Regularization parameters for the self-force of a moving particle and coordinate transformations

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Overview

1. What is self-force?

2. Review of general formal schemes for calculating self-force

- Dirac (1938), Dewitt and Brehme (1960), Mino, Sasaki and Tanaka (1997), Quinn and Wald (1997), Quinn (2000).
- Difficulty with these schemes.
- Analytic implementation: Dewitt and Dewitt (1964), Pfenning and Poisson (2001).

3. Recent approach to evaluation of self-force: Barack and Ori (2000, 2002, 2003), Burko (2000), Barack and Burko (2000)

- Subtraction method (half analytical and half numerical).
- Spherical harmonic decomposition.
- Mode-sum regularization.

4. My approach to self-force of a scalar field in Schwarzschild spacetime (I): Detweiler and Whiting (2002), Detweiler, Messaritaki, and Whiting (2003), Kim (2004)

- Subtraction method*, spherical harmonic decomposition, mode-sum regularization.
- Elementary implementation of calculating dominant "regularization parameters" - A and B terms.
- Extending to self-forces of an electromagnetic vector field and a gravitational tensor field .
- Extending to other cases of spacetime.

5. My approach to self-force of a scalar field in Schwarzschild spacetime (II): Kim (2004), Detweiler and Kim (in progress)

- Structural analyses of C and D terms.
- Analysis of geometry.
- Implentation of calculating D terms (in progress).
- Why interesting?

1. What is self-force?

- The self-force is the interaction of a particle with the perturbation of the spacetime geometry created by the particle itself.

- It changes the worldline of the particle.

- e.g. Self-force of a scalar field:

$$F^a_{\rm self} = q\nabla^a\psi,$$

q: charge of a particle,

ψ: field that results from the particle itself but does not include the ***"singular"*** part.

2. Review of general formal schemes for calculating self-force

• Dirac (1938): an electron moving in ***flat spacetime***

1) $m\dot{v}^a = ev^b F^a{}_{b\,\mathrm{in}} + \frac{2}{3}e^2\left(\ddot{v}^a - \dot{v}^2 v^a\right).$

2) $F^{ab}_{\mathrm{ret}} = \frac{1}{2}\left(F^{ab}_{\mathrm{ret}} + F^{ab}_{\mathrm{adv}}\right)_{(i)} + \frac{1}{2}\left(F^{ab}_{\mathrm{ret}} - F^{ab}_{\mathrm{adv}}\right)_{(ii)}$, where $F^{ab} = \nabla^a A^b - \nabla^b A^a$,

- (i) solution of $\eta^{bc}\partial_b\partial_c A_a = 4\pi j_a \Rightarrow$ ***singular field*** $\sim$ ***Coulomb*** q/r ***piece*** near particle.
- (ii) soultion of $\eta^{bc}\partial_b\partial_c A_a = 0 \Rightarrow$ ***radiation field*** $\sim$ responsible for ***ALD force***.

3) ***No self-force*** on the particle in the absence of external fields ($\leftarrow$ ***physical*** solution $\dot{v} = 0$ when $F^{ab}_{\mathrm{in}} = 0$).

• Dewitt and Brehme (1960): Dirac's problem extended to ***curved spacetime***

1) $$m\dot{v}^a = ev^b F^a{}_{b\,\mathrm{in}} + \frac{2}{3}e^2(\ddot{v}^a - \dot{v}^2 v^a) + \frac{1}{3}e^2(R^a{}_b v^b + v^a R_{bc} v^b v^c)$$
$$+\lim_{\epsilon\to 0} e^2 v_b \int_{-\infty}^{\tau-\epsilon} \nabla^{[b} G^{\mathrm{ret}\,a]}{}_{c'}[p(\tau), p'(\tau')] v^{c'}(\tau') d\tau'.$$

2) ***Self-force exists***; ***radiation damping*** occurs even for a particle in ***free fall***.

• Quinn (2000): a ***scalar*** point particle in ***curved spacetime***

1) $$m\dot{v}^a = q\nabla^a \phi_{\mathrm{in}} + \frac{1}{3}q^2(\ddot{v}^a - \dot{v}^2 v^a) + \frac{1}{6}q^2(R^a{}_b v^b + v^a R_{bc} v^b v^c)$$
$$-\frac{1}{12}q^2 R v^a + \lim_{\epsilon\to 0} q^2 \int_{-\infty}^{\tau-\epsilon} \nabla^a G^{\mathrm{ret}}[p(\tau), p'(\tau')] d\tau'.$$

2) ***Self-force exists***; ***radiation reaction*** occurs even for a particle in ***free fall***.

• **Difficulty with these schems:** ***extremely difficult*** to determine the ***retarded Green's functions*** precisely for ***general curved spacetime***.

