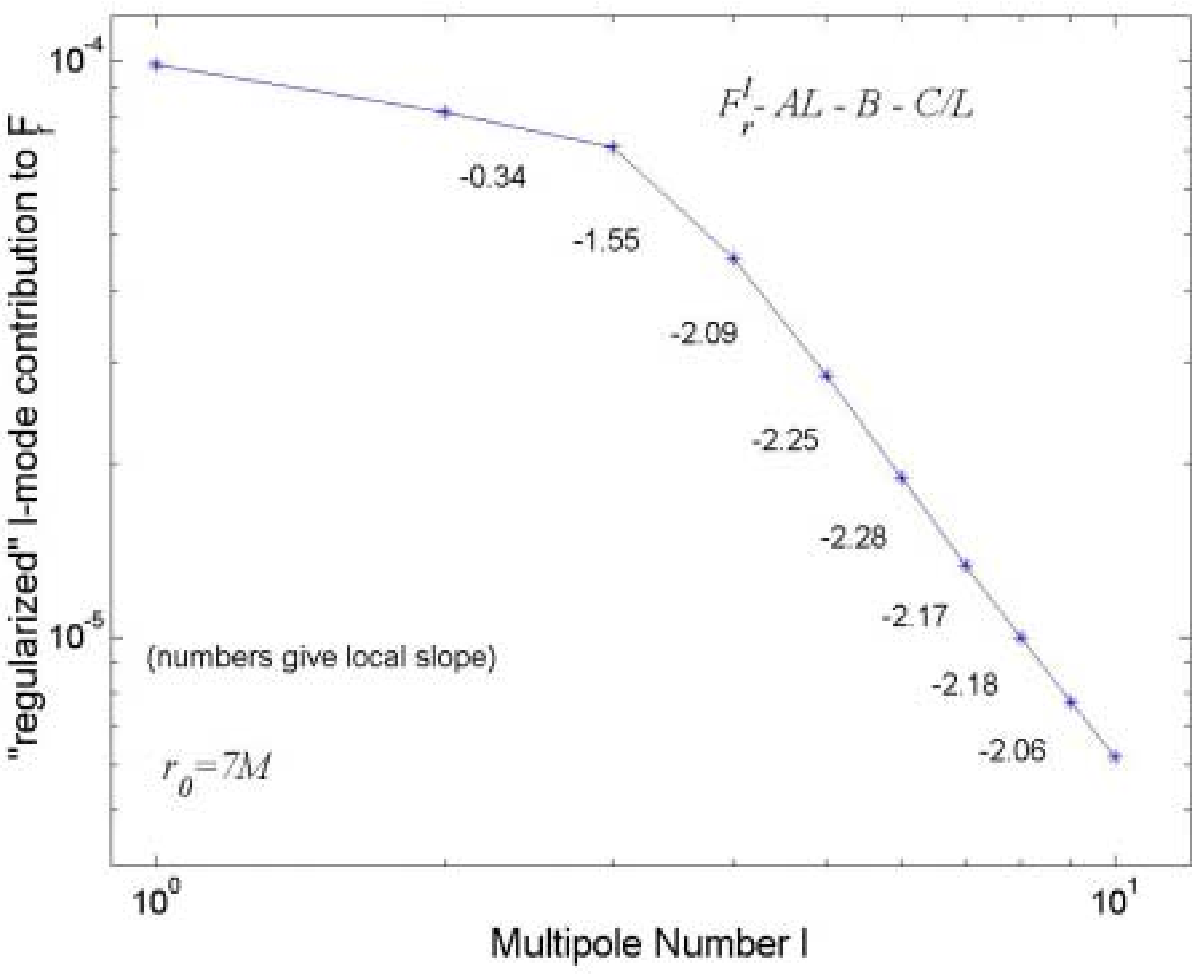


5. Results: Scalar force (sample output, screen capture)

```
[leor@hercules scalar]$ ./a.out
 l=
1
 Evolution time (# of orbits)=
5
 Initial Resolution (steps per M in r*,t)=
5
 ITERATION #           1
 ITERATION #           2
 ITERATION #           3
 ----------------------------------------
 Phi(r0):
 Cycle #           2 :  0.136805155563096
 Cycle #           3 :  0.136805162960639
 Cycle #           4 :  0.136805163931519
 ----------------------------------------
 F_r(r0+):
 Cycle #           2 :  -4.31956375723335D-002
 Cycle #           3 :  -4.31956353278614D-002
 Cycle #           4 :  -4.31956342813805D-002
 ----------------------------------------
 F_r(r0-):
 Cycle #           2 :   2.17335149052099D-002
 Cycle #           3 :   2.17335151321284D-002
 Cycle #           4 :   2.17335143854717D-002
 ----------------------------------------
```

```
 ----------------------------------------
[F_r(r0+)-F_r(r0-)]/(2L): ("A_r")
 -2.16430508258478D-002
 -2.16430501533300D-002
 -2.16430495556174D-002
 ----------------------------------------
[F_r(r0+)+F_r(r0-)]/2: ("B_r")
 -1.07310613335618D-002
 -1.07310600978665D-002
 -1.07310599479544D-002
 ----------------------------------------
 ----------------------------------------
|F_r(r0+)-F_r(r0-)|/2-|A|*L: ("C_r")
  6.76214094093648D-005
  6.76204006326203D-005
  6.76195040637838D-005
 ----------------------------------------
F_r(REG):
 -9.90421617212026D-005
 -9.90409250297095D-005
 -9.90407736144393D-005
 ----------------------------------------
F_t(r0):
  1.09151357977453D-004
  1.09149895091873D-004
  1.09148592668721D-004
 ----------------------------------------
```

6. Results: Regularized scalar self force



Numerical Implementation:

II. Gravitational field (odd-parity part);
circular orbit in Schwarzschild

1. Evolution equations

$$\Box^{2d}_{\rm sc}\bar{h}^{(8)} + \frac{1}{4}ff'\left(\bar{h}^{(8)}_{,r} - \frac{3}{r}\bar{h}^{(8)} - if^{-1}\bar{h}^{(9)}_{,t}\right) = \bar{S}^{(8)}$$

$$\Box^{2d}_{\rm sc}\bar{h}^{(9)} + \frac{f}{r^2}\left(1 - 4.5M/r\right)\bar{h}^{(9)} - \frac{f}{r^2}(\lambda/2)^{1/2}\,\bar{h}^{(10)} = 0$$

$$\Box^{2d}_{\rm sc}\bar{h}^{(10)} - \frac{f}{2r^2}\left[\bar{h}^{(10)} + (\lambda/2)^{1/2}\,\bar{h}^{(9)}\right] = \bar{S}^{(10)}$$

where

$$\begin{cases} \bar{S}^{(8)} = \frac{8\pi f E\omega}{\sqrt{2l(l+1)}}\,\delta(r-r_0)\,b_{lm}\,e^{-im\omega t_p} \\ \bar{S}^{(10)} = -\frac{4\pi m E\omega^2 r_0}{\sqrt{\lambda l(l+1)}}\,\delta(r-r_0)\,b_{lm}\,e^{-im\omega t_p} \end{cases}$$

2. Symmetry $m \leftrightarrow -m$

Under $m \to -m$:

$$\left.\begin{array}{l} h^{(8)} \to h^{(8)*}, \quad h^{(9,10)} \to -h^{(9,10)*} \\ Y^{(8)}_{\alpha\beta} \to Y^{(8)*}_{\alpha\beta}, \; Y^{(9,10)}_{\alpha\beta} \to -Y^{(9,10)*}_{\alpha\beta} \end{array}\right\} \Rightarrow h^{(i)} Y^{(i)}_{\alpha\beta} \to \left[h^{(i)} Y^{(i)}_{\alpha\beta}\right]^*$$

$$\bar{h}^{(\mathrm{odd})}_{\alpha\beta} = \sum_{l,m} \sum_{i=8}^{10} \frac{R^{(i)}(r)}{r} \bar{h}^{(i)lm} \, Y^{(i)lm}_{\alpha\beta}$$

$$= (1/r) \sum_{l} \left\{ \left[h^{(8)} Y^{(8)}_{\alpha\beta}\right]_{m=0} + 2 \sum_{i=8}^{10} R^{(i)} \sum_{m>0}^{l} \mathrm{Re}\left[h^{(i)} Y^{(i)}_{\alpha\beta}\right] \right\}$$

