A New Analytical Method for Self-Force Regularization

Yukawa Institute for Theoretical Physics,
Kyoto University
Wataru Hikida (D2)

in collaboration with
S.Jhingan, H.Nakano, N.Sago, M.Sasaki, T.Tanaka

§ 1.Introduction

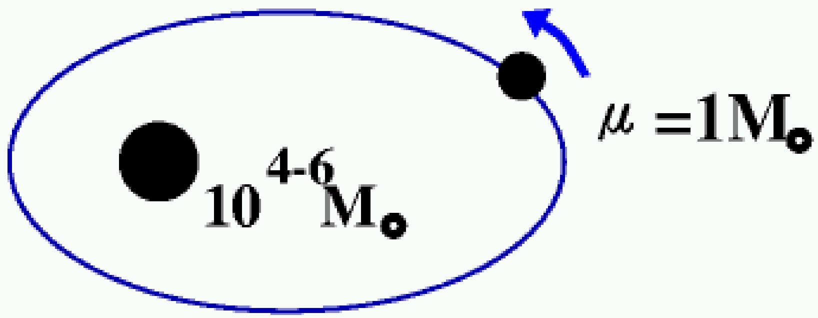
GW observatory

- LIGO, TAMA300 and GEO600 are currently in the early stage of their operations
- Space-based interferometer project (LISA) is in rapid progress

Targets for these detectors

- Binary systems, Pulsar, Big Bang, etc

Here we focus on an extreme mass-ratio binary system.



Effective method

- BH perturbation approach + point-particle approximation

And

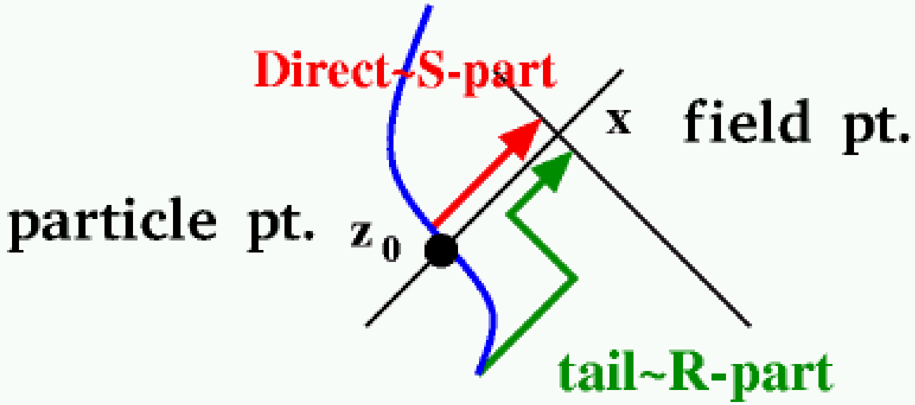
It is then important to consider the self-force.

§ 1.Introduction

Self-Force=R-part

However, the self-force diverges at $x \to z_0$ and hence should be regularized.

Split the retarded Green function into two pieces



Self-force = R-part (but it is hard to calculate directly)

$$F^{\text{Self-Force}}(z_0) = F^R(z_0) = \lim_{x\to z_0}\left(F^{\text{full}}(x) - F^S(x)\right)$$

However
$$\lim_{x\to z_0} F^{\text{full}}(x) = \lim_{x\to z_0} F^S(x) = \infty$$

It is necessary to develop a regularization scheme to calculate this subtraction.

Mode-Sum Reg. and Problems

The most successful method developed so far is
mode-sum (or mode-decomposition) regularization.
[Mino, Nakano & Sasaki ('03), Barack et al.('02)]

$$F_\alpha^{\mathrm{R}}(z_0) = \sum_\ell \lim_{x \to z_0} \left(F_{\alpha,\ell}^{\mathrm{full}}(x) - F_{\alpha,\ell}^{S}(x) \right)$$

However there are some **problems** in this method.

i) Gauge Mismatch (Gauge problem)

- each force of r.h.s has the different gauge

ii) Domain Mismatch (Subtraction problem)

- F^{full} : the Fourier decomposition of time-dependence, in order to treat easily.

→ the full-part is calculated in the frequency domain.

- F^{S} : only determined near the particle, so it cannot perform the Fourier decomposition.

→ the S-part is only calculated in the time domain.

Let's consider about these problems in details.