• **Analytic implementation:** ***direct computation*** of the ***retarded Green's functions***

1) Dewitt and Dewitt (1964); a charged particle falling freely with the nonrelativistic small velocities in a static weak gravitational field.

2) Pfenning and Poisson (2001); self-forces of scalar, electromagnetic, and gravitational in the weakly curved spacetime.

3. Recent approach to evaluation of self-force (for Schwarzshild spacetime): Barack and Ori (2000, 2002, 2003), Burko (2000), Barack and Burko (2000)

- Self-force is obtained by ***subtraction***

$$\mathcal{F}_a^{\mathrm{self}} = \lim_{p \to p'} \left[\mathcal{F}_a^{\mathrm{ret}}(p) - \mathcal{F}_a^{\mathrm{dir}}(p') \right] .$$

- Use ***spherical harmonic decompositions***

$$\psi^{\mathrm{ret/dir}} = \sum_{\ell m} \psi^{\mathrm{ret/dir}}(r,\, t) Y_{\ell m}(\theta,\, \phi),$$

where $\psi^{\mathrm{ret}}(r,\, t)$ $(\to \mathcal{F}_{\ell\, a}^{\mathrm{ret}})$ is determined ***numerically*** and $\psi^{\mathrm{dir}}(r,\, t)$ $\left(\to \mathcal{F}_{\ell\, a}^{\mathrm{dir}}\right)$ ***analytically***.

- Self-force is finally computed by ***mode-sum regularization***

$$\begin{aligned}
\mathcal{F}_a^{\mathrm{self}} &= \lim_{p \to p'} \nabla_a \left\{ \sum_{\ell m} \left[\psi^{\mathrm{ret}}(r,\, t) - \psi^{\mathrm{dir}}(r,\, t) \right] Y_{\ell m}(\theta,\, \phi) \right\} \\
&= \sum_{\ell=0}^{\infty} \left(\lim_{p \to p'} \mathcal{F}_{\ell\, a}^{\mathrm{ret}} - \lim_{p \to p'} \mathcal{F}_{\ell\, a}^{\mathrm{dir}} \right) \\
&= \sum_{\ell=0}^{\infty} \left[\lim_{p \to p'} \mathcal{F}_{\ell\, a}^{\mathrm{ret}} - \left(\ell + \frac{1}{2} \right) A_a - B_a - \frac{C_a}{\ell + \frac{1}{2}} \right] - D_a,
\end{aligned}$$

where A_a, B_a, C_a, D_a are termed ***"regularization parameters"***.

- Implemented self-forces of a ***scalar*** field for ***radial*** and for ***circular*** orbits of ***Schwarzschild*** .

4. My approach to self-force of a scalar field in Schwarzschild spacetime (I): Detweiler and Whiting (2002), Detweiler, Messaritaki and Whiting (2003), Kim (2004)

• Similar method of implementation to Barack and Ori's, except that for ***subtraction*** we use

$$\mathcal{F}_a^{\rm self} = \lim_{p\to p'} \left[\mathcal{F}_a^{\rm ret}(p) - \mathcal{F}_a^{\rm S}(p')\right] \quad \text{(S: } \textbf{Singular Source}\text{)}.$$

• Elementary implementation of calculating A_a and B_a terms

1) Describe the ***singular*** field in ***co-moving normal*** coordinates

$$\psi^{\rm S} \approx \frac{q}{\rho} \quad \text{with} \quad \rho = \sqrt{X^2+Y^2+Z^2}$$

$\Longleftarrow \rho^2$ can be obtained as follows;

$$X^{A'} = \Lambda^{A'}{}_A \left[B^A{}_a(x^a - x^a_{\rm o}) + \frac{1}{2} B^A{}_a\, \Gamma^a_{bc}|_{\rm o}\, (x^b - x^b_{\rm o})(x^c - x^c_{\rm o})\right] + O[(x-x_{\rm o})^3],$$

$$\left(\to\ g^{A'B'} = \quad \eta^{A'B'} \quad + \quad 0 \quad + \quad \frac{1}{2}\frac{\partial^2 g^{A'B'}}{\partial X^{C'}\partial X^{D'}} X^{C'} X^{D'}\right),$$

with

$$B^A{}_a = \begin{bmatrix} f^{1/2} & & & \\ & f^{-1/2} & & \\ & & r_{\rm o} & \\ & & & -r_{\rm o} \end{bmatrix} \quad \left(f = 1 - \frac{2M}{r_{\rm o}}\right) \text{ and}$$

$$\Lambda^{A'}{}_A = \begin{bmatrix} f^{-1/2}E & -f^{-1/2}\dot r & -J/r_{\rm o} & 0 \\ & 1 + \dot r^2/(f^{1/2}E + f) & J\dot r/[r_{\rm o}(E+f^{1/2})] & 0 \\ & \textbf{SYM} & 1 + J^2/[r_{\rm o}^2(f^{-1/2}E+1)] & 0 \\ & & & 1 \end{bmatrix},$$