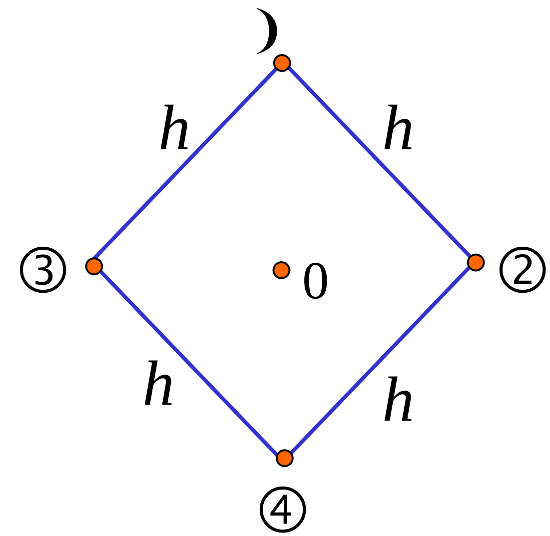
3. Finite difference scheme

$$h_1^{(8)} = -h_4^{(8)} + f_1 h_+^{(8)} + f_2 h_-^{(8)} + m\omega f_3 h_+^{(9)} + hZ^{(8)}$$

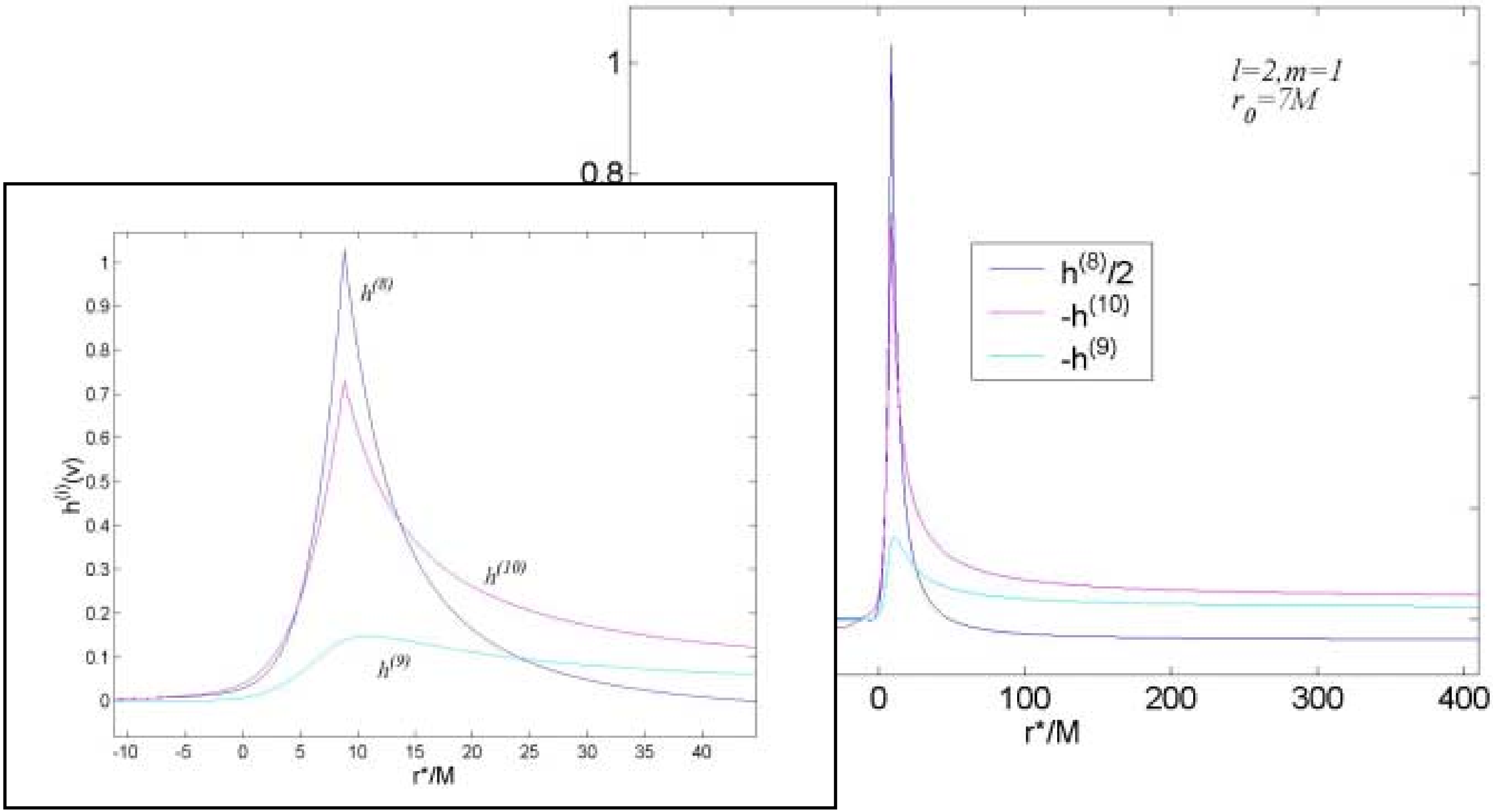
$$h_1^{(9)} = -h_4^{(9)} + f_4 h_+^{(9)} + f_5 h_+^{(10)}$$

$$h_1^{(10)} = -h_4^{(10)} + f_6 h_+^{(10)} + f_7 h_+^{(9)} + hZ^{(10)}$$

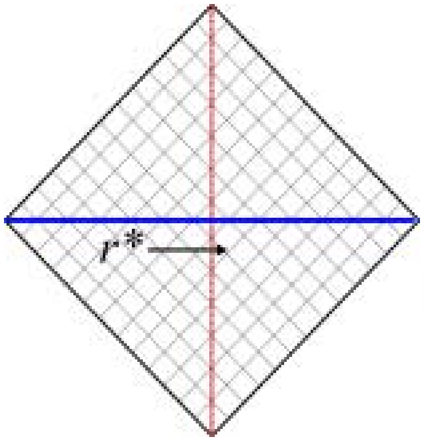
$$h_\pm^{(i)} \equiv h_2^{(i)} \pm h_3^{(i)}$$

