Gravitational radiation from Kerr parabolic orbits

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Introduction

About this talk:

- Motivation
- Teukolsky vs Sasaki-Nakamura
- Regularising the Teukolsky equation
- A useful approximation ?
- EMRI inventory: “kludged” and Teukolsky waveforms/fluxes

Motives

- EMRI: small body initially captured in an $e \approx 1$ orbit.
- Initial periastron can be as small as $r_p \sim 4M$ (Freitag) (even smaller for a rapidly spinning BH) $\Rightarrow$ Quadrupole formula expected to fail.
- Provide accurate $\{\dot{E}, \dot{L}\} \Rightarrow$ firm up capture rates/background confusion noise estimates
- Compare with/complement time-domain calculations, especially for Kerr where they face a serious challenge.
- Useful benchmark problem when the self-force calculations will become a practical tool.

Dealing with $e \approx 1$ orbits

- Naive first attempt with the Teukolsky equation $\Rightarrow$ integral over source term is divergent !
- Use the S-N equation instead which has a well-behaved source term $\Rightarrow$ GWs from parabolic/plunging orbits in Kerr discussed by Kojima & Nakamura in 1984.
- **Drawback of the S-N approach**: extra radial integration required to obtain source term (discussed in more detail later on) $\Rightarrow$ S-N code inherently much slower than a Teukolsky code...
- There is nothing intrinsically wrong with the solution of the Teukolsky equation - rather one has to regularise certain integrals over the source term when the latter has no compact support.
- Two regularisation schemes : S. Detweiler & E.Szedenits (already used in practise many years ago) and E. Poisson. The latter scheme was extended for Kerr by M. Campanelli & C.Lousto.

Teukolsky vs Sasaki-Nakamura

Traditionally, GW radiation from eccentric, plunging and parabolic orbits has been computed by solving the S-N equation instead of the original Teukolsky equation. For a given $\{\ell, m, \omega\}$ mode,

$$\left[\frac{d^2}{dr_*^2} - F\frac{d}{d_{r_*}} - U\right] X = S$$

where the source term is related to the Teukolsky source term T as,

$$S = \frac{\gamma \Delta W}{r^2(r^2+a^2)^{3/2}} \exp\left(-i\int \frac{K}{\Delta} dr\right)$$

and

$$\frac{d^2 W}{dr^2} = \frac{r^2 T}{\Delta^2} \exp\left(i\int \frac{K}{\Delta} dr\right)$$

- The nice property of the S-N equation is that $S \sim r^{-5/2}$ as opposed to $T \sim r^{-3/2}$. Then, the solution of the Teukolsky equation is,

$$R(r \to \infty) = \frac{R^{up}}{W_T} \int_{-\infty}^{+\infty} dr_* \frac{X^{in}}{\gamma} S$$

  where the integral is convergent.

- On the other hand, we can formally write (following E.Poisson),

$$\begin{aligned} R &= \frac{1}{W_T} \Big[R^{up} \{ A + \int_a^r dr' \, e^{i\omega T(r') - im\Phi(r')} T(r') R^{in}(r') \} \\ &+ R^{in} \{ B + \int_r^b dr' \, e^{i\omega T(r') - im\Phi(r')} T(r') R^{up}(r') \} \Big] \end{aligned}$$

  This leads to divergent integrals if, naively, we set $a = r_+$, $b = +\infty$ and disregard the constants A, B.

Regularising the Teukolsky equation (I)

- We are asked to regularise integrals of the form,

$$I^{in,up}(a,b) = \int_a^b dr\ e^{i\omega T - im\Phi} R^{in,up} T$$

when $a \to r_+$ and $b \to +\infty$. Take the following steps (drop in, up):

- Change wavefunction: $R = \mathcal{C}Y = F\mathcal{L}[Y] + GY$ where $\mathcal{L} = d/dr_* + i\omega$
- The function Y satisfies an equation of the form,

$$d^2Y/dr_*^2 + V_Y Y = 0$$

One attractive choice is $Y = \sqrt{\eta}X$.

Regularising the Teukolsky equation (II)

- Then:

$$I = I_{conv} + I_{div} = \int_a^b dr\ e^{i\omega T - im\Phi}\Gamma_{conv} Y + \int_a^b dr\ e^{i\omega T - im\Phi}\Gamma_{div}\mathcal{L}[Y]$$

  Only second integral is divergent now.