then

$$
\begin{aligned}
\rho^2 &= X^2+Y^2+Z^2 \\
&= (E^2-f)(t-t_{\rm o})^2 - \frac{2E\dot r}{f}(t-t_{\rm o})(r-r_{\rm o}) - 2EJ(t-t_{\rm o})(\phi-\phi_{\rm o}) \\
&\quad +\frac{1}{f}\left(1+\frac{\dot r^2}{f}\right)(r-r_{\rm o})^2 + \frac{2J\dot r}{f}(r-r_{\rm o})(\phi-\phi_{\rm o}) + (r_{\rm o}^2+J^2)(\phi-\phi_{\rm o})^2 + r_{\rm o}^2\left(\theta-\frac{\pi}{2}\right)^2 \\
&\quad -\frac{ME\dot r}{r_{\rm o}^2}(t-t_{\rm o})^3 + \frac{M}{r_{\rm o}^2}\left(-1+\frac{2E^2}{f}+\frac{\dot r^2}{f}\right)(t-t_{\rm o})^2(r-r_{\rm o}) + \frac{MJ\dot r}{r_{\rm o}^2}(t-t_{\rm o})]^2(\phi-\phi_{\rm o}) \\
&\quad -\frac{ME\dot r}{f^2 r_{\rm o}^2}(t-t_{\rm o})(r-r_{\rm o})^2 - \frac{2(r_{\rm o}-M)EJ}{f r_{\rm o}}(t-t_{\rm o})(r-r_{\rm o})(\phi-\phi_{\rm o}) \\
&\quad + r_{\rm o}E\dot r(t-t_{\rm o})(\phi-\phi_{\rm o})^2 + r_{\rm o}E\dot r(t-t_{\rm o})\left(\theta-\frac{\pi}{2}\right)^2 \\
&\quad -\frac{M}{f^2 r_{\rm o}^2}\left(1+\frac{\dot r^2}{f}\right)(r-r_{\rm o})^3 + \frac{(2r_{\rm o}-5M)J\dot r}{f^2 r_{\rm o}^2}(r-r_{\rm o})^2(\phi-\phi_{\rm o}) \\
&\quad + r_{\rm o}\left(1-\frac{\dot r^2}{f}+\frac{2J^2}{r_{\rm o}^2}\right)(r-r_{\rm o})(\phi-\phi_{\rm o})^2 + r_{\rm o}\left(1-\frac{\dot r^2}{f}\right)(r-r_{\rm o})\left(\theta-\frac{\pi}{2}\right)^2 \\
&\quad - r_{\rm o}J\dot r(\phi-\phi_{\rm o})^3 - r_{\rm o}J\dot r(\phi-\phi_{\rm o})\left(\theta-\frac{\pi}{2}\right)^2 + O[(x-x_{\rm o})^4].
\end{aligned}
$$

2) ***Rotate*** the Schwarzschild coordinates

$$
\begin{aligned}
\sin\theta\cos(\phi-\phi') &= \cos\Theta \\
\sin\theta\sin(\phi-\phi') &= \sin\Theta\cos\Phi \\
\cos\theta &= \sin\Theta\sin\Phi,
\end{aligned}
$$

where $\phi' \equiv \phi_{\rm o} - \frac{J\dot r}{f(r_{\rm o}^2+J^2)}(r-r_{\rm o})$, such that

$$
Y_{\ell m}(\theta,\,\phi) = \sum_{m'=-\ell}^{\ell} \alpha^{\ell}_{mm'} Y_{\ell m'}(\Theta,\,\Phi) \;\rightarrow\; \text{only } Y_{\ell 0}(\Theta,\,\Phi) \neq 0 \text{ in } p \rightarrow p'.
$$

3) Based on rotated coordinates, express

$$
\rho^2 = \epsilon^2\tilde\rho^2 + \epsilon^3\mathcal{B} + O(\epsilon^4) \quad (\epsilon = \text{order parameter}),
$$

then

$$
\partial_a\left(\frac{1}{\rho}\right)\bigg|_{t=t_{\rm o}} = -\frac{1}{2}\frac{\partial_a\left(\tilde\rho^2\right)\big|_{t=t_{\rm o}}}{\tilde\rho_{\rm o}^3}\epsilon^{-2} + \left\{-\frac{1}{2}\frac{\partial_a\mathcal{B}\big|_{t=t_{\rm o}}}{\tilde\rho_{\rm o}^3} + \frac{3}{4}\frac{\left[\partial_a\left(\tilde\rho^2\right)\right]\mathcal{B}\big|_{t=t_{\rm o}}}{\tilde\rho_{\rm o}^5}\right\}\epsilon^{-1} + O(\epsilon^0),
$$

where in particular, $\tilde\rho_{\rm o}^2 \equiv \tilde\rho^2\big|_{t=t_{\rm o}} = 2\left(r_{\rm o}^2+J^2\right)\chi\left(\delta^2+1-\cos\Theta\right)$

with $\chi \equiv 1 - J^2\sin^2\Phi/\left(r_{\rm o}^2+J^2\right)$ and $\delta^2 \equiv r_{\rm o}^2E^2(r-r_{\rm o})^2/2f^2\left(r_{\rm o}^2+J^2\right)^2\chi$.