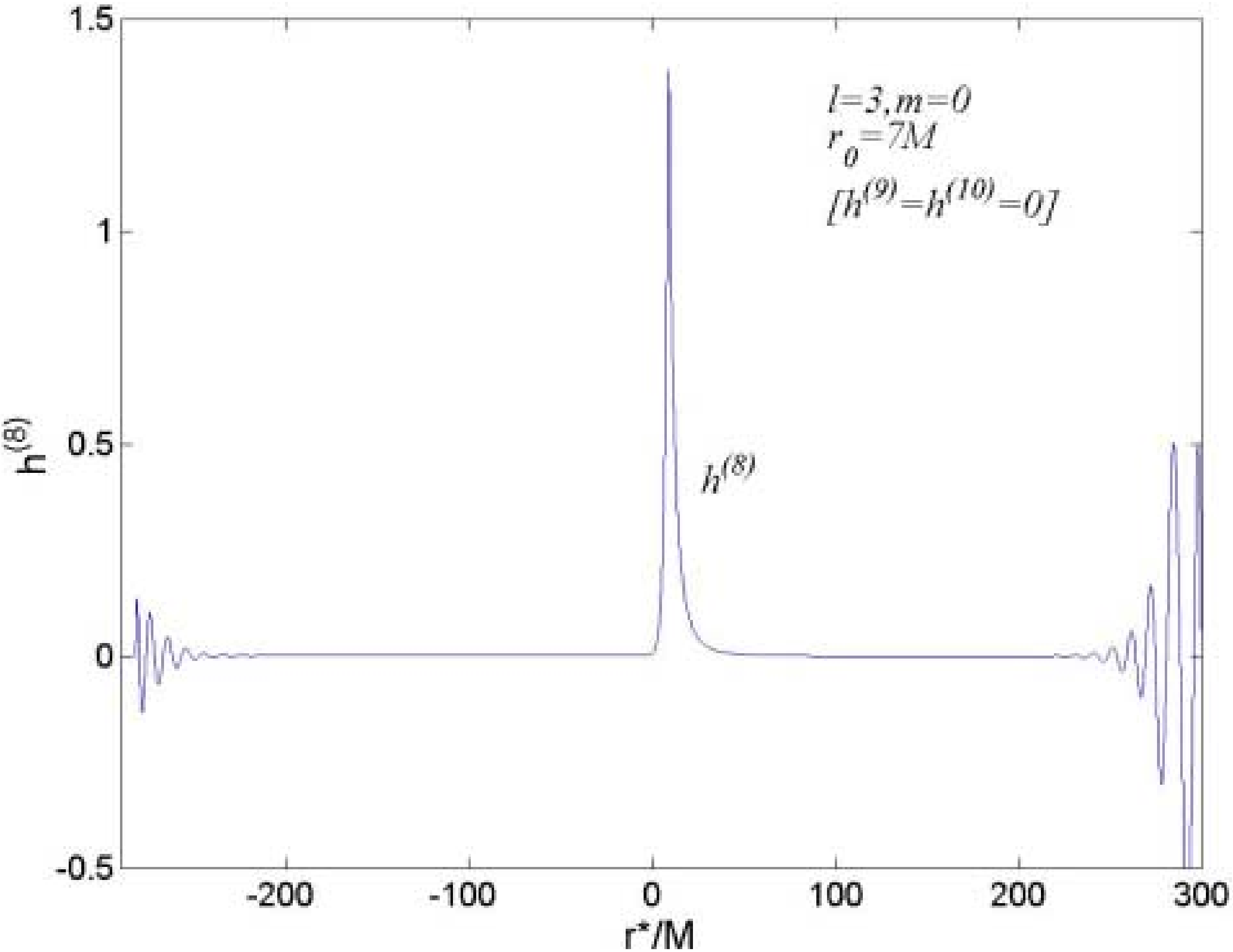


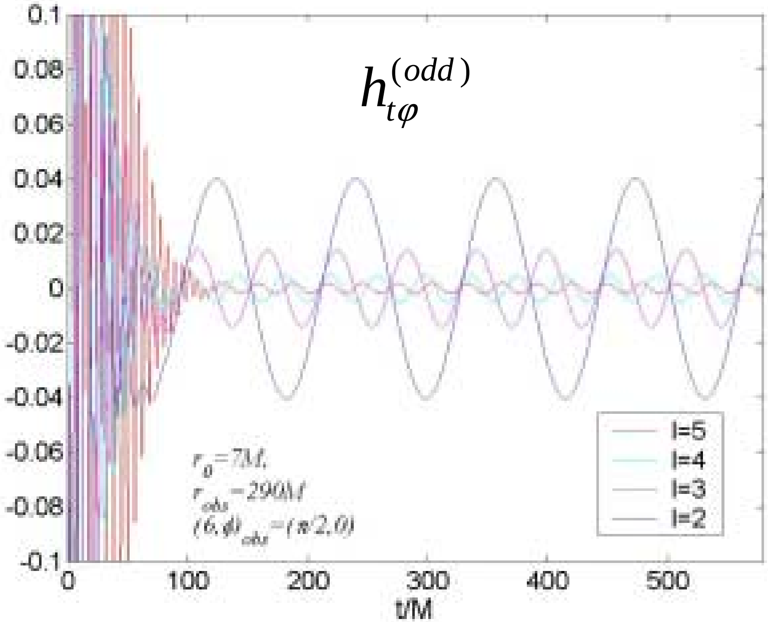
4. Results: MP

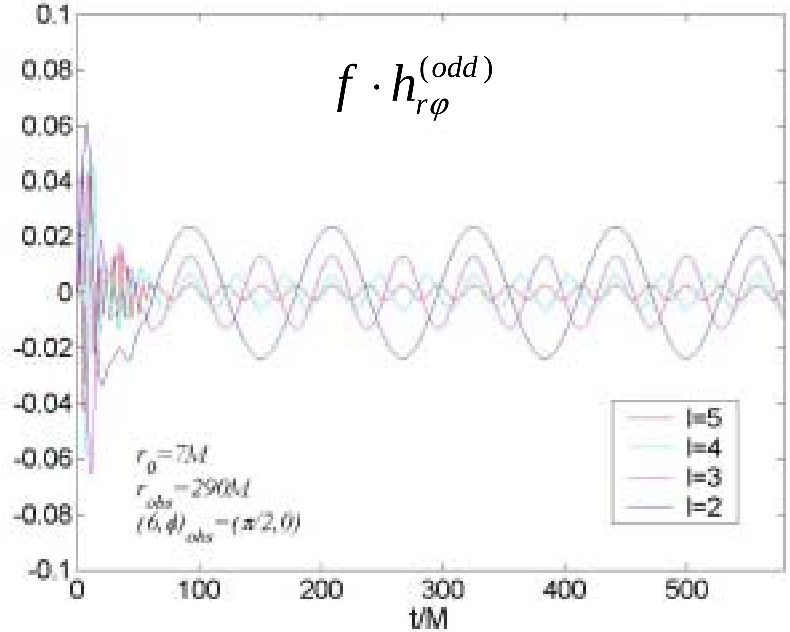


Results: MP

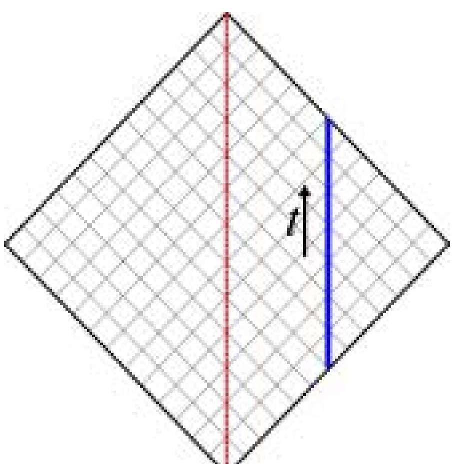


