Gravitational Self-Force under Regge-Wheeler Gauge

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§1. Introduction

Space-based gravitational wave observatory

Super massive black hole − Compact object binary

Black hole perturbation

Background: Black hole (mass: M)

+ Perturbation: Point particle (μ)

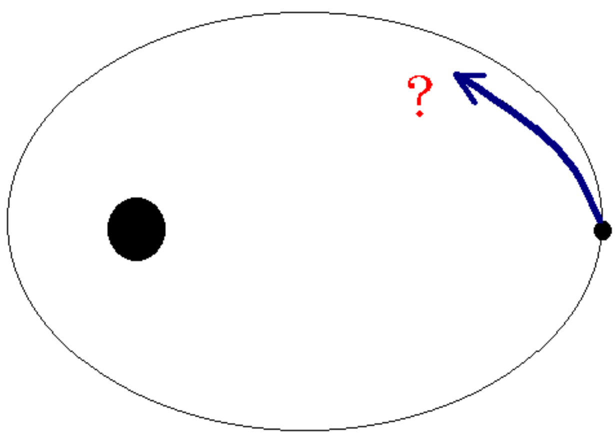
$$g_{\alpha\beta} = g^{(\mathrm{b})}_{\alpha\beta} + h^{\mathrm{full}}_{\alpha\beta}$$

Lowest order in the mass ratio $(\mu/M)^0$:

Geodesic on the background geometry

Next order: Orbit deviates from the geodesic.

Self-force



Point particle =====> Singular

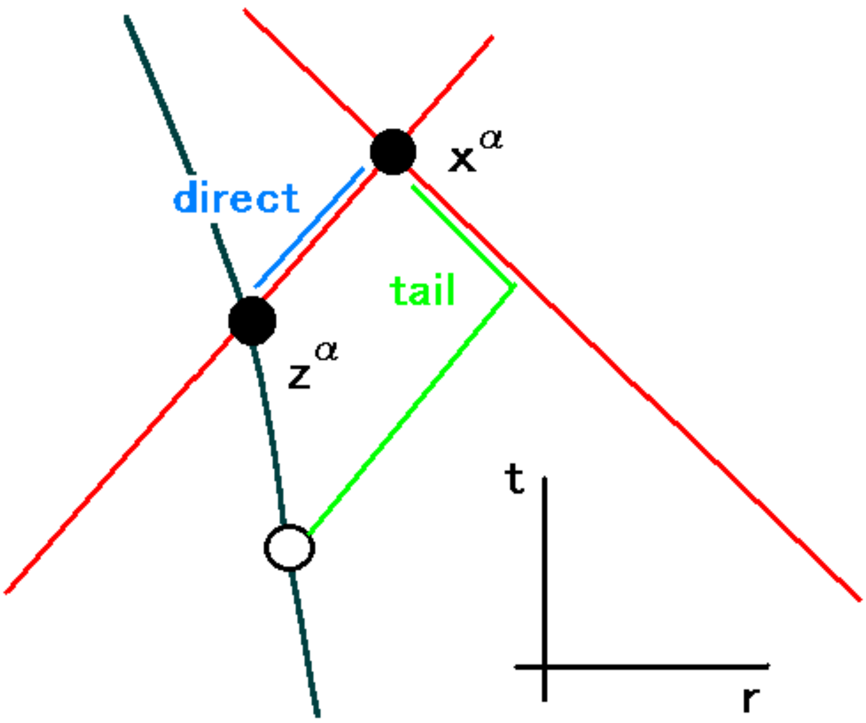
Need Regularization!!

* Mino, Sasaki and Tanaka ('97)

Quinn and Wald ('97) [harmonic (H) gauge]

MiSaTaQuWa

$$\boxed{h_{\alpha\beta}^{\mathrm{full,H}} = h_{\alpha\beta}^{\mathrm{dir,H}} + h_{\alpha\beta}^{\mathrm{tail,H}}}$$



$\{z^\alpha\}$: Orbit of a particle

$\{x^\alpha\}$: Regularization point

Gravitational self-force $O(\mu/M)$:

Deviation from geodesic

$$\mu \frac{D^2 z^\mu(\tau)}{d\tau^2} = F^\mu(z)$$

$$F^\mu = -\frac{\mu}{2}(g_{(\mathrm{b})}^{\mu\nu} + u^\mu u^\nu)\left(2h_{\nu\beta;\alpha}^{\mathrm{tail,H}} - h_{\alpha\beta;\nu}^{\mathrm{tail,H}}\right) u^\alpha u^\beta$$

$\{u^\alpha\}$: Four velocity

$h^{\text{dir,H}}$: **Direct part**

support only on the past light cone

evaluated locally

$h^{\text{tail,H}}$: **Tail part**

Support inside the past null cones

Depend on the whole history of the particle

*** Gives the physical self-force**

It is difficult to calculate the tail part directly.