4) Determining the $\ell-$***mode expansion*** of $q\,\partial_a\,(1/\rho)|_{t=t_{\rm o}}$ in the ***coincidence limit***

$$\lim_{p\to p'} \mathcal{F}^{\rm S}_{\ell\,a} = \left(\ell+\frac{1}{2}\right) A_a + B_a + O(\ell^{-1}),$$

where

$$A_t = {\rm sgn}(r-r_{\rm o})\frac{q^2}{r_{\rm o}^2}\frac{\dot r}{1+J^2/r_{\rm o}^2},\;\; A_r = -{\rm sgn}(r-r_{\rm o})\frac{q^2}{r_{\rm o}^2}\frac{E(1-\frac{2M}{r_{\rm o}})^{-1}}{(1+J^2/r_{\rm o}^2)},\;\; A_\phi = 0,$$

$$B_t = \frac{q^2}{r_{\rm o}^2}E\dot r\left[\frac{F_{3/2}}{(1+J^2/r_{\rm o}^2)^{3/2}} - \frac{3F_{5/2}}{2(1+J^2/r_{\rm o}^2)^{5/2}}\right],$$

$$B_r = \frac{q^2}{r_{\rm o}^2}\left\{-\frac{F_{1/2}}{(1+J^2/r_{\rm o}^2)^{1/2}} + \frac{[1-2(1-\frac{2M}{r_{\rm o}})^{-1}\dot r^2]F_{3/2}}{2(1+J^2/r_{\rm o}^2)^{3/2}} + \frac{3(1-\frac{2M}{r_{\rm o}})^{-1}\dot r^2 F_{5/2}}{2(1+J^2/r_{\rm o}^2)^{5/2}}\right\},$$

$$B_\phi = \frac{q^2}{J}\dot r\left[\frac{F_{1/2}-F_{3/2}}{(1+J^2/r_{\rm o}^2)^{1/2}} + \frac{3(F_{5/2}-F_{3/2})}{2(1+J^2/r_{\rm o}^2)^{3/2}}\right],$$

$$A_\theta = B_\theta = 0,$$

where E = conserved energy, J = conserved angular momentum, $\dot r \equiv u^r$, $F_p \equiv {}_2F_1\left(p,\,\frac{1}{2}\,;1;\frac{J^2}{r_{\rm o}^2+J^2}\right)$.

→ The results agree with Barack and Ori (2002) and Mino, Nakano, and Sasaki (2002).

• **Extending to self-forces of an electromagnetic vector field and a gravitational tensor field:** use ***analyses of Green's functions*** from Dewitt and Brehme (1960), Mino, Sasaki and Tanaka (1997), Detweiler and Whiting (2002)

1) Scalar field;

$$\mathcal{F}_a^{\mathrm{S}(\mathbf{sc})} = q\nabla_a \left(\frac{1}{\rho}\right),$$

2) Electromagnetic field;

$$\mathcal{F}_a^{\mathrm{S}(\mathbf{em})} = u^b(\nabla_a A_b^{\mathrm{S}} - \nabla_b A_a^{\mathrm{S}}) = -P^b{}_a \mathcal{F}_b^{\mathrm{S}(\mathbf{sc})},$$

3) Gravitational field;

$$\mathcal{F}_a^{\mathrm{S}(\mathbf{gr})} = -\mu(\delta^b{}_a + u^b u_a)u^c u^d \left(\nabla_c h_{db}^{\mathrm{S}} - \nabla_b h_{cd}^{\mathrm{S}}\right) = P^b{}_a \mathcal{F}_b^{\mathrm{S}(\mathbf{sc})},$$

where $P^b{}_a = \delta^b{}_a + u^b u_a$.

$\Longrightarrow$ Their regularization parameters are ***"proportional to each other"***.

4) Find $\mathcal{F}_a^{\mathrm{ret}(\mathbf{em})}$ and $\mathcal{F}_a^{\mathrm{ret}(\mathbf{gr})}$ using ***vector and tensor spherical harmonics***, respectively. Then, by ***subtraction*** and ***mode-sum regularization***, determine their self-forces finally.