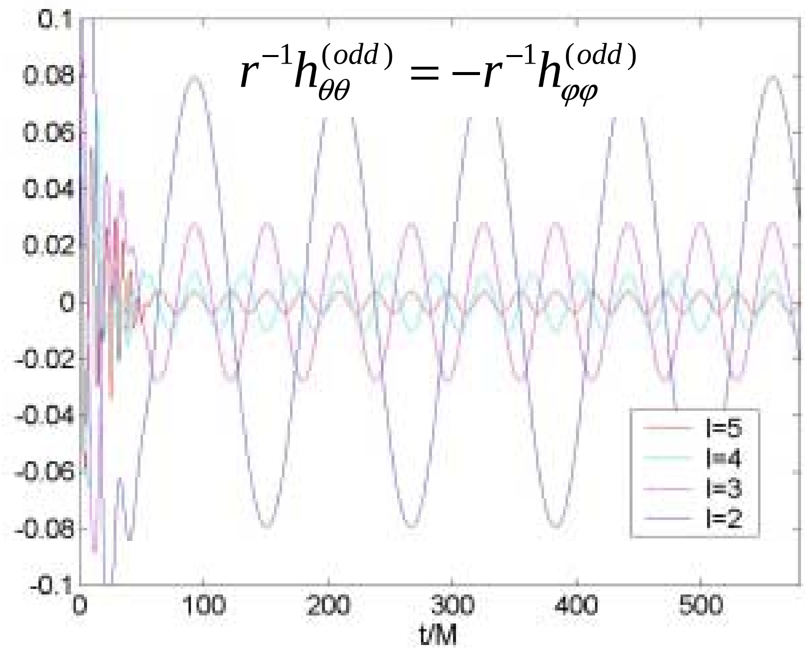






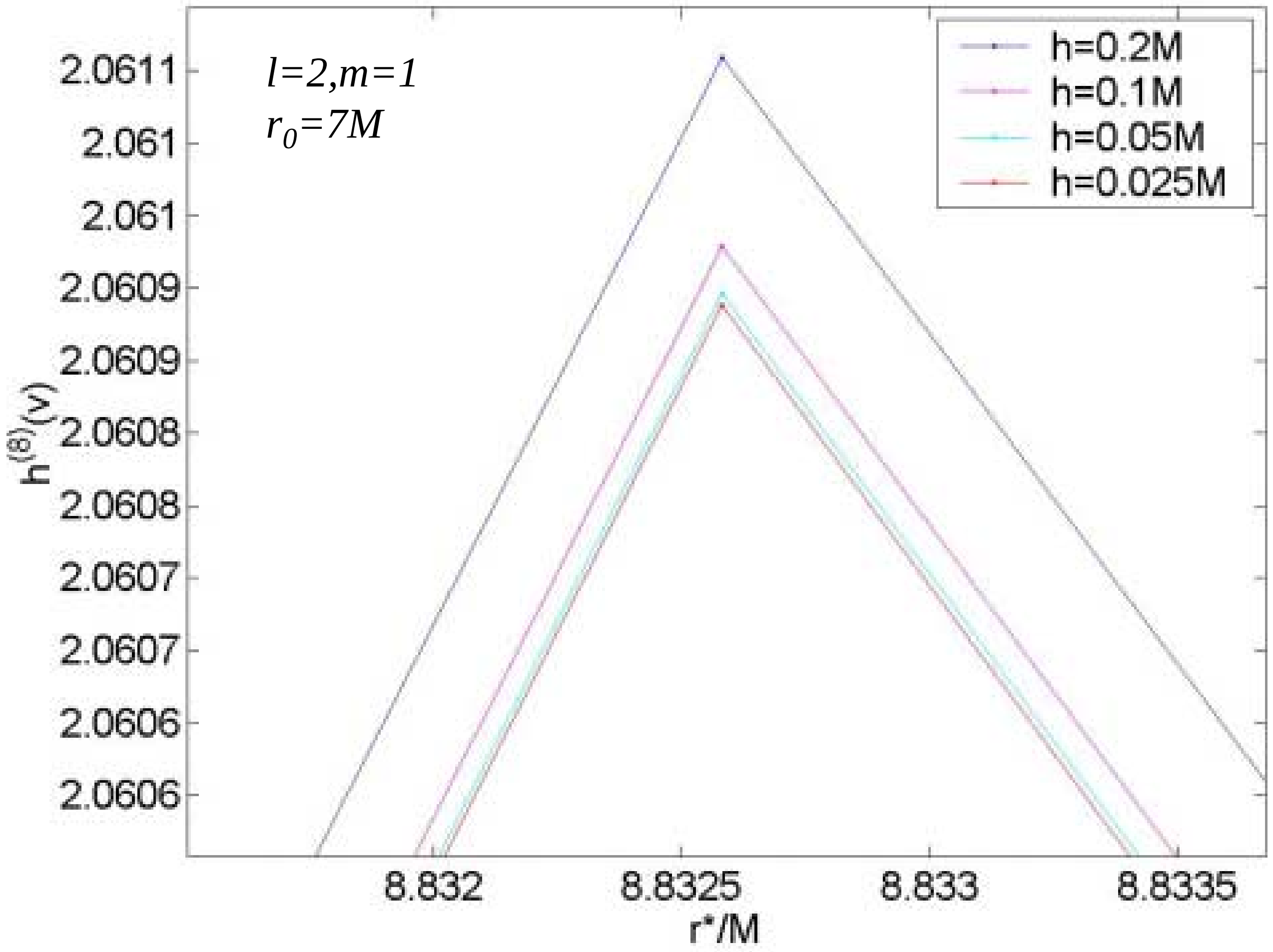
Odd part of Full MP (summed over m):
Static observer at r=290M, θ=π/2, φ=0