§ 2 Method for calculation (S-part)

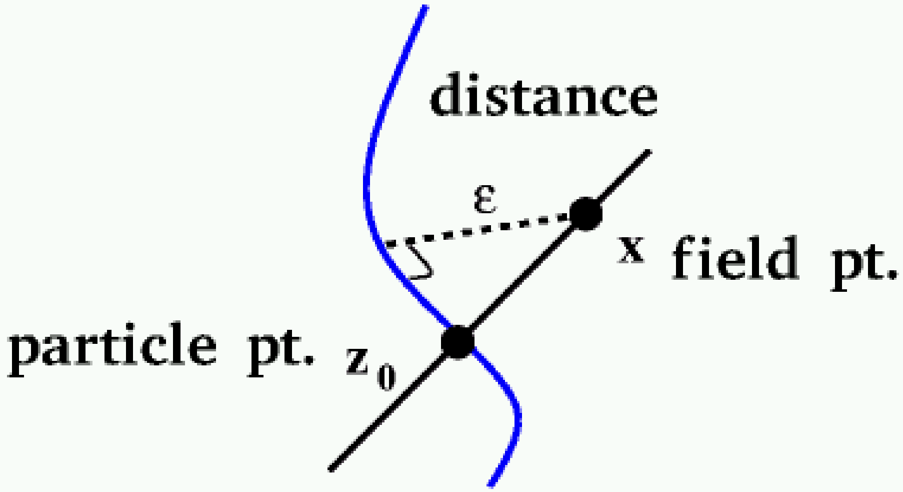
The S-part is determined by local expansion near the particle and can be expressed by

$$h^{S,H}_{\alpha\beta} = \mu \sum C^{m,n,p,q,r}_{\alpha\beta} \frac{T^m R^n \Theta^p \Phi^q}{\epsilon^r} + O(y^2)$$

$$\epsilon \equiv (r_0^2 + r^2 - 2 r_0 r \cos\Theta \cos\Phi)^{1/2}$$

$$(x - z_0)^\alpha \equiv (T, R, \Theta, \Phi), \quad \epsilon \sim T \sim R \sim \Theta \sim \Phi \sim O(y)$$

Here 'H' represents harmonic gauge.



§ 2 Method for calculation

S-part (II)

Force

$$F_\alpha^{S,H} = \mathcal{L}\,[h_{\alpha\beta,\ell}^{S,H}], \quad \text{where } \mathcal{L} \text{ :some differential operator}$$

mode-sum (or mode decomposition) regularization

[Mino, Nakano & Sasaki ('03), Barack et al.('02)]

- ℓ -mode of this Force

$$F_{\alpha,\ell}^{S,H} = A_\alpha L + B_\alpha + C_\alpha / L + D_{\alpha,\ell}, \quad L \equiv \ell + 1/2,$$

In harmonic gauge, it is known that

$$C_\alpha = \sum_\ell D_{\alpha,\ell} = 0$$

This property is called "Standard Form"

§ 2 Method for calculation

Full part (e.g. Schwarzschild case)

Master equation for BH perturbations (RW gauge), after the Fourier Harmonic expansion.

$$\mathcal{L}_{RW}\, R^{RW}_{\ell m\omega} = S_{\ell m\omega}$$

Metric Reconstruction

$$h^{\text{full,RW}}_{\mu\nu,\ell m\omega} = \mathcal{L}'\,[\Psi^{RW}_{\ell m\omega}], \quad \text{where } \Psi^{RW}_{\ell m\omega} = R^{RW}_{\ell m\omega}(r) Y_{\ell m}(\theta,\varphi) e^{-i\omega t}$$

Force

$$F^{\text{full,RW}}_{\alpha,\ell} = \sum_{m\omega} \left(\mathcal{L}\,[h^{\text{full,RW}}_{\mu\nu,\ell m\omega}] \right)$$

(cf. S-part**)**

$$F^{S,H}_{\alpha,\ell} = A_\alpha L + B_\alpha + C_\alpha / L + D_{\alpha,\ell}$$

Mismatch

- **Gauge** $\cdots$ gauge transformation $\quad (F^{S,H}_{\alpha} \to F^{S,RW}_{\alpha})$
- **Domain** $\cdots$ integration over ω $\quad (\sum_{m\omega} F^{\text{full,RW}}_{\alpha,\ell m\omega} \to F^{\text{full,RW}}_{\alpha,\ell})$

full part calculation (numerical approach)

To this time, the calculation of the full part and the regularization have been done numerically.

e.g. Schwarzschild + Scalar

- Radial orbits [Barack & Burko ('00)]
- Circular orbits [Burko('00), Detweiler et al.('03)]

etc.

But

- It is necessary to get the (small) = (large) - (large). So it is hard to calculate accurately.
- It is necessary to decide to cut off the number of ℓ . But there is no systematic method for controling the error.

↓

So we would like to solve the RW eq., perform the integration over ω and subtract more analytically.