• **Extending to other cases of spacetime (e.g. Kerr spacetime):** find ways of ***diagonalizing*** the background geometry locally to obtain the form

$$X^{A'} = \Lambda^{A'}{}_A \left[B^A{}_a(y^a - y_{\mathrm{o}}^a) + \frac{1}{2}B^A{}_a\, \Gamma^a_{bc}|_{\mathrm{o}}\, (y^b - y_{\mathrm{o}}^b)(y^c - y_{\mathrm{o}}^c)\right] + O[(y - y_{\mathrm{o}})^3],$$

where

$$(y^a - y_{\mathrm{o}}^a) = \text{diagonalization of } (x^a - x_{\mathrm{o}}^a).$$

Then, the rest process is the same as for the Schwarzschild case.

5. My approach to self-force of a scalar field in Schwarzschild spacetime (II): in progress

- ***Structural analyses*** of C_a and D_a terms

1) In a *"particular"* normal coordinate system,

$$\lim_{p\to p'} \partial_a \psi^{\rm S} \text{ "=" } q \lim_{p\to p'} \partial_a \left(\frac{1}{\rho}\right),$$

and we may express to higher orders

$$\rho^2 = \epsilon^2 \tilde{\rho}^2 + \epsilon^3 \mathcal{B} + \epsilon^4 \mathcal{C} + \epsilon^5 \mathcal{D} + O(\epsilon^6),$$

such that

$$\begin{aligned}\partial_a \left(\frac{1}{\rho}\right)\Big|_{t=t_{\rm o}} &= -\frac{1}{2}\frac{\partial_a\left(\tilde{\rho}^2\right)\big|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^3}\epsilon^{-2} + \left\{-\frac{1}{2}\frac{\partial_a \mathcal{B}|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^3} + \frac{3}{4}\frac{\left[\partial_a\left(\tilde{\rho}^2\right)\right]\mathcal{B}\big|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^5}\right\}\epsilon^{-1} \\ &\quad + Q_a^{(0)}\epsilon^0 + Q_a^{(1)}\epsilon^1 + O(\epsilon^2),\end{aligned}$$

where

$$\begin{aligned}Q_a^{(0)} &\equiv -\frac{1}{2}\frac{\partial_a \mathcal{C}|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^3} + \frac{3}{4}\frac{\left(\partial_a\mathcal{B}\right)\mathcal{B}|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^5} \\ &\quad + \frac{3}{4}\frac{\left[\partial_a\left(\tilde{\rho}^2\right)\right]\mathcal{C}\big|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^5} - \frac{15}{16}\frac{\left[\partial_a\left(\tilde{\rho}^2\right)\right]\mathcal{B}^2\big|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^7}\end{aligned}$$

and

$$\begin{aligned}Q_a^{(1)} &\equiv -\frac{1}{2}\frac{\partial_a \mathcal{D}|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^3} + \frac{3}{4}\frac{\left\{\left[\partial_a\left(\tilde{\rho}^2\right)\right]\mathcal{D}\big|_{t=t_{\rm o}} + \left(\partial_a\mathcal{B}\right)\mathcal{C}|_{t=t_{\rm o}} + \left(\partial_a\mathcal{C}\right)\mathcal{B}|_{t=t_{\rm o}}\right\}}{\tilde{\rho}_{\rm o}^5} \\ &\quad - \frac{15}{16}\frac{\left\{2\left[\partial_a\left(\tilde{\rho}^2\right)\right]\mathcal{B}\mathcal{C}\big|_{t=t_{\rm o}} + \left(\partial_a\mathcal{B}\right)\mathcal{B}^2\big|_{t=t_{\rm o}}\right\}}{\tilde{\rho}_{\rm o}^7} + \frac{35}{32}\frac{\left[\partial_a\left(\tilde{\rho}^2\right)\right]\mathcal{B}^3\big|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^9}.\end{aligned}$$

Then,

$$\lim_{p\to p'} \mathcal{F}^{\rm S}_{\ell\, a} = \left(\ell + \frac{1}{2}\right) A_a + B_a + \frac{C_a}{(\ell+\frac{1}{2})} - \frac{2\sqrt{2}D_a}{(2\ell-1)(2\ell+3)} + O(\ell^{-4}).$$

2) C_a terms

$$
\begin{aligned}
Q_a^{(0)} &= \sum_{n=1}^{3}\sum_{k=0}^{2n+1}\sum_{p=0}^{[k/2]} \frac{c_{kp(a)}\Delta^{2n+1-k}\left(\phi-\phi_{\rm o}\right)^{k-2p}\left(\theta-\frac{\pi}{2}\right)^{2p}}{\tilde{\rho}_{\rm o}^{2n+1}}, \\
&\sim \sum_{n=1}^{3}\sum_{k,p,i} c_{kpi(a)}\tilde{\rho}_{\rm o}^{-(2n+1)}\Delta^{2n+1-2p-i}\left(\sin\Theta\right)^{2p+i}\left(\sin\Phi\right)^{2p}\left(\cos\Phi\right)^{i}
\end{aligned}
$$

where $\Delta = (r - r_{\rm o})$ and $(\phi - \phi') \equiv (\phi - \phi_{\rm o}) - \frac{J\dot{r}}{f(r_{\rm o}^2+J^2)}\Delta$. Then,

(i) $i =$ odd integer; $\left\langle Q_a^{(0)}\right\rangle_{\Phi} = 0$ ($\langle\ \rangle_{\Phi}$ =average by integration over Φ),

(ii) $i =$ even integer; $Q_a^{(0)} \underset{\Delta\to 0}{\sim} \Delta$ or $\Delta^2 \longrightarrow 0.$

Hence, C_a terms ***always vanish***.