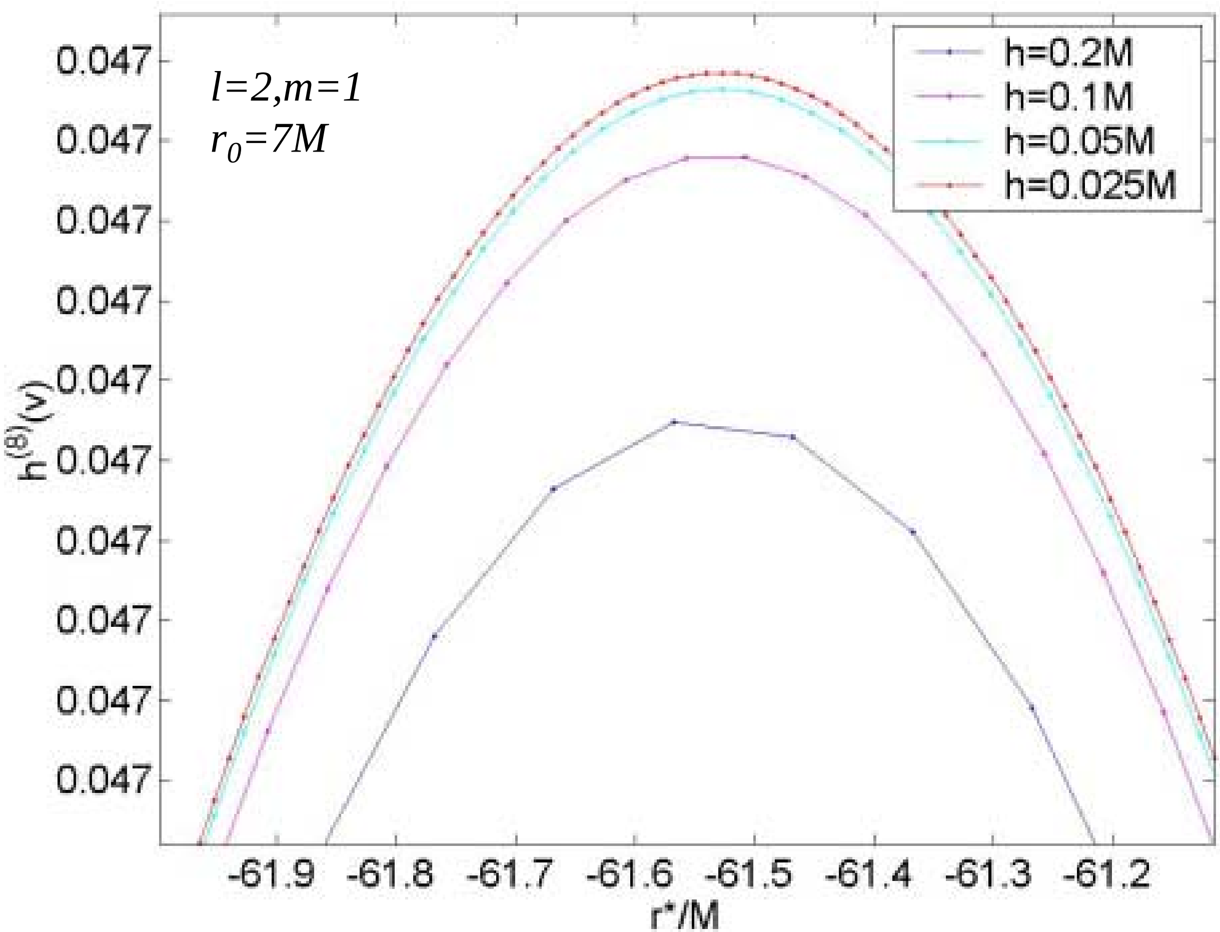




Tests of Code

Test #1: 2^{nd}-order Numerical Convergence





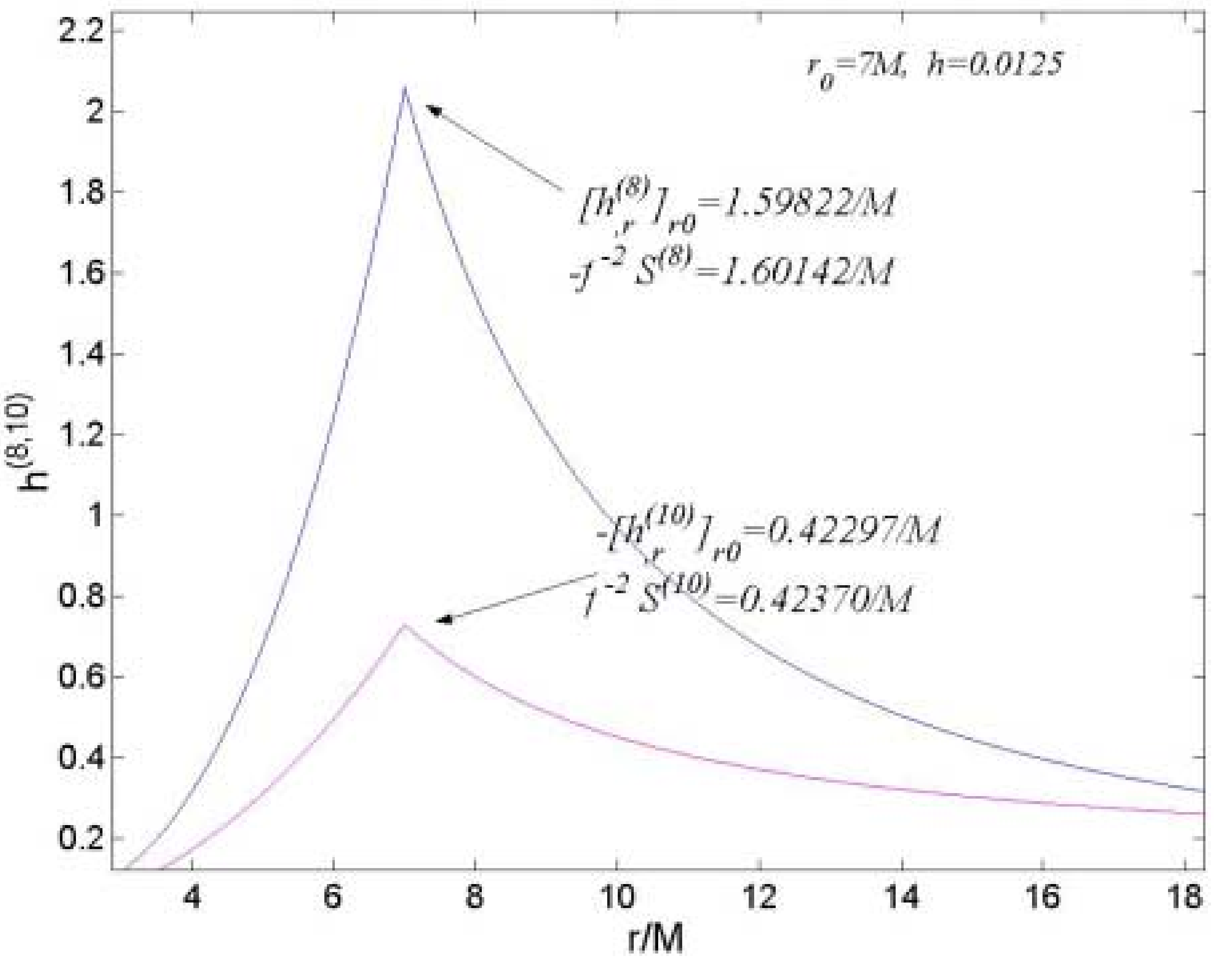
l=2,m=1
r0=7M
h=0.2M
h=0.1M
h=0.05M
h=0.025M
0.047
0.047
0.047
0.047
0.047
0.047
0.047
0.047
0.047
0.047
h(8)(v)
-61.9
-61.8
-61.7
-61.6
-61.5
-61.4
-61.3
-61.2
r*/M

Test #2: Jump in MP derivative across the particle

$$\bar{h}^{(i)}_{,tt} - \bar{h}^{(i)}_{,r_* r_*} + a_{(j)}\bar{h}^{(j)}_{,t} + b_{(j)}\bar{h}^{(j)}_{,r} + c_{(j)}\bar{h}^{(j)} = \hat{S}^{(i)}\delta(r - r_0)$$

$$\lim_{\Delta r \to 0} \int_{r_0 - \Delta r}^{r_0 + \Delta r} (\mathrm{EQS})\, dr \quad \to \qquad -f(r_0)\left[h^{(i)}_{,r_*}\right]_{r_0} = \hat{S}^{(i)}$$

$$\boxed{\left[h^{(i)}_{,r}\right]_{r_0} = -f^{-2}(r_0)\,\hat{S}^{(i)}}$$



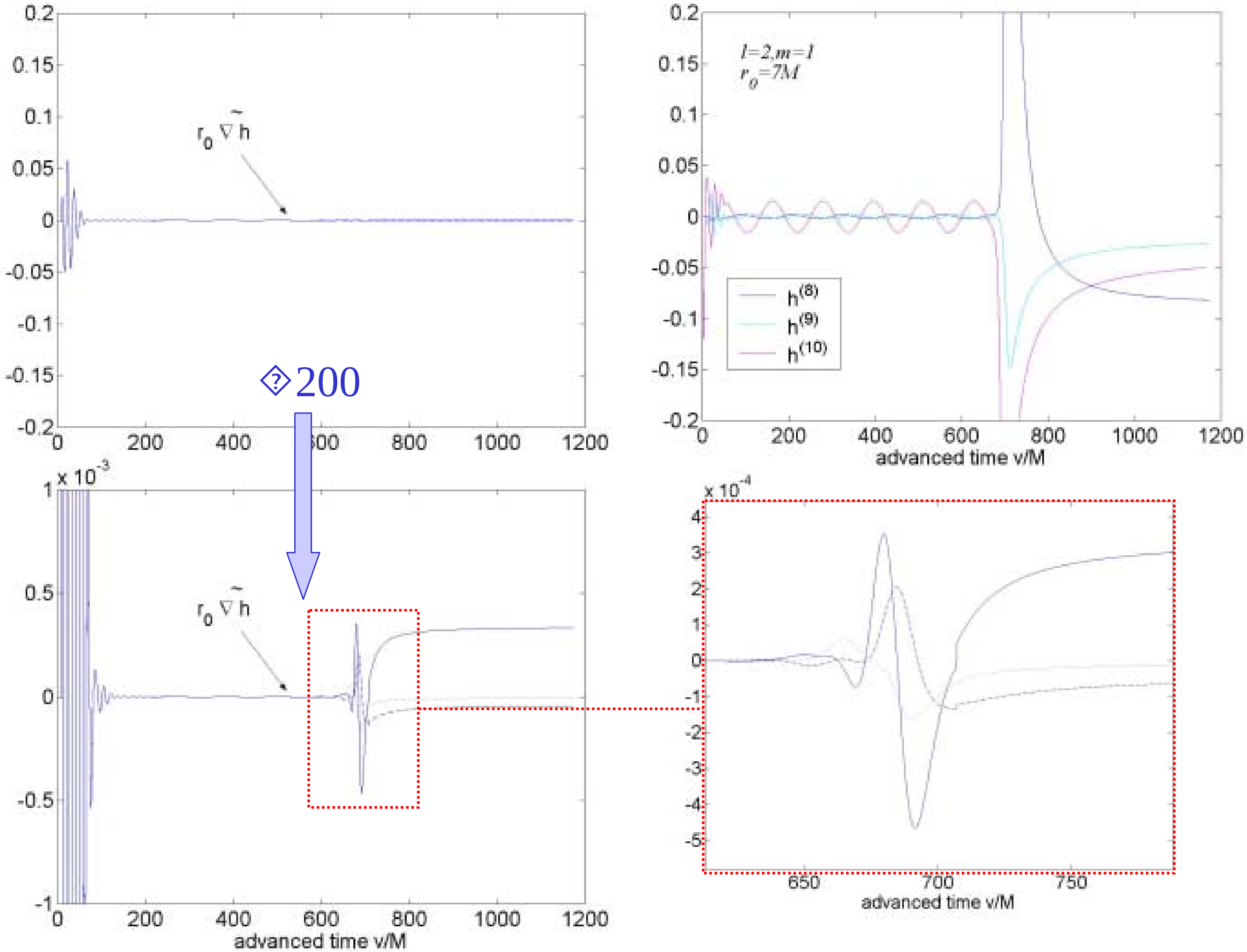
r_0=7M, h=0.0125
$[h^{(8)}_{,r}]_{r0}$=1.59822/M
$-f^{-2}S^{(8)}$=1.60142/M
$-[h^{(10)}_{,r}]_{r0}$=0.42297/M
$f^{-2}S^{(10)}$=0.42370/M
$h^{(8,10)}$
r/M