§ 2 Method for calculation

full part calculation (analytical approach)

Recently, analytical BH perturbation approach is developing.
[Review Sasaki & Tagoshi ('03)]

Solving Master equation by the Green function method

$$\psi^{\rm full}(x) = q \int d^4x' G^{\rm full}(x,x')T(x')$$

Fourier-Harmonic Transformation:

$$G^{\rm full}(x,x') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \sum g^{\rm full}_{\ell m\omega}(r,r') Y_{\ell m}(\theta,\varphi) Y^*_{\ell m}(\theta',\varphi'),$$

$$\left[\partial^2_{r^*} + \omega^2 - V\right] g^{\rm full}_{\ell m\omega}(r,r') = -\frac{\delta(r-r')}{r^2}, \quad : \text{RW eq.}$$

The Green function is represented by in- and up-going homogeneous solutions.

$$g^{\rm full}_{\ell m\omega}(r,r') = \frac{-1}{W(\phi_{\rm in},\phi_{\rm up})} \left(\phi^{\nu}_{\rm in}(r)\, \phi^{\nu}_{\rm up}(r')\theta(r'-r) + (r \leftrightarrow r')\right)$$

Character of the analytical approach

i) Mano-Suzuki-Takasugi's solution

- Homogeneous solution of Regge-Wheeler/Teukolsky eq.
- use ν , in place of ℓ
  - (called as renormalized angular momentum)
- The explanation in details is at the next page.

ii) Slow motion approximation

- assumption that $z \equiv r\omega \sim O(v)$ and $\epsilon \equiv 2M\omega \sim O(v^3)$, which are used by MST's solution, are small
- In principle, it is possible to expand up to the order you want

§ 2 Method for calculation

Decomposition by MST's solution

The in- and up-going homogeneous solutions can be decomposed by Mano-Suzuki-Takasugi's solutions.

$$\phi_{\rm in}^{\nu} = \phi_{\rm c}^{\nu} + \tilde{\beta}_{\nu}\phi_{\rm c}^{-\nu-1}, \quad \phi_{\rm up}^{\nu} = \tilde{\gamma}_{\nu}\phi_{\rm c}^{\nu} + \phi_{\rm c}^{-\nu-1},$$

where ϕ_c^{ν} is the Coulomb wave function calculated by MST;

$$\phi_{\rm c}^{\nu} \approx \sum_{n=-\infty}^{\infty} a_n^{\nu} e^{-iz}(2z)^{n+\nu} F(\alpha_1, \alpha_2; 2iz),$$

$$\nu = \ell + O(\epsilon^2) \quad (\nu = \ell \text{ up to 2.5PN})$$

here,

$a_n^{\nu}, \tilde{\beta}, \tilde{\gamma}, \nu$: the functions of ϵ

α_1, α_2 : the functions of n and ν

$F(\alpha_1, \alpha_2; 2iz)$: the confluent hypergeometric function

Relation of the coefficients

$$\begin{cases} a_{n+1}^{\nu}/a_n^{\nu} \approx O(\epsilon) & (n > 0) \\ a_n = 1 & (n = 0) \\ a_{n-1}^{\nu}/a_n^{\nu} \approx O(\epsilon) & (n < 0) \end{cases}$$

Problems of analytical approach

We can easily expand ϕ_c^ν, up to the order you want, with respect to $z \equiv r\omega$ and $\epsilon \equiv 2M\omega$.

The results are

$$\phi_c^\nu = (2z)^\nu \, \Phi^\nu,$$

$$\Phi^\nu = 1 - \frac{z^2}{2(2\ell+3)} - \frac{\ell\epsilon}{2z} + \frac{z^4}{8(2\ell+3)(2\ell+5)} + \frac{(\ell^2 - 5\ell - 10)\epsilon z}{4(2\ell+3)(\ell+1)} + \cdots,$$

$$\nu = \ell - \frac{15\ell^2 + 15\ell - 11}{2(2\ell-1)(2\ell+1)(2\ell+3)}\epsilon^2 + \cdots .$$

and $\phi_c^{-\nu-1}$ is obtained using the $\ell \to -\ell - 1$.

In order to integrate over ω, we have to re-expand with respect to ω. Then

$$(2z)^\nu \sim (2r\omega)^{\ell-(2M\omega)^2} \sim (2r\omega)^\ell \left(1 - 4M \ln(2r\omega) + \cdots\right)$$

Due to this $\ln\omega$, it is hard to transform from the frequency domain to the time domain analytically.

§ 3 New method (new decomposition)

Here, we propose a new decomposition of the full part into two parts, the $\tilde{S}$-part and the $\tilde{R}$-part

This decomposition naturally arises from the developed by Mano, Suzuki and Takasugi.

Property

- $\tilde{S}$-part ⋯ contains all the singular terms to be subtracted
- $\tilde{R}$-part ⋯ remaining term, hence finite (not need to regularize)

The part that we need to regularize is only the $\tilde{S}$-part

Therefore

$$\begin{aligned} F_\alpha^{\text{S.F.}}(z_0) = F_\alpha^R(z_0) &= \sum_\ell \lim_{x\to z_0} \left(F_{\alpha,\ell}^{\text{full}}(x) - F_{\alpha,\ell}^S(x)\right) \\ &= \sum_\ell \lim_{x\to z_0} \left(F_{\alpha,\ell}^{\tilde{S}}(x) + F_{\alpha,\ell}^{\tilde{R}}(x) - F_{\alpha,\ell}^S(x)\right) \\ &= \sum_\ell \lim_{x\to z_0} \left(F_{\alpha,\ell}^{\tilde{S}}(x) - F_{\alpha,\ell}^S(x)\right) + \sum_\ell F_{\alpha,\ell}^{\tilde{R}}(z_0). \end{aligned}$$