3) D_a terms

$$
\begin{aligned}
Q_a^{(1)} &= \sum_{n=1}^{4}\sum_{k=0}^{2n+2}\sum_{p=0}^{[k/2]} \frac{d_{kp(a)}\Delta^{2n+2-k}\left(\phi-\phi_{\rm o}\right)^{k-2p}\left(\theta-\frac{\pi}{2}\right)^{2p}}{\tilde{\rho}_{\rm o}^{2n+1}} \\
&\sim \sum_{n=1}^{4}\sum_{k,p,i} d_{kpi(a)}\tilde{\rho}_{\rm o}^{-(2n+1)}\Delta^{q+k-2p-i}\left(\sin\Theta\right)^{2p+i}\left(\sin\Phi\right)^{2p}\left(\cos\Phi\right)^{i}
\end{aligned}
$$

where $2n + 2 - k \equiv q$. Then,

(i) $q > 0$; ***always vanishing*** since

(i-1) $i =$ odd integer; $\left\langle Q_a^{(1)}\right\rangle_{\Phi} = 0,$

(i-2) $i =$ even integer; $Q_a^{(1)} \underset{\Delta\to 0}{\sim} \Delta^2$ or $\Delta^3 \longrightarrow 0,$

(ii) $q = 0$; ***non-vanishing*** only as

$$
\begin{aligned}
Q_a^{(1)} &= \sum_{n=1}^{4}\sum_{p=0}^{n+1} \frac{e_{np(a)}\left(\phi-\phi_{\rm o}\right)^{2(n-p)+2}\left(\theta-\frac{\pi}{2}\right)^{2p}}{\tilde{\rho}_{\rm o}^{2n+1}} \\
&\underset{\Delta\to 0}{\sim} \tilde{\rho}_{\rm o}\sum_{n=1}^{4}\sum_{p=0}^{n+1} e_{np(a)}\left(\sin\Phi\right)^{2p}\left(\cos\Phi\right)^{2(n-p)+2} \Rightarrow \textbf{provides } D_a \textbf{ terms}.
\end{aligned}
$$

• **Analysis of geometry:** ***more precise description of*** ρ^2 and hence ***more precise prescription of normal coordinates*** are required to compute D_a terms.

1) ***Instant initial normal coordinates*****:** Weinberg (1972)

$$\hat{X}^A = B^A{}_a(x^a - x^a_{\rm o}) + \frac{1}{2}B^A{}_a\,\Gamma^a_{bc}\big|_{\rm o}\,(x^b - x^b_{\rm o})(x^c - x^c_{\rm o}) + O\left((x - x_{\rm o})^3\right).$$

2) ***Fermi normal coordinates***

$$T_{\bf FN} = -\hat{u}_{{\rm o}A}\hat{X}^A - \hat{u}_{{\rm o}A}\left[\xi^A{}_{BCD}\hat{X}^B\hat{X}^C\hat{X}^D + \zeta^A{}_{BCDE}\hat{X}^B\hat{X}^C\hat{X}^D\hat{X}^E + O(\hat{X}^5)\right],$$

$$X^I_{\bf FN} = \hat{n}^I_{{\rm o}A}\hat{X}^A + \hat{n}^I_{{\rm o}A}\left[\xi^A{}_{BCD}\hat{X}^B\hat{X}^C\hat{X}^D + \zeta^A{}_{BCDE}\hat{X}^B\hat{X}^C\hat{X}^D\hat{X}^E + O(\hat{X}^5)\right],$$

where

$$\xi^A{}_{BCD} = \frac{1}{6}\,\hat{\Gamma}^A_{PQ,R}\Big|_{\rm o}\left(\pi^P{}_B\pi^Q{}_C\pi^R{}_D + 3\pi^Q{}_B\pi^R{}_C h^P{}_D + 3\pi^R{}_B h^P{}_C h^Q{}_D + h^P{}_B h^Q{}_C h^R{}_D\right),$$