Test #3: gauge condition

$$\mathrm{div}\ \bar{h}^{\mathrm{odd}} \equiv g^{\beta\gamma} \begin{pmatrix} \bar{h}_{t\beta;\gamma} \\ \bar{h}_{r\beta;\gamma} \\ \bar{h}_{\theta\beta;\gamma} \\ \bar{h}_{\varphi\beta;\gamma} \end{pmatrix}$$

$$= \sum_{lm} \frac{f^{-1}}{\sqrt{2l(l+1)}} \times \begin{pmatrix} 0 \\ 0 \\ (\sin\theta)^{-1} Y^{lm}_{,\varphi}(\theta,\varphi) \\ \sin\theta\, Y^{lm}_{,\theta}(\theta,\varphi) \end{pmatrix} \times \left[\bar{h}^{(8)}_{,t} - if\left(\bar{h}^{(9)}_{,r} + \frac{2}{r}\bar{h}^{(9)} - \frac{\sqrt{2\lambda}}{r}\bar{h}^{(10)} \right) \right] = 0$$

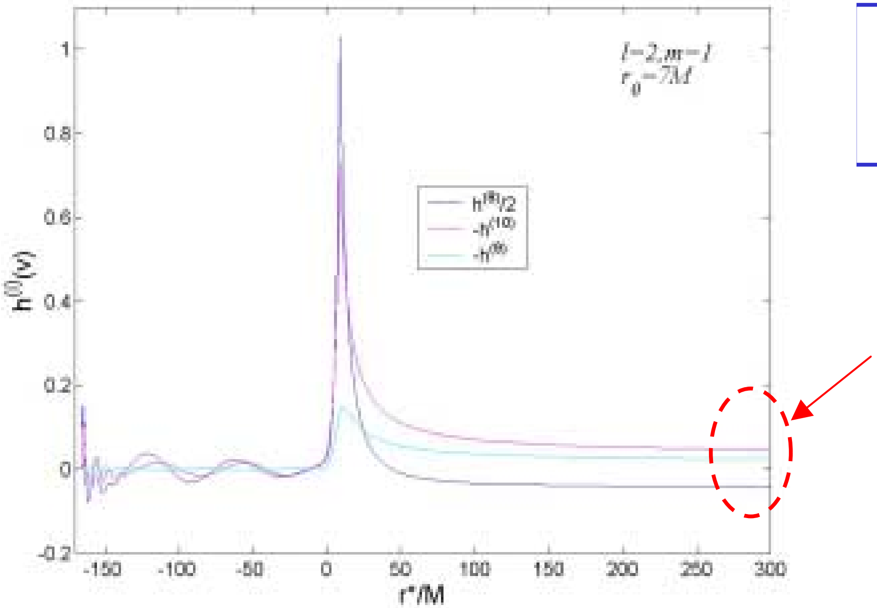
$$\equiv i\tilde{\nabla}\,\overline{h}\,/\,r_0$$



$r_0 \nabla \tilde{h}$
l=2,m=1
r_0=7M
$h^{(8)}$
$h^{(9)}$
$h^{(10)}$
200
x 10^-3
$r_0 \nabla \tilde{h}$
x 10^-4
advanced time v/M
advanced time v/M
advanced time v/M

Test #4: Flux of radiated energy

$$\mu^{-1}\dot{E}^{\infty}_{lm} = \int \frac{d\Omega}{16\pi r^2}\left(\left|(\sin\theta)^{-1}\dot{h}^{lm}_{\theta\varphi}\right|^2 + \frac{1}{4}\left|\dot{h}^{lm}_{\theta\theta} - (\sin\theta)^{-2}\dot{h}^{lm}_{\varphi\varphi}\right|^2\right)$$



$$\mu^{-1}\dot{E}^{\infty}_{lm} = \frac{m^2\omega^2}{32\pi}\left|\bar{h}^{(10)}_{lm}\right|^2_{\mathrm{w.z.}}$$

l	m	**Ours:** t-domain, from $h_{\alpha\beta}$	**Poisson's:** f-domain, from ψ	**Martel's:** t-domain, from ψ
2	1	$8.1636e{-}07$	$8.1633e{-}07$ [0.00%]	$8.1623e{-}07$ [0.02%]
3	2	$2.5246e{-}07$	$2.5199e{-}07$ [0.19%]	$2.5164e{-}07$ [0.36%]
4	1	$8.3825e{-}13$	$8.3956e{-}13$ [0.16%]	$8.3507e{-}13$ [0.38%]
	3	$5.7828e{-}08$	$5.7751e{-}08$ [0.13%]	$5.7464e{-}08$ [0.63%]
5	2	$2.7897e{-}12$	$2.7896e{-}12$ [0.00%]	$2.7587e{-}12$ [1.12%]
	4	$1.2296e{-}08$	$1.2324e{-}08$ [0.23%]	$1.2193e{-}08$ [0.84%]

Energy flux $\dot{E}^{\infty}_{lm}$: Comparison of 3 methods (Circular orbit at r_0=7.9456M)

What’s next?

Even-parity part of MP

$$\Box^{2d}_{\rm sc}\bar{h}^{(i)lm} + \mathcal{M}^{(i)}_{(j)}\bar{h}^{(j)lm} = \tilde{S}^{(i)lm},$$

$$\mathcal{M}^{(1)}_{(j)}\bar{h}^{(j)} = \frac{1}{2}ff'\bar{h}^{(1)}_{,r} - \frac{1}{2}(f'/r)(1-1.5M/r)\left(\bar{h}^{(1)} - \bar{h}^{(3)}\right) - \frac{1}{4}\sqrt{2}f'\left(i\bar{h}^{(2)}_{,t} + r^{-1}f^2\bar{h}^{(6)}\right),$$

$$\mathcal{M}^{(2)}_{(j)}\bar{h}^{(j)} = \frac{1}{2}ff'\bar{h}^{(2)}_{,r} + \frac{1}{2}(f^2/r^2)\bar{h}^{(2)} + \frac{1}{4}\sqrt{2}if'\left(\bar{h}^{(1)}_{,t} + \bar{h}^{(3)}_{,t}\right) - \frac{1}{2}\sqrt{l(l+1)}(f^2/r^2)\bar{h}^{(4)},$$