Regularization

$$\boxed{h^{\text{tail,H}}_{\alpha\beta} = h^{\text{full,H}}_{\alpha\beta} - h^{\text{dir,H}}_{\alpha\beta}}$$

<Two Problems>

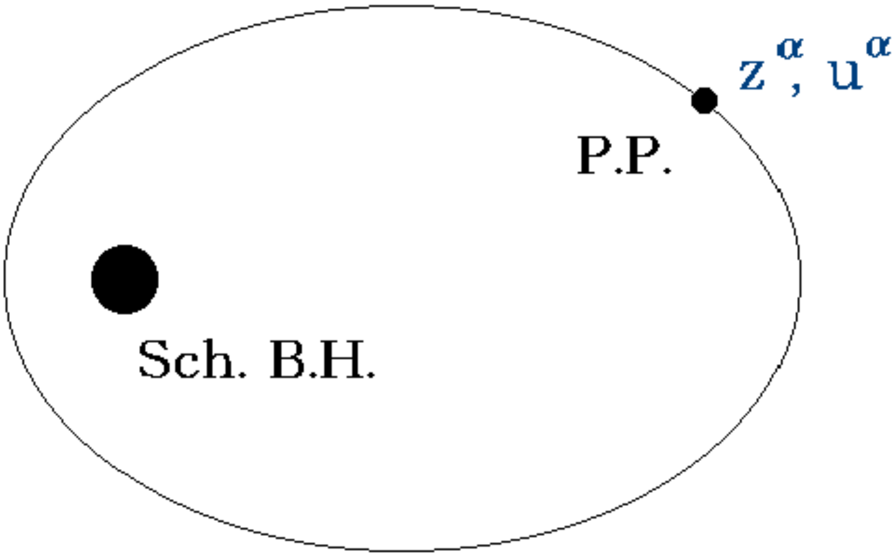
1) **Subtraction problem**

2) **Gauge problem**

Background: Schwarzschild black hole

$$g_{\mu\nu}^{(\mathrm{b})} dx^\mu dx^\nu = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$f(r) = 1 - \frac{2M}{r}$$



Black hole: mass M

Point particle: mass μ

<Subtraction problem>

$$\boxed{h^{\rm tail,H}_{\alpha\beta} = h^{\rm full,H}_{\alpha\beta} - h^{\rm dir,H}_{\alpha\beta}}$$

Full metric perturbation:

Form of spherical harmonic series

in the Regge-Wheeler/Teukolsky formalism

Fourier expansion: In order to treat analytically

Summation of (Fourier-)harmonic series

→ In general, numerical calculation

Direct part:

Given only locally

around the location of the particle

Difficult transformation

to the (Fourier-)harmonic series form

Two different form → Same form

*** We use spherical harmonics expansion!**

Mode sum (decomposition) regularization

Barack and Ori ('00)
Barack, Mino, Nakano, Ori and Sasaki ('02)

Full force: $F^\mu_{\rm full} = F^\mu[h^{\rm full}]$

$$F^\mu_{\rm full}(x) = \sum_{\ell m\omega} F^\mu_{\rm full}(\ell m\omega;\, x) = \sum_\ell F^\mu_{\rm full}(\ell;\, x)\,.$$

Direct part: $F^\mu_{\rm dir} = F^\mu[h^{\rm dir}]$

$$F^\mu_{\rm dir}(x) = \sum_{\ell m} F^\mu_{\rm dir}(\ell m;\, x) = \sum_\ell F^\mu_{\rm dir}(\ell;\, x)\,.$$

*** Derived by Legendre expansion directly.**

Each ℓ mode is finite at the particle location.