$$\begin{aligned}\zeta^A{}_{BCDE} &= \frac{1}{24}\,\hat{\Gamma}^A_{PQ,RS}\Big|_{\rm o}\left(\pi^P{}_B\pi^Q{}_C\pi^R{}_D\pi^S{}_E + 4\pi^Q{}_B\pi^R{}_C\pi^S{}_D h^P{}_E + 6\pi^R{}_B\pi^S{}_C h^P{}_D h^Q{}_E\right.\\ &\quad \left.+\,4\pi^R{}_B h^P{}_C h^Q{}_D h^R{}_E + h^P{}_B h^Q{}_C h^R{}_D h^S{}_E\right),\end{aligned}$$

with $\pi^A{}_B \equiv -\hat{u}^A_{\rm o}\hat{u}_{{\rm o}B}$(time-projection tensor) and $h^A{}_B = \delta^A{}_B + \hat{u}^A_{\rm o}\hat{u}_{{\rm o}B}$(space-projection tensor) ($\leftarrow \hat{u}^A_{\rm o} = (f^{-1/2}E,\ f^{-1/2}\dot{r},\ J/r_{\rm o},\ 0)$ for geodesic in equatorial plane).

3) ***Thorne-Hartle-Zhang normal coordinates*****: *de Donder gauge*** $\partial_A\mathfrak{g}^{AB} = 0$ ($\mathfrak{g}^{AB} \equiv \sqrt{-g}g^{AB}$) is satisfied via the following transformation from Fermi normal coordinates to THZ normal coordinates

$$\begin{aligned}T_{\bf THZ} &= T_{\bf FN} - \frac{5}{168}\dot{\mathcal{E}}_{KL}X^K_{\bf FN}X^L_{\bf FN}\rho^2_{\bf FN},\\ X^I_{\bf THZ} &= X^I_{\bf FN} - \frac{1}{6}\mathcal{E}^I{}_K X^K_{\bf FN}\rho^2_{\bf FN} + \frac{1}{3}\mathcal{E}_{KL}X_{\bf FN}X^L_{\bf FN}X^I_{\bf FN}\\ &\quad -\frac{1}{6}\dot{\mathcal{E}}^I{}_K X^K_{\bf FN}\rho^2_{\bf FN}T_{\bf FN} + \frac{1}{3}\dot{\mathcal{E}}_{KL}X^K_{\bf FN}X^L_{\bf FN}X^I_{\bf FN}T_{\bf FN}\\ &\quad -\frac{1}{24}\mathcal{E}^I{}_{KL}X^K_{\bf FN}X^L_{\bf FN}\rho^2_{\bf FN} + \frac{1}{12}\mathcal{E}_{KLM}X^K_{\bf FN}X^L_{\bf FN}X^M_{\bf FN}X^I_{\bf FN}\\ &\quad -\frac{2}{63}\epsilon^I{}_{PK}\dot{\mathcal{B}}^P{}_L X^K_{\bf FN}X^L_{\bf FN}\rho^2_{\bf FN},\end{aligned}$$

where $\mathcal{E}_{IJ} = R_{TITJ}\big|_{\rm o}$, $\mathcal{B}_{IJ} = \frac{1}{2}\,\epsilon_I{}^{PQ}R_{PQJT}\big|_{\rm o}$, $\mathcal{E}_{IJK} = \left[\nabla_K R_{TITJ}\big|_{\rm o}\right]^{\rm STF}$, and $\mathcal{B}_{IJ} = \frac{3}{8}\left[\epsilon_I{}^{PQ}\nabla_K R_{PQJT}\big|_{\rm o}\right]^{\rm STF}$.

3) ***Why THZ normal coordinates? -*** description of ***singular field*** becomes ***very simple***: Detweiler, Messaritaki and Whiting (2003)

$$\psi^{\rm S} = \frac{q}{\rho} + O\left(\frac{\rho^3}{\mathcal{R}^4}\right) \;\Rightarrow\; \lim_{\vec{X}\to\vec{X}_{\rm o}} \partial_a \psi^{\rm S} = q\partial_a\left(\frac{1}{\rho}\right) + 0.$$

However, ***for regularization parameters*** A_a ***and*** B_a ***terms***, Fermi normal and THZ normal coordinates make ***no*** difference: they are determined already at the stage of ***Instant initial normal coordinates*** $\hat{X}^A$.

4) Relationship between ***Fermi normal*** and ***THZ normal coordinates*** in terms of their ***Local tetrad structures***

$$\begin{aligned} dT_{\bf THZ} &= dT_{\bf FN} + \varphi_P dX^P_{\bf FN}, \\ dX^I_{\bf THZ} &= \vartheta^I{}_0 dT_{\bf FN} + (\delta^I{}_P + \vartheta^I{}_P) dX^P_{\bf FN}, \end{aligned}$$

where $|\varphi_P| \sim \left|\vartheta^I{}_0\right| \sim \left|\vartheta^I{}_P\right| \sim O\left(\rho^3/\mathcal{R}^3\right)$.