$\tilde{S}$ -part

Important fact about the $\tilde{S}$ -part

- As long as we use the slow motion approximation, the $\tilde{S}$ -part of the Green function in the frequency domain is given by

$$G^{\tilde{S}}(x,x') = \int \frac{d\omega}{2\pi} \sum e^{-i\omega(t-t')} Y_{\ell m}(\theta,\varphi) Y^{*}_{\ell m}(\theta',\varphi') \sum_{n=0}^{\infty} C^{(n)}(r,r')\omega^{n}$$

expanded by only positive power of ω

So, using the formula

$$\int d\omega\, e^{-i\omega(t-t')}\omega^{n} = 2\pi\delta^{(n)}(t-t'),$$

we are able to perform integration over ω (namely, inverse Fourier transformation) , for a general orbit analytically.

$(\tilde{S} - S)$-part

The S-part under RW gauge is given by

$$F^{S,RW}_{\alpha,\ell} = A^{RW}_{\alpha} L + B^{RW}_{\alpha} + C^{RW}_{\alpha}/L + D^{RW}_{\alpha,\ell}$$

Sincethe $\tilde{S}$-part contains all the singular terms to be subtracted, the $\tilde{S}$-part under RW gauge is given by

$$F^{\tilde{S},RW}_{\alpha,\ell} = A^{RW}_{\alpha} L + B^{RW}_{\alpha} + C^{RW}_{\alpha}/L + \tilde{D}^{RW}_{\alpha,\ell}$$

Then regularization is given by

$$\sum_{\ell=0}^{\infty} \lim_{x \to z_0} \left(F^{\tilde{S}}_{\alpha,\ell}(x) - F^{S}_{\alpha,\ell}(x) \right) = \sum_{\ell=0}^{\infty} \left(\tilde{D}^{RW}_{\alpha,\ell}(z_0) - D^{RW}_{\alpha,\ell}(z_0) \right)$$

If the S-part under RW gauge becomes "Standard Form", then

$$\sum_{\ell=0}^{\infty} \lim_{x \to z_0} \left(F^{\tilde{S}}_{\alpha,\ell}(x) - F^{S}_{\alpha,\ell}(x) \right) = \sum_{\ell=0}^{\infty} \tilde{D}^{RW}_{\alpha,\ell}(z_0)$$

§ 3 New method (new decomposition)

Summary

Self-force

$$F^{\mathrm{R}}_{\alpha}(z_0) = \sum_{\ell} \lim_{x \to z_0} \left(F^{\mathrm{full}}_{\alpha,\ell}(x) - F^{S}_{\alpha,\ell}(x) \right)$$

method for calculating the full part:

numerical **analytical**

New decomposition:

$(\tilde{S} - S)$-part $\tilde{R}$-part

$$\sum_{\ell=0}^{\infty} \left(\tilde{D}_{\alpha,\ell}(z_0) - D_{\alpha,\ell}(z_0) \right) \qquad \sum_{\ell} F^{\tilde{R}}_{\alpha,\ell}(z_0)$$

Recover the "Standard Form" under RW gauge

Yes **No**

$$F^{R}_{\alpha,\ell}(z_0) = \sum_{\ell} \tilde{D}_{\alpha,\ell}(z_0) + \sum_{\ell} F^{\tilde{R}}_{\alpha,\ell}(z_0) \qquad F^{R}_{\alpha,\ell} = \sum_{\ell} \left(\tilde{D}_{\alpha,\ell}(z_0) - D_{\alpha,\ell}(z_0) \right) + \sum_{\ell} F^{\tilde{R}}_{\alpha,\ell}(z_0)$$

§ 4 Example (Schwarzschild + Scalar case)

For simplicity, we consider the scalar case.

$$\psi^{\rm full}(x) = -q \int d\tau \, G^{\rm full}(x, z(\tau)), \quad \nabla^\alpha \nabla_\alpha G^{\rm full}(x, x') = -\frac{\delta^{(4)}(x - x')}{\sqrt{-g}}$$

Fourier-harmonic decomposition

$$G^{\rm full}(x, x') = \int \frac{d\omega}{2\pi} \sum_{\ell m \omega} g^{\rm full}_{\ell m \omega}(r, r') Y_{\ell m}(\theta, \varphi) Y^*_{\ell m \omega}(\theta', \varphi'),$$

$$\left[\partial^2_{r^*} + \omega^2 - V\right] g^{\rm full}_{\ell m \omega}(r, r') = -\frac{\delta(r - r')}{r^2}, \quad : \text{RW eq.}(s = 0)$$

The Green function is represented by in-going and up-going homogeneous solutions.