$\to$ **Possible take the coincidence limit $x \to z_0$.**

$$F^\mu(\ell;\, z_0) = F^\mu[h^{\rm tail}](\ell;\, z_0) = F^\mu_{\rm full}(\ell;\, z_0) - F^\mu_{\rm dir}(\ell;\, z_0)\,.$$

Regularized self-force:

$$F^\mu(z_0) = \sum_\ell F^\mu(\ell;\, z_0)\,.$$

<Standard form>

Direct part of the self-force under H gauge

Regularization parameter: $(L = \ell + 1/2)$

$$\boxed{F^{\mu(\pm)}_{\rm dir,H}|_\ell = \pm A^\mu L + B^\mu + \frac{C^\mu}{L} + D^\mu_\ell}$$

A^μ**-term: Quadratic divergence**

B^μ**-term: Linear divergence**

A^μ **and** B^μ **are independent of** ℓ

$\pm$ **denotes that the limit to** r_0 **is taken from the greater or smaller value of** r

C^μ**-term: Logarithmic divergence**

$$C^\mu = 0\,.$$

D^μ_ℓ**-term: Remaining finite contribution**

[Ambiguity]

$$\begin{aligned} D^\mu_\ell &= \frac{d^\mu}{(2\ell-1)(2\ell+3)} \\ &\quad + \frac{e^\mu}{(2\ell-1)(2\ell+3)(2\ell-3)(2\ell+5)} + \cdots . \\ &= \frac{d^\mu}{4(L^2-1)} + \frac{e^\mu}{16(L^2-1)(L^2-4)} + \cdots . \end{aligned}$$

The summation of D^μ_ℓ **over** ℓ **(from** $\ell = 0$ **to** ∞**) vanishes.**

<The difference of time/frequency domain>

“Analytic calculation”

$h_{\alpha\beta}^{\rm full,H}$: **in the frequency domain**

Slow motion approximation

considered as the post-Newtonian (PN) Approx.

*** Arbitrarily higher order calculation**

$h_{\alpha\beta}^{\rm dir,H}$: **in the time domain**

evaluated locally

To use mode sum regularization

$\rightarrow$ Need the integration w.r.t. the frequency.

Numerical calculation for general orbits.

New regularization method : W. Hikida’s Talk

Decomposition of the Green function

Subtraction analytically for general orbits

Convergency of the slow motion Approx.

Useful for numerical regularization!

<Gauge problem>

Direct part: $h_{\alpha\beta}^{\mathrm{dir,H}}$

Hadamard prescription in the harmonic gauge

Full metric perturbation: $h_{\alpha\beta}^{\mathrm{full,H}}$

in the Regge-Wheeler/Teukolsky formalism

NOT the one in the harmonic gauge

Regge-Wheeler/radiation gauge

*** Appropriate gauge transformation is needed.**

Harmonic gauge approach:

Sago, Nakano and Sasaki ('03)

From RW gauge to harmonic gauge

for full metric perturbation

In practice, it is difficult to calculate.

Need double integrals.

From harmonic to RW?

Regularization:

$$\boxed{h_{\alpha\beta}^{\rm tail,H} = h_{\alpha\beta}^{\rm full,H} - h_{\alpha\beta}^{\rm dir,H}}$$

Slight modification, but very important!

by Detweiler & Whiting ('03)

$$\boxed{h_{\alpha\beta}^{\rm R,H} = h_{\alpha\beta}^{\rm full,H} - h_{\alpha\beta}^{\rm S,H}}$$

$h_{\mu\nu}^{\rm S,H}$ **: S part**

Inhomogeneous solution of
the linearized Einstein equation under H gauge

$$\bar{h}^{\rm H}_{\mu\nu;\alpha}{}^{;\alpha} + 2R_{\mu\alpha\nu\beta}\bar{h}^{{\rm H}\alpha\beta} = -16\pi T_{\mu\nu}$$
$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}h_\mu{}^\mu$$

$h_{\mu\nu}^{\rm R,H}$ **: R part**

Homogeneous solution

Tail force = R force

$$\boxed{F_\alpha\left[h_{\mu\nu}^{\rm tail,H}\right] = F_\alpha\left[h_{\mu\nu}^{\rm R,H}\right]}$$

Let's consider a Finite Gauge Transformation.

Mino's idea

$$\boxed{x^{\alpha}_{\rm RW} = x^{\alpha}_{\rm H} - \xi^{\alpha}_{\rm H\to RW}\left[h^{\rm R,H}_{\mu\nu}\right] .}$$