- **Implementation of calculating D_a terms: (in progress)**

1) Recall the structural analysis

$$
\begin{aligned}
Q_a^{(1)} &= \sum_{n=1}^{4}\sum_{p=0}^{n+1} \frac{e_{np(a)}\,(\phi-\phi_{\rm o})^{2(n-p)+2}\left(\theta-\frac{\pi}{2}\right)^{2p}}{\tilde{\rho}_{\rm o}^{2n+1}} \\
&\underset{\Delta\to 0}{\sim} \tilde{\rho}_{\rm o}\sum_{n=1}^{4}\sum_{p=0}^{n+1} e_{np(a)}\,(\sin\Phi)^{2p}\,(\cos\Phi)^{2(n-p)+2}.
\end{aligned}
$$

2) Also

$$
\begin{aligned}
\tilde{\rho}_{\rm o} &= \sqrt{2}\left(r_{\rm o}^2+J^2\right)^{1/2}\chi^{1/2}\left(\delta^2+1-\cos\Theta\right)^{1/2} \\
&\underset{\Delta\to 0}{\longrightarrow} -4\left(r_{\rm o}^2+J^2\right)^{1/2}\chi^{1/2}\sum_{\ell=0}^{\infty}\frac{P_\ell(\cos\Theta)}{(2\ell-1)(2\ell+3)}.
\end{aligned}
$$

where $\chi \equiv 1-J^2\sin^2\Phi/\left(r_{\rm o}^2+J^2\right)$.

3) Then, combine 1) and 2), and take the process $\langle\,\rangle_\Phi$ in the limit $\Theta\to 0$ to obtain

$$
D_a = \sqrt{2}\left(r_{\rm o}^2+J^2\right)^{1/2}\sum_{n=1}^{4}\sum_{p=0}^{n+1}\sum_{s=0}^{n-p+1}(-1)^s e_{np(a)}\left(\frac{r_{\rm o}^2+J^2}{J^2}\right)^{p+s}F_{-(p+s+1/2)},
$$

where $e_{np(a)}$ is to be determined from the ***pure Schwarzschild angular dependence*** in the numerators of the series expansion of $\partial_a\left(1/\rho_{\bf THZ}\right)|_{t=t_{\rm o}}$, i.e.

$$
\partial_a\left(\frac{1}{\rho_{\bf THZ}}\right)\bigg|_{t=t_{\rm o}} = \sum_{n=1}^{4}\sum_{p=0}^{n+1}\frac{e_{np(a)}\,(\phi-\phi_{\rm o})^{2(n-p)+2}\left(\theta-\frac{\pi}{2}\right)^{2p}}{\tilde{\rho}_{\rm o}^{2n+1}}.
$$

4) Circular orbit limit : Detweiler, Messaritaki and Whiting (2003)

$$
\begin{aligned}
D_a &= \left[\frac{2r_{\rm o}^2(r_{\rm o}-2M)}{r_{\rm o}-3M}\right]^{1/2}\left[-\frac{M(r_{\rm o}-2M)F_{-1/2}}{2r_{\rm o}^4(r_{\rm o}-3M)}-\frac{(r_{\rm o}-2M)(r_{\rm o}-4M)F_{1/2}}{8r_{\rm o}^4(r_{\rm o}-2M)}\right. \\
&\quad\left.+\frac{(r_{\rm o}-3M)(5r_{\rm o}^2-7r_{\rm o}M-14M^2)F_{3/2}}{16r_{\rm o}^4(r_{\rm o}-2M)^2}-\frac{3(r_{\rm o}-3M)^2(r_{\rm o}+M)F_{5/2}}{16r_{\rm o}^4(r_{\rm o}-2M)^2}\right].
\end{aligned}
$$

- **Why interesting?**

$$\sum_{\ell=0}^{N} \frac{1}{(2\ell-1)(2\ell+3)} = -\frac{N+1}{(2N+1)(2N+3)}.$$

$$\Rightarrow \lim_{N\to\infty} \sum_{\ell=0}^{N} \frac{1}{(2\ell-1)(2\ell+3)} = 0.$$

$$\Rightarrow \text{For } N = 1000, \; \sum_{\ell=0}^{N} \frac{1}{(2\ell-1)(2\ell+3)} \sim 10^{-4}.$$

$$\Rightarrow \text{For } N = 100, \; \sum_{\ell=0}^{N} \frac{1}{(2\ell-1)(2\ell+3)} \sim 10^{-3}.$$

$$\Rightarrow \text{For } N = 40, \; \sum_{\ell=0}^{N} \frac{1}{(2\ell-1)(2\ell+3)} \sim 10^{-2}.$$

$$\text{Typically, for N} = 40, \; \mathcal{F}_r^{\mathrm{R}} \approx 1.37817 \times 10^{-5}.$$