$$g^{\rm full}_{\ell m \omega}(r, r') = \frac{-1}{W(\phi_{\rm in}, \phi_{\rm up})} \left(\phi^{\nu}_{\rm in}(r) \, \phi^{\nu}_{\rm up}(r') \theta(r' - r) + (r \leftrightarrow r') \right)$$

New decomposition

The Green functions is represented by MST's solutions and decomposition by whether there is step function or not.

$$g^{\text{full}}_{\ell m\omega}(r,r') = \frac{-1}{W(\phi_{\text{in}},\phi_{\text{up}})}\left(\phi^{\nu}_{\text{in}}(r)\,\phi^{\nu}_{\text{up}}(r')\theta(r'-r) + (r \leftrightarrow r')\right)$$

$$= \frac{-1}{(1-\tilde{\beta}_\nu\tilde{\gamma}_\nu)W(\phi^{\nu}_{\text{c}},\phi^{-\nu-1}_{\text{c}})}$$

$$\times\left[\left(\phi^{\nu}_{\text{c}}(r) + \tilde{\beta}_\nu\phi^{-\nu-1}_{\text{c}}(r)\right)\left(\tilde{\gamma}_\nu\phi^{\nu}_{\text{c}}(r') + \phi^{-\nu-1}_{\text{c}}(r')\right)\theta(r'-r) + (r \leftrightarrow r')\right]$$

$$\equiv g^{\tilde{S}}_{\ell m\omega}(r,r') + g^{\tilde{R}}_{\ell m\omega}(r,r').$$

where

$$g^{\tilde{S}}_{\ell m\omega}(r,r') = \frac{-1}{W}\left[\phi^{\nu}_{\text{c}}(r)\phi^{-\nu-1}_{\text{c}}(r')\theta(r'-r) + (r \leftrightarrow r')\right],$$

$$g^{\tilde{R}}_{\ell m\omega}(r,r') = \frac{-1}{(1-\tilde{\beta}\,\tilde{\gamma})W}\left[\tilde{\beta}\,\tilde{\gamma}\,\left(\phi^{\nu}_{\text{c}}(r)\phi^{-\nu-1}_{\text{c}}(r') + \phi^{-\nu-1}_{\text{c}}(r)\phi^{\nu}_{\text{c}}(r)\right) + \cdots\right]$$

§ 4 Example (Schwarzschild + Scalar case)

$\tilde{S}$-part

The only term to be regularized of full part is the $\tilde{S}$-part.

$$
\begin{aligned}
g^{\tilde{S}}_{\ell m\omega}(r,r') &= \frac{-1}{W}\left[\phi^{\nu}_{\mathrm{c}}(r)\phi^{-\nu-1}_{\mathrm{c}}(r')\theta(r'-r) + (r \leftrightarrow r')\right] \\
&= \frac{-1}{W}\left[(2z)^{-1}\Phi^{\nu}(r)\Phi^{-\nu-1}(r')\theta(r'-r) + (r \leftrightarrow r')\right]
\end{aligned}
$$

Except for the overall fractional powers z^{ν} and $z^{-\nu-1}$, they contain only the terms with positive integer powers of ω.

Because z^{ν} is cancel out, the $\tilde{S}$-part of the Green functions are represented by positive series of ω.

$$
\begin{aligned}
\phi^{\nu}_{\mathrm{c}} &= (2z)^{\nu}\,\Phi^{\nu}, \\
\Phi^{\nu} &= 1 - \frac{z^2}{2(2\ell+3)} - \frac{\ell\epsilon}{2z} + \frac{z^4}{8(2\ell+3)(2\ell+5)} + \frac{(\ell^2-5\ell-10)\epsilon z}{4(2\ell+3)(\ell+1)} + \cdots, \\
\nu &= \ell - \frac{15\ell^2+15\ell-11}{2(2\ell-1)(2\ell+1)(2\ell+3)}\epsilon^2 + \cdots, \\
\omega W &= -\frac{2\ell+1}{2} + \frac{496\ell^6+1488\ell^5+1336\ell^4+192\ell^3-757\ell^2-605\ell+338}{16(2\ell-1)^2(2\ell+1)(2\ell+3)^2}\epsilon^2 + \cdots
\end{aligned}
$$

§ 4 Example (Schwarzschild + Scalar case)