- (E.Poisson): introduce an unspecified function $h(r)$ such as,

$$I_{div} = \int_a^b dr\ e^{i\omega T - im\Phi}\{\tilde{\Gamma}_{div}\mathcal{L}[Y] + \tilde{\Gamma}_{conv} Y\} - [e^{i\omega T - im\Phi} h\mathcal{L}[Y]]_a^b$$

- We have that,

$$\tilde{\Gamma}_{div} = dh/dr + (i\omega dT/dr - imd\Phi/dr)\, h + \Gamma_{div}$$

Regularising the Teukolsky equation (III)

- Choose: $\tilde{\Gamma}_{div} = 0 \Rightarrow$ an ODE for $h(r)$. Drawback: typically the above ODE cannot be solved analyticaly...
- We have ended up with convergent integrals; infinities have been “transfered” to surface terms $\Rightarrow$ absorb them into A, B.
- *Then*, we are allowed to take $r \to \infty, r_+$. Enforcing the physically required boundary conditions, we eliminate both A, B.
- Final well-behaved result for $r \to \infty$:

$$R(r) = \frac{R^{up}(r)}{W_T} \int_{r_+}^{r} dr' \; e^{i\omega T - im\Phi} g(r') X^{in}$$

where $g(r)$ denotes all the mess!

An alternative scheme (I)

- Write,

$$\Gamma_{div} = \{i\omega dT/dr - imd\Phi/dr\}h(r) + \beta(r)$$

- Then

$$\begin{aligned} I_{div} &= -\int_a^b dr\ e^{i\omega T - im\Phi}\{dh/dr + i\omega dr_*/drh - \beta\}\mathcal{L}[Y]\} \\ &- \int_a^b dr\ e^{i\omega T - im\Phi}\tilde{\Gamma}_{conv}Y + [e^{i\omega T - im\Phi}h\mathcal{L}[Y]]_a^b \end{aligned}$$

- Choose $\beta = i\omega h dr_*/dr \Rightarrow$ fixes h too. No need to solve any ODE now.
- If $\Gamma_{div} \sim r^{-n}$ then new integrand goes $\sim r^{n-3/2}$.
- After performing this procedure twice we end up with convergent integrals $+$ divergent surface terms. Retracing the steps outlined above we eliminate these terms.

An alternative scheme (II)

- The final result looks like,

$$R(r) = \frac{R^{up}(r)}{W_T} \int_{r_+}^{r} dr' \; e^{i\omega T - im\Phi} \{ G_{conv} \mathcal{L}[Y] + F_{conv} Y \}$$

- The functions $G_{conv}(r)$,$F_{conv}(r)$ contain angular functions $S^{a\omega}_{lm}$, orbital functions $T(r), \Phi(r), \Theta(r)$ and part of the projections $T_{nn}, T_{\bar{m}\bar{m}}, T_{\bar{m}n}$.
- This regularisation method was first introduced by S. Detweiler & E. Szedenits in 1979 in their study of Schwarzschild plunging orbits.
- Although they simply dropped the surface terms with no formal justification, they got correct results. As we discussed, one *is allowed*, after all, to drop these terms following part of Poisson's prescription.
- This “new” source integral should be the fastest to compute in a Teukolsky code [coding in progress - need to wait for Capra 8 !]

Near-periastron approximations (I)

- Waveforms from $e \approx 1$ orbits exhibit extreme zoom-whirl features: long "silent" intervals separated by "bursts" when the small body is at the periastron vicinity.
- This behaviour is encoded in the source term integral which can be scaled in terms of the radial velocity component as,

$$I = \int_{r_p}^{r_a} dr\ e^{i\omega T - im\Phi} \left\{ \frac{F_0}{u^r} + F_1(r,\omega) + \mathcal{O}(u^r) \right\}$$

- One may be tempted to truncate the basic integral as,

$$I \approx \int_{r_p}^{r_o} dr\ e^{i\omega T - im\Phi} \frac{F_{res}(r,\omega)}{r - r_p}$$

This could still give a good result while saving significant amount of computational time.

Near-periastron approximations (II)

- One can deduce (Oohara & Nakamura) that the emitted spectrum will have a maximum at the frequency,

$$\omega_{max} \approx \ell\,\Omega_\phi(r_p) = \frac{2L}{r_p^2}(1 - 2M/r_p) \quad \text{for} \quad a/M = 0,\ \ell = m = 2$$

- Simulations (Freitag) suggest that initial $r_p \sim 4M - 60M$, which translates to,

$$f_{max} \approx (0.3 - 7)mHz \gtrsim \text{ LISA's } \quad f_{peak}$$

The upper limit can become much higher for Kerr BHs.

- Note: the above frequency is much higher than the naive "circular-orbit" estimate

$$f_{max} \approx \frac{2}{T_{period}} \sim 10^{-6} mHz$$

Quick and dirty: kludged waveforms/fluxes

- Basic idea (essentially this is a rediscovery of Sasaki-Ruffini's semi-relativistic approximation) : Feed the quadrupole formula for waveform/fluxes with the **exact** Kerr geodesic motion, pretending that B-L coordinates are Cartesian spherical coordinates.
- Hence, contributions due to backscattering higher multipoles are ignored from the very beginning.
- Results accurate to $\sim 10\%$ down to $r \sim 5 - 6M$.
- Can be considered as the "poor man's time-domain code" !
- Can be adapted to any background metric once the geodesics are known $\Rightarrow$ potential data analysis tool for EMRI around non-Kerr objects (in which case we do not enjoy the luxury of having a Teukolsky-like equation) [work in progress].

Sample kludged waveform