$$\mathcal{M}^{(3)}_{(j)}\bar{h}^{(j)} = \frac{1}{2}ff'\bar{h}^{(3)}_{,r} + r^{-2}\left[1-5M/r+5.5(M/r)^2\right]\bar{h}^{(3)} + \frac{1}{2}(f'/r)(1-1.5M/r)\bar{h}^{(1)} - \frac{1}{4}\sqrt{2}if'\bar{h}^{(2)}_{,t}$$
$$-\frac{1}{2}\sqrt{2}(f^2/r^2)\left[\sqrt{l(l+1)}\,\bar{h}^{(5)} + (1-3M/r)\bar{h}^{(6)}\right],$$

$$\mathcal{M}^{(4)}_{(j)}\bar{h}^{(j)} = \frac{1}{4}f'f\bar{h}^{(4)}_{,r} - \frac{3}{4}f'(f/r)\bar{h}^{(4)} + \frac{1}{4}f'i\bar{h}^{(5)}_{,t} - \frac{1}{2}\sqrt{l(l+1)}(f/r^2)\bar{h}^{(2)},$$

$$\mathcal{M}^{(5)}_{(j)}\bar{h}^{(j)} = \frac{1}{4}ff'\bar{h}^{(5)}_{,r} + (f/r^2)(1-3.5M/r)\bar{h}^{(5)} - \frac{1}{2}(f/r^2)\sqrt{2l(l+1)}\,\bar{h}^{(3)} - \frac{1}{4}f'i\bar{h}^{(4)}_{,t}$$
$$+\frac{1}{2}(f^2/r^2)\left[\sqrt{l(l+1)}\,\bar{h}^{(6)} - \sqrt{2\lambda}\,\bar{h}^{(7)}\right],$$

$$\mathcal{M}^{(6)}_{(j)}\bar{h}^{(j)} = \frac{1}{2}(f/r^2)(1-4M/r)\bar{h}^{(6)} - \frac{\sqrt{2}}{4}(f'/r)\bar{h}^{(1)} - \frac{\sqrt{2}}{2}r^{-2}(1-3M/r)\bar{h}^{(3)} + \frac{1}{2}\sqrt{l(l+1)}(f/r^2)\bar{h}^{(5)},$$

$$\mathcal{M}^{(7)}_{(j)}\bar{h}^{(j)} = -\frac{1}{2}(f/r^2)\left(\bar{h}^{(7)} + \sqrt{2\lambda}\,\bar{h}^{(5)}\right),$$

Self Force

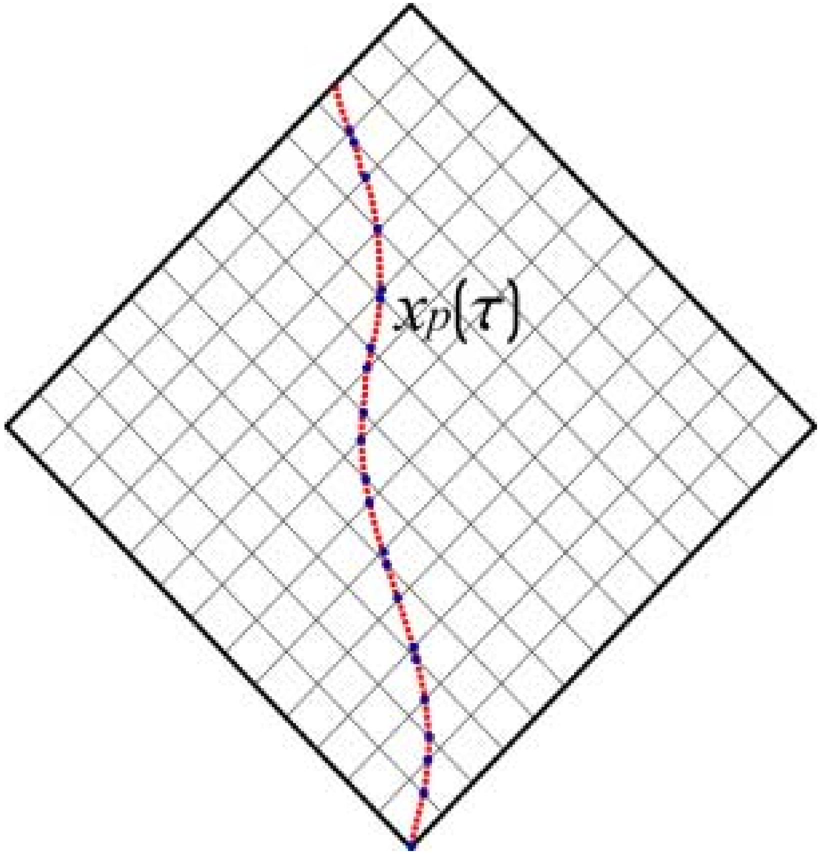
- Getting the self force is now straightforward:
  - No need to take 3 derivatives, just one – as in scalar case
  - Modes l=0,1 given (almost) analytically, in the harmonic gauge
  - Most crucial: no gauge problem; mode-sum implemented as is:

$$F_{\rm self}(x_p) = \sum_{l=0}^{\infty}\left[F_{\rm full}(h_{\alpha\beta})\big|_{x\to x_p} - A(l+1/2) - B - C/(l+1/2)\right] - D$$

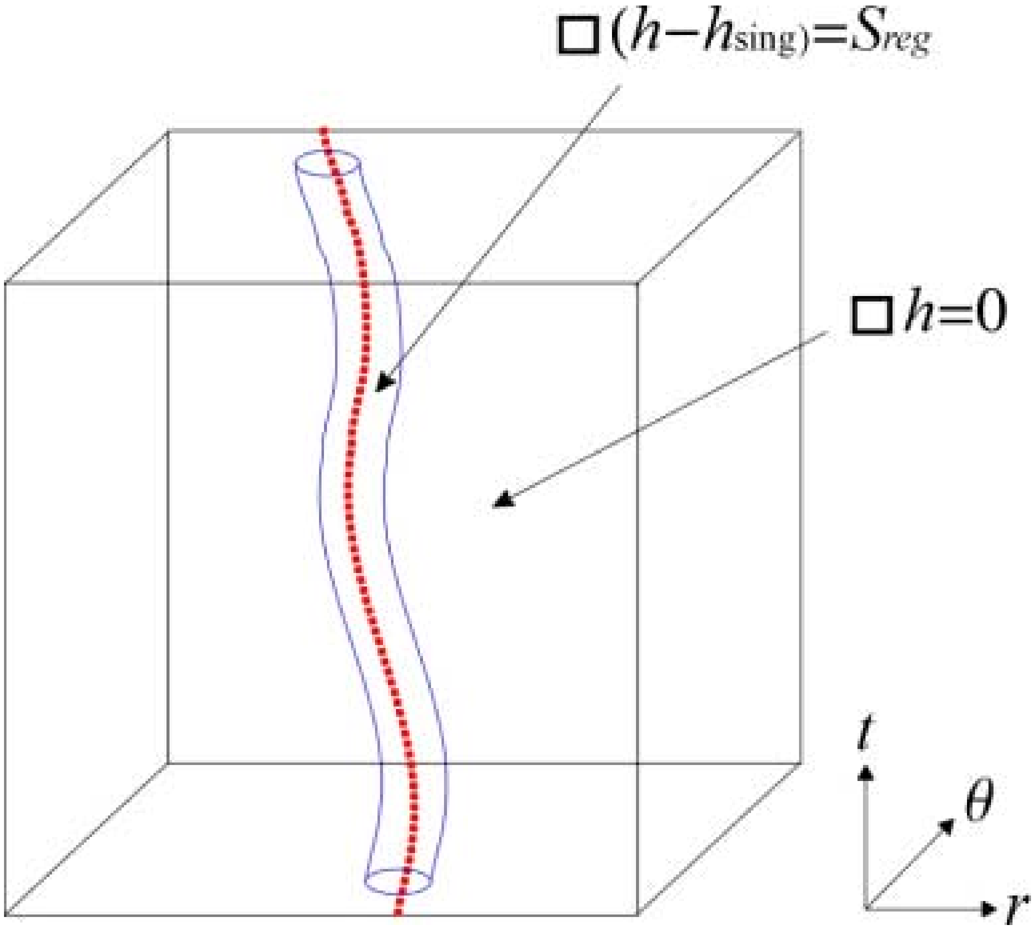
- Two options:
  - Either decompose $F_{\rm full}$ in Scalar harmonics & use the “standard” RP;
  - Or decompose $F_{\rm full}$ in Vector harmonics and re-derive the RP accordingly.