$(\tilde{S} - S)$-part

This part of the self-force is obtained by

$$F_t^{\tilde{S}-S} = \frac{q^2 u^r}{4\pi r_0^2}\sum_{n=0}^{3} C_t^{\tilde{S}-S(n)}, \quad F_\theta^{\tilde{S}-S} = 0, \quad F_\varphi^{\tilde{S}-S} = \frac{q^2 u^r \mathcal{L}}{4\pi r_0^2}\sum_{n=0}^{2} C_\varphi^{\tilde{S}-S(n)},$$

where n indicates the order of PN expansion and

$$C_t^{\tilde{S}-S(0)} = \frac{73}{133},$$

$$C_t^{\tilde{S}-S(1)} = -\frac{610}{31521}\delta_{\mathcal{E}} + \frac{282}{1501}\frac{\mathcal{L}^2}{r_0^2} - \frac{59590}{31521}U,$$

$$C_t^{\tilde{S}-S(2)} = -\frac{2296958}{8878415}\delta_{\mathcal{E}}^2 + \left[\frac{14127898}{8878415}\frac{\mathcal{L}^2}{r_0^2} - \frac{20571064}{26635245}U\right]\delta_{\mathcal{E}} - \frac{5579893}{1775683}\frac{\mathcal{L}^4}{r_0^4} - \frac{59116}{253669}\frac{\mathcal{L}^2 U}{r_0^2} - \frac{18112}{10507}U^2,$$

$$\begin{aligned}C_t^{\tilde{S}-S(3)} = & -\frac{115291414894}{269415503175}\delta_{\mathcal{E}}^3 + \left[\frac{43471970326}{17961033545}\frac{\mathcal{L}^2}{r_0^2} - \frac{48448379368}{89805167725}U\right]\delta_{\mathcal{E}}^2 \\ & + \left[-\frac{22584903396}{2565861935}\frac{\mathcal{L}^4}{r_0^4} - \frac{508295808}{2565861935}\frac{\mathcal{L}^2 U}{r_0^2} - \left(\frac{244692415685084}{8336485426815} + \frac{21\pi^2}{32}\right)U^2\right]\delta_{\mathcal{E}} \\ & + \frac{16156048}{1301367}\frac{\mathcal{L}^6}{r_0^6} - \frac{291581166}{32479265}\frac{\mathcal{L}^4 U}{r_0^4} + \left(\frac{122513312775814}{1667297085363} + \frac{105\pi^2}{64}\right)\frac{\mathcal{L}^2 U^2}{r_0^2} \\ & - \left(\frac{170795214491529 4}{25009456280445} + \frac{7\pi^2}{4}\right)U^3,\end{aligned}$$

$$C_\varphi^{\tilde{S}-S(0)} = \frac{960}{10507},$$

§ 5 Simple case (Schwarzschild + Scalar + Circular)

We use the slow motion approximation. So it is necessary to answer how fast the convergence of the PN expansion is. We investigate the Simplest case (Sch.+Scalar+Circular).

- Orbit

$$z^\alpha(\tau) = (u^t\tau, r_0, \frac{\pi}{2}, u^\varphi\tau); \quad u^t = \sqrt{\frac{r_0}{r_0 - 3M}}, \ u^\varphi = \frac{1}{r_0}\sqrt{\frac{M}{r_0 - 3M}}.$$

- Result of the $(\tilde{S} - S)$-part

$$F_r^{\tilde{S}-S} = \frac{q^2}{4\pi r_0^2}\Big[-\frac{73}{133} + \frac{16151}{21014}V^2 + \frac{395567}{106808}V^4 + \Big(\frac{1107284037660637}{400151300487120} + \frac{7}{64}\pi^2\Big)V^6 + \Big(-\frac{18211898191137768997827 1}{854863070735138617152 0} + \frac{29\pi^2}{1024}\Big)V^8\Big] + \cdots$$

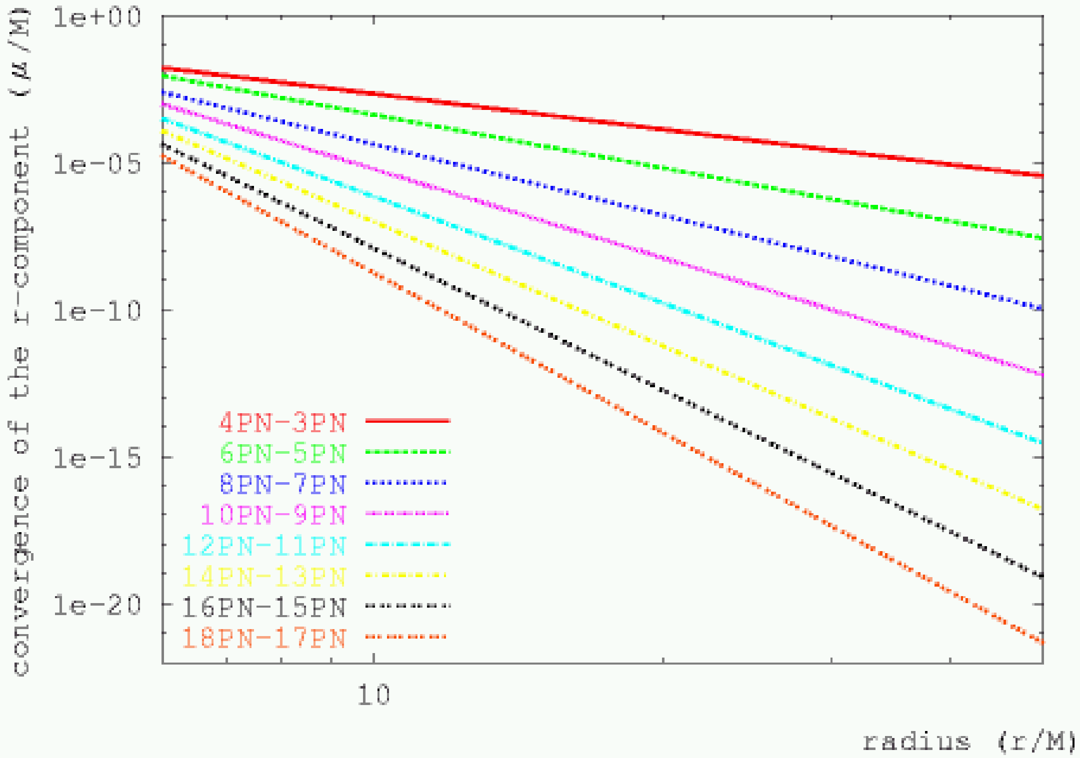
where $V = \sqrt{M/r_0}$ and we show our results up to the 4PN order.