From harmonic to RW gauge

$$\lim_{x\to z(\tau)} F_{\alpha}\left[h^{\rm R,H}_{\mu\nu} + 2\,\xi^{\rm H\to RW}_{(\mu;\nu)}\left[h^{\rm R,H}_{\mu\nu}\right]\right](x)$$

$$= \lim_{x\to z(\tau)} F_{\alpha}[h^{\rm full,H}_{\mu\nu} - h^{\rm S,H}_{\mu\nu} \\ +2\,\xi^{\rm H\to RW}_{(\mu;\nu)}\left[h^{\rm full,H}_{\mu\nu} - h^{\rm S,H}_{\mu\nu}\right]](x)$$

$$= \lim_{x\to z(\tau)} F_{\alpha}[h^{\rm full,H}_{\mu\nu} + 2\,\xi^{\rm H\to RW}_{(\mu;\nu)}\left[h^{\rm full,H}_{\mu\nu}\right] \\ -h^{\rm S,H}_{\mu\nu} - 2\,\xi^{\rm H\to RW}_{(\mu;\nu)}\left[h^{\rm S,H}_{\mu\nu}\right]](x)$$

$$= \lim_{x\to z(\tau)} F_{\alpha}\left[h^{\rm full,RW}_{\mu\nu} - h^{\rm S,H}_{\mu\nu} - 2\,\xi^{\rm H\to RW}_{(\mu;\nu)}\left[h^{\rm S,H}_{\mu\nu}\right]\right](x)$$

$$= \lim_{x\to z(\tau)} \Big(F_{\alpha}\left[h^{\rm full,RW}_{\mu\nu}\right](x) \\ -F_{\alpha}\left[h^{\rm S,H}_{\mu\nu} + 2\,\xi^{\rm H\to RW}_{(\mu;\nu)}\left[h^{\rm S,H}_{\mu\nu}\right]\right](x)\Big)$$

$$= F^{\rm RW}_{\alpha}(\tau)$$

This is a Regge-Wheeler self-force.

§2. Regularization under RW gauge

⋆ **First, we consider the full metric perturbation.**

Metric perturbation in the Schwarzschild B.G.:

Using tensor harmonics,

$$\begin{aligned}\boldsymbol{h} = \sum_{\ell m} \Big[& f(r) H_{0\ell m}(t,r) \boldsymbol{a}^{(0)}_{\ell m} - i\sqrt{2} H_{1\ell m}(t,r) \boldsymbol{a}^{(1)}_{\ell m} \\ & + \frac{1}{f(r)} H_{2\ell m}(t,r) \boldsymbol{a}_{\ell m} \\ & - \frac{i}{r}\sqrt{2\ell(\ell+1)} h^{(\mathrm{e})}_{0\ell m}(t,r) \boldsymbol{b}^{(0)}_{\ell m} + \frac{1}{r}\sqrt{2\ell(\ell+1)} h^{(\mathrm{e})}_{1\ell m}(t,r) \boldsymbol{b}_{\ell m} \\ & + \sqrt{\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)} G_{\ell m}(t,r) \boldsymbol{f}_{\ell m} \\ & + \left(\sqrt{2} K_{\ell m}(t,r) - \frac{\ell(\ell+1)}{\sqrt{2}} G_{\ell m}(t,r)\right) \boldsymbol{g}_{\ell m} \\ & - \frac{\sqrt{2\ell(\ell+1)}}{r} h_{0\ell m}(t,r) \boldsymbol{c}^{(0)}_{\ell m} + \frac{i\sqrt{2\ell(\ell+1)}}{r} h_{1\ell m}(t,r) \boldsymbol{c}_{\ell m} \\ & + \frac{\sqrt{2\ell(\ell+1)(\ell-1)(\ell+2)}}{2r^2} h_{2\ell m}(t,r) \boldsymbol{d}_{\ell m} \Big] \,. \end{aligned}$$

Source:

$$\begin{aligned}\boldsymbol{T} = \sum_{\ell m} \Big[& A^{(0)}_{\ell m} \boldsymbol{a}^{(0)}_{\ell m} + A^{(1)}_{\ell m} \boldsymbol{a}^{(1)}_{\ell m} + A_{\ell m} \boldsymbol{a}_{\ell m} + B^{(0)}_{\ell m} \boldsymbol{b}^{(0)}_{\ell m} + B_{\ell m} \boldsymbol{b}_{\ell m} \\ & + Q^{(0)}_{\ell m} \boldsymbol{c}^{(0)}_{\ell m} + Q_{\ell m} \boldsymbol{c}_{\ell m} + D_{\ell m} \boldsymbol{d}_{\ell m} + G^{(s)}_{\ell m} \boldsymbol{g}_{\ell m} + F_{\ell m} \boldsymbol{f}_{\ell m} \Big] \,, \end{aligned}$$