Eccentric orbits in Schwarzschild

- Generalization is easy, as code is time-domain



Orbits in Kerr



Conclusion

When doing local calculations, near a point particle (e.g., for SF):

- Let go of Moncrief-Zerilli-Regge/Wheeler-Teukolsky variables and RW/Radiation gauges.

- Solve directly for Metric perturbation, in the harmonic gauge.

Reason: It is possible!

Basis of tensor harmonics (in t, r, θ, φ coordinates):
I. Even parity

$$Y^{(1)}_{lm} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Y_{lm}, \qquad Y^{(2)}_{lm} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Y_{lm}, \qquad Y^{(3)}_{lm} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Y_{lm},$$

$$Y^{(4)}_{lm} = \frac{ir}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & \partial_\theta & \partial_\varphi \\ 0 & 0 & 0 & 0 \\ \partial_\theta & 0 & 0 & 0 \\ \partial_\varphi & 0 & 0 & 0 \end{pmatrix} Y_{lm}, \qquad Y^{(5)}_{lm} = \frac{r}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_\theta & \partial_\varphi \\ 0 & \partial_\theta & 0 & 0 \\ 0 & \partial_\varphi & 0 & 0 \end{pmatrix} Y_{lm},$$

$$Y^{(6)}_{lm} = \frac{r^2}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s^2 \end{pmatrix} Y_{lm}, \qquad Y^{(7)}_{lm} = \frac{r^2}{2\sqrt{\lambda l(l+1)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{L}_2 & \mathbb{L}_1 \\ 0 & 0 & \mathbb{L}_1 & -s^2\mathbb{L}_2 \end{pmatrix} Y_{lm},$$

$$\mathbb{L}_1 \equiv 2(\partial_{\theta\varphi} - \cot\theta\,\partial_\varphi), \quad , \mathbb{L}_2 \equiv \partial_{\theta\theta} - \cot\theta\,\partial_\theta - s^{-2}\partial_{\varphi\varphi}, \quad s \equiv \sin\theta, \quad \lambda(l-1)(l+2)/2$$

Basis of tensor harmonics (in t,r,θ,φ coordinates):
II. Odd parity

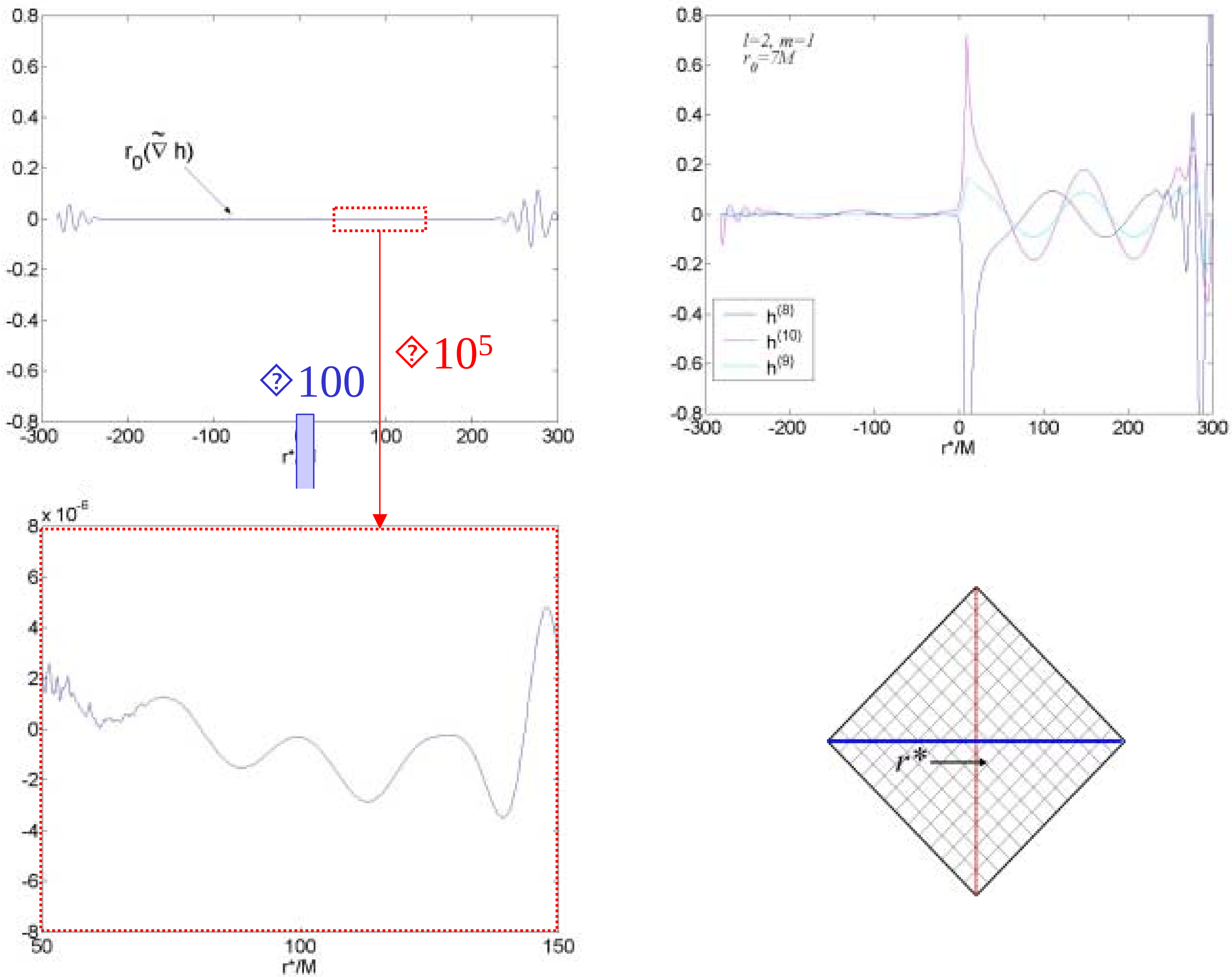
$$Y_{lm}^{(8)} = \frac{r}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & s^{-1}\partial_\varphi & -s\,\partial_\theta \\ 0 & 0 & 0 & \\ s^{-1}\partial_\varphi & 0 & 0 & 0 \\ -s\,\partial_\theta & 0 & 0 & 0 \end{pmatrix} Y_{lm},$$

$$Y_{lm}^{(9)} = \frac{ir}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & s^{-1}\partial_\varphi & -s\,\partial_\theta \\ 0 & s^{-1}\partial_\varphi & 0 & 0 \\ 0 & -s\,\partial_\theta & 0 & 0 \end{pmatrix} Y^{lm},$$

$$Y_{lm}^{(10)} = \frac{-ir^2}{2\sqrt{\lambda l(l+1)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -s^{-1}\mathbb{L}_1 & s\,\mathbb{L}_2 \\ 0 & 0 & s\,\mathbb{L}_2 & s\,\mathbb{L}_1 \end{pmatrix} Y_{lm},$$

10 Basis tensors are orthonormal:

$$\int d\Omega\, \eta^{\alpha\mu}\eta^{\beta\nu}[Y_{\mu\nu}^{(i)lm}]^* Y_{\alpha\beta}^{(j)l'm'} = \delta_{ij}\delta_{ll'}\delta_{mm'} \qquad [\text{where } \eta^{\alpha\beta} = \mathrm{diag}(-1, 1, r^{-2}, r^{-2}s^{-2})]$$



$r_0(\tilde{\nabla} h)$
100
10^5
x 10^-6
r*/M
l=2, m=1
r_0=7M
h^(8)
h^(10)
h^(9)
r*/M
r*