§ 5 Simple case (Schwarzschild + Scalar + Circular)

Convergence test

$$(\text{convergence}) = \left| \frac{F_r^{\tilde{S}-S}|_n - F_r^{\tilde{S}-S}|_{n-1}}{F_r^{\text{Newton}} + F_r^{\tilde{S}-S} + F_r^{R}} \right| \approx \frac{\left| F_r^{\tilde{S}-S}|_n - F_r^{\tilde{S}-S}|_{n-1} \right|}{M\mu/r_0^2}$$

where n indicates the order of the PN expansion.



- **We can see that the convergence is good in the far zone**
- **In the ISCO, the convergence is very slow.**

$\tilde{R}$-part

In circular case, the spectrum is discrete. So it is possible to calculate the $\tilde{R}$-part of the self-force analytically.

$$F_r^{\tilde{R}} = \frac{q^2}{4\pi r_0^2}\Big[\frac{73}{133} - \frac{16151}{21014}V^2 - \frac{395567}{106808}V^4 + \left(-\frac{4}{3}\gamma - \frac{4}{3}\ln(2V) - \frac{1196206548879997}{400151300487120}\right)V^6 + \left(\frac{5937212059223214798 4979}{1709726141470277234304} - \frac{14}{3}\ln(V) - \frac{66}{5}\ln(2) - \frac{14\gamma}{3}\right)V^8 + O(v^9)\Big]$$

Here, $V = \sqrt{\dfrac{M}{r_0}}$, γ: the Euler's constant

Self-force in case of a circular orbit:

$$F_r^{R} = \frac{q^2}{4\pi r_0^2}\Big[\left(-\frac{4}{3}\gamma + \frac{7}{64}\pi^2 - \frac{4}{3}\ln(2V) - \frac{2}{9}\right)V^6 + \left(\frac{604}{45} + \frac{29\pi^2}{1024} - \frac{66}{5}\ln(2) - \frac{14}{3}\ln(V) - \frac{14}{3}\gamma\right)V^8\Big] + O(v^9)$$

§ 5 Simple case (Schwarzschild + Scalar + Circular)

Comparison with previous results

The regularized self-force obtained by Detweiler et al. is

$$F_r^R = 1.37844828(2) \times 10^{-5} \quad (r_0 = 10M).$$

The most accurate self-force in our calculation is

$$F_r^R = 1.378448203 \times 10^{-5} \quad (r_0 = 10M).$$

coincidence at the accuracy 10^{-8}!!

- **Table: the r-component of the self-force**

PN order	$F_r^{\mathrm{R}}(r_0 = 6M)$	$F_r^{\mathrm{R}}(r_0 = 10M)$	$F_r^{\mathrm{R}}(r_0 = 20M)$
4	$-3.698897009 \times 10^{-4}$	$5.438965544 \times 10^{-8}$	$4.009204942 \times 10^{-7}$
6	$3.900997486 \times 10^{-5}$	$1.215734502 \times 10^{-5}$	$4.900744665 \times 10^{-7}$
8	$1.469034988 \times 10^{-4}$	$1.370724270 \times 10^{-5}$	$4.937547086 \times 10^{-7}$
10	$1.634644402 \times 10^{-4}$	$1.377874928 \times 10^{-5}$	$4.937898906 \times 10^{-7}$
12	$1.665705633 \times 10^{-4}$	$1.378392510 \times 10^{-5}$	$4.937905702 \times 10^{-7}$
14	$1.674516681 \times 10^{-4}$	$1.378443247 \times 10^{-5}$	$4.937905862 \times 10^{-7}$
16	$1.676513985 \times 10^{-4}$	$1.378447488 \times 10^{-5}$	$4.937905865 \times 10^{-7}$
18	$1.677456783 \times 10^{-4}$	$1.378448203 \times 10^{-5}$	$4.937905865 \times 10^{-7}$

§ 5 Simple case (Schwarzschild + Scalar + Circular)

The error of cycle of the GW

We can roughly estimate the cycle of the gravitational wave by regarding scalar charge as mass of the particle.