Ten tensor harmonics:

$$\mathbf{a}_{\ell m}{}^{(0)} = \begin{pmatrix} Y_{\ell m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{a}_{\ell m}{}^{(1)} = (i/\sqrt{2}) \begin{pmatrix} 0 & Y_{\ell m} & 0 & 0 \\ Sym & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{a}_{\ell m} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & Y_{\ell m} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{b}_{\ell m}{}^{(0)} = ir[2\ell(\ell+1)]^{-1/2} \begin{pmatrix} 0 & 0 & (\partial/\partial\theta)Y_{\ell m} & (\partial/\partial\phi)Y_{\ell m} \\ 0 & 0 & 0 & 0 \\ Sym & 0 & 0 & 0 \\ Sym & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{b}_{\ell m} = r[2\ell(\ell+1)]^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & (\partial/\partial\theta)Y_{\ell m} & (\partial/\partial\phi)Y_{\ell m} \\ 0 & Sym & 0 & 0 \\ 0 & Sym & 0 & 0 \end{pmatrix},$$

$$\mathbf{c}_{\ell m}{}^{(0)} = r[2\ell(\ell+1)]^{-1/2} \times \begin{pmatrix} 0 & 0 & (1/\sin\theta)(\partial/\partial\phi)Y_{\ell m} & -\sin\theta(\partial/\partial\theta)Y_{\ell m} \\ 0 & 0 & 0 & 0 \\ Sym & 0 & 0 & 0 \\ Sym & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{c}_{\ell m} = ir[2\ell(\ell+1)]^{-1/2} \times \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & (1/\sin\theta)(\partial/\partial\phi)Y_{\ell m} & -\sin\theta(\partial/\partial\theta)Y_{\ell m} \\ 0 & Sym & 0 & 0 \\ 0 & Sym & 0 & 0 \end{pmatrix},$$

$$\mathbf{d}_{\ell m} = -ir^2[2\ell(\ell+1)(\ell-1)(\ell+2)]^{-1/2} \times \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(1/\sin\theta)X_{\ell m} & \sin\theta W_{\ell m} \\ 0 & 0 & Sym & \sin\theta X_{\ell m} \end{pmatrix},$$

$$\mathbf{g}_{\ell m} = (r^2/\sqrt{2}) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Y_{\ell m} & 0 \\ 0 & 0 & 0 & \sin^2\theta Y_{\ell m} \end{pmatrix},$$

$$\mathbf{f}_{\ell m} = r^2[2\ell(\ell+1)(\ell-1)(\ell+2)]^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & W_{\ell m} & X_{\ell m} \\ 0 & 0 & Sym & -\sin^2\theta W_{\ell m} \end{pmatrix}.$$

$$X_{\ell m} = 2\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial\theta} - \cot\theta\right) Y_{\ell m},$$

$$W_{\ell m} = \left(\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) Y_{\ell m}.$$

Odd parity: $(-1)^{\ell+1}$

Even parity: $(-1)^{\ell}$

Regge-Wheeler-Zerilli formalism [Zerilli ('70)]

$$\boldsymbol{h} = \sum_{\ell m} \Big[f(r) H_{0\ell m}(t,r) \boldsymbol{a}^{(0)}_{\ell m} - i\sqrt{2} H_{1\ell m}(t,r) \boldsymbol{a}^{(1)}_{\ell m} + \frac{1}{f(r)} H_{2\ell m}(t,r) \boldsymbol{a}_{\ell m} \cdots \Big]$$

Linearized Einstein equation →

*** Introduce new radial functions:** $R^{(\mathrm{odd/even})}_{\ell m\omega}$

Regge-Wheeler Equation:

(Using Chandrasekhar transformation)

$$\frac{d^2 R^{(\mathrm{odd/even})}_{\ell m\omega}}{dr^{*2}} + [\omega^2 - V_\ell(r)] R^{(\mathrm{odd/even})}_{\ell m\omega} = S^{(\mathrm{odd/even})}_{\ell m\omega}$$

$V_\ell(r)$: **Regge-Wheeler potential**

$S^{(\mathrm{odd/even})}_{\ell m\omega}$: **Source**

Radial function: $R^{(\mathrm{odd/even})}_{\ell m\omega}$

By a Green function method

Homogeneous solutions [Mano et al. ('96)]

Slow motion approximation:

$$z = \omega r \sim v$$

$$\epsilon = 2M\omega \sim v^3$$

* $O(v^n)$: $\left(\frac{n}{2}\right)$ **post-Newtonian (PN) order**

We can calculate arbitrarily higher order.