Result in the mass ratio = 10^6 case

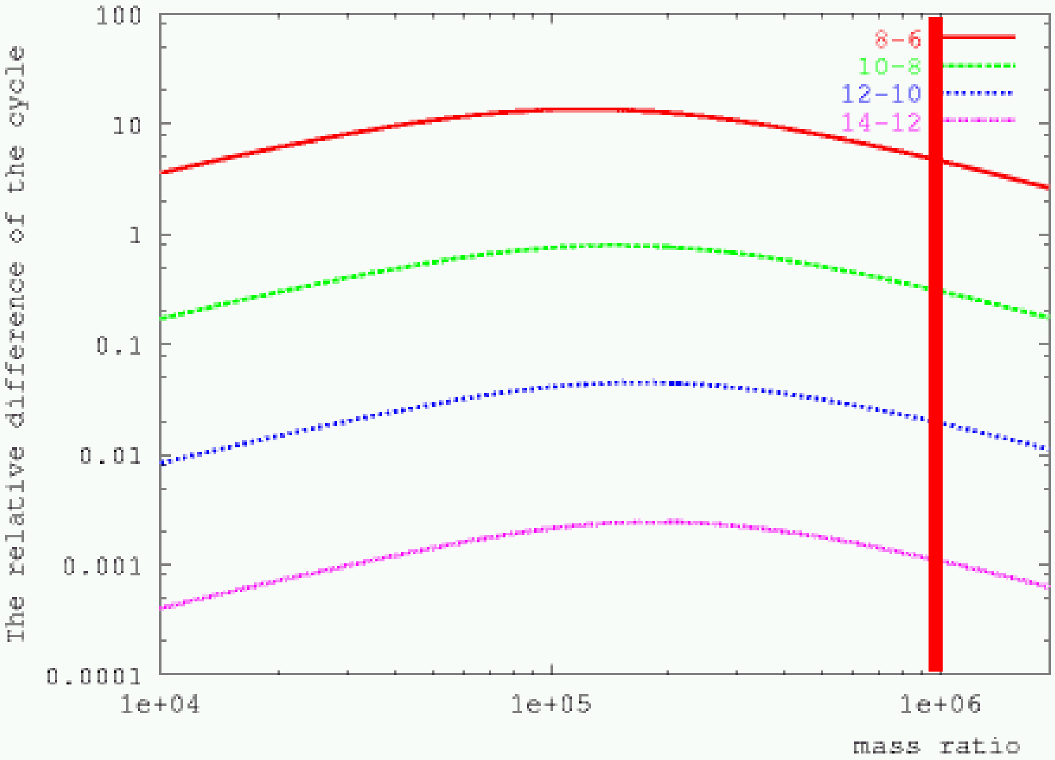
$$N = 2\frac{\Delta\phi}{2\pi} = \frac{1}{\pi}\int_{r_i}^{r_f} \Omega \frac{dE/dr_0}{dE/dt} dr_0 = \frac{1}{\pi}\int_{r_i}^{r_f} \Omega \frac{dE/dr_0}{F_t^R} dr_0$$

Order	$(6.2M, 6M)$	$(10M, 9.995M)$	$(20M, 19.9991M)$
$N^{(6)}$	1.345475356×10^5	6.881110147×10^4	2.265486877×10^4
$N^{(8)}$	1.345382272×10^5	6.881094334×10^4	2.265486831×10^4
$N^{(10)}$	1.345388366×10^5	6.881094827×10^4	2.265486831×10^4
$N^{(12)}$	1.345387979×10^5	6.881094812×10^4	2.265486831×10^4
$N^{(14)}$	1.345388001×10^5	6.881094812×10^4	2.265486831×10^4
$N^{(16)}$	1.345388000×10^5	6.881094812×10^4	2.265486831×10^4

- The n-th order accuracy means that they are expanded in terms of ω to O(ω^n).
- We choose each set of (r_i, r_f) such that the time interval during the particle moves is nearly equal 1 year.

§ 5 Simple case (Schwarzschild + Scalar + Circular)

$\tilde{R}$-part



Cycle error vs Mass ratio

- We choose each set of (r_i, 6M) such that the time interval during the particle moves is nearly equal 1 year.

§ 6 Summary & Conclusion

We propose the new analytical method for Self-Force regularization. ($\tilde{S}-\tilde{R}$ decomposition)

We calculate the $(\tilde{S}-S)$-part of the self-force for a general orbit in the Sch.+Scalar case.

In order to investigate the convergence of the PN expansion, we calculate the self-force in the Sch.+Scalar+Circular case.

For a circular orbit, we calculate the $\tilde{R}$-part of the self-force analytically. But it is easy to calculate numerically. So this method isapplicable to general orbital case.

In the future, we want to calculate in the grav. and Kerr case.