Regge-Wheeler gauge:

$$\boxed{h_{0\ell m\omega}^{(e)\mathrm{RW}} = h_{1\ell m\omega}^{(e)\mathrm{RW}} = G_{\ell m\omega}^{\mathrm{RW}} = h_{2\ell m\omega}^{\mathrm{RW}} = 0}$$

Reconstruction of metric perturbation:

$$\begin{aligned} h_{1\ell m\omega}^{\mathrm{RW}} &= \frac{r^2}{(r-2M)} R_{\ell m\omega}^{(\mathrm{odd})} \\ h_{0\ell m\omega} &= \frac{i}{\omega}\frac{d}{dr^*}(r R_{\ell m\omega}^{(\mathrm{odd})}) \\ &\quad - \frac{8\pi r(r-2M)}{\omega[\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)]^{1/2}} D_{\ell m\omega}\,. \end{aligned}$$

Regge-Wheeler-Zerilli formalism $\rightarrow$ **for** $\ell \geq 2$ **modes**

For $\ell = 0$ **and** 1 **modes**

[Zerilli ('70), Detweiler and Poisson ('04)]

S-Part of Metric Perturbation under H gauge

$$\bar{h}^{\rm S,H}_{\mu\nu} = 4\mu \left[\frac{\bar{g}_{\mu\alpha}(x,z(\tau_{\rm ret}))\bar{g}_{\nu\beta}(x,z(\tau_{\rm ret}))u^{\alpha}(\tau_{\rm ret})u^{\beta}(\tau_{\rm ret})}{\sigma_{;\gamma}(x,z(\tau_{\rm ret}))u^{\gamma}(\tau_{\rm ret})}\right]$$

$$+2\mu\int_{\tau_{\rm ret}}^{\tau_{\rm adv}} \bar{g}_{\mu}{}^{\alpha}(x,z(\tau))\bar{g}_{\nu}{}^{\beta}(x,z(\tau))R_{\gamma\alpha\delta\beta}(z(\tau))u^{\gamma}(\tau)u^{\delta}(\tau)\,d\tau$$

$$+O(y^2)\,.$$

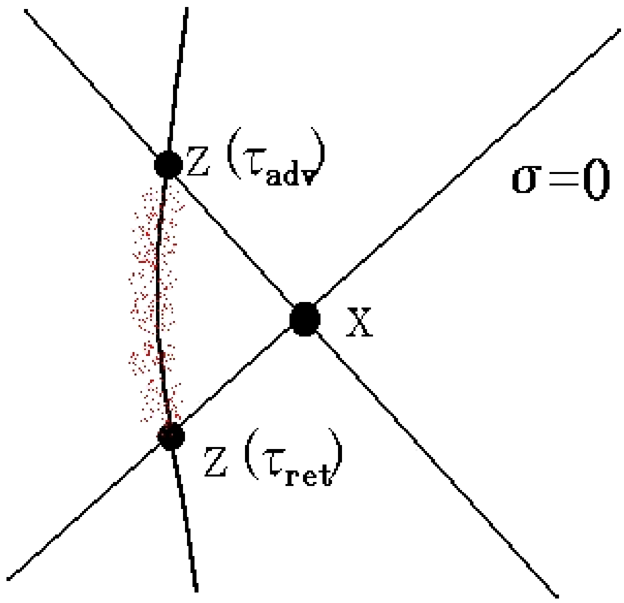
$\sigma(x,z)$: **Bi-scalar of half the squared geodesic distance**

$\bar{g}_{\alpha\beta}(x,z)$: **Parallel displacement bi-vector**

$\tau_{\rm ret}(x)$: **Retarded time for x**

y: **Coordinate difference between x and z_0**

z_0: **Location of the particle**



Metric components:

(Local coordinate expansion)

$$h_{\alpha\beta}^{\mathrm{S,H}} = \mu \sum_{m,n,p,q,r} C_{\alpha\beta}^{m,n,p,q,r} \frac{T^m R^n \Theta^p \Phi^q}{\epsilon^r} + O(y^2)$$

$$\begin{aligned}
&\epsilon := \left(r_0^2 + r^2 - 2\, r_0\, r\ \cos\Theta \cos\Phi\right)^{1/2}\,, \\
&T := t - t_0\,, \qquad R := r - r_0\,, \\
&\Theta := \theta - \frac{\pi}{2}\,, \qquad \Phi := \phi - \phi_0\,.
\end{aligned}$$

When we use PN expansion,

→ the above quantities are useful.

e.g.) $\displaystyle \frac{1}{\epsilon} = \sum_{\ell m} \frac{1}{r_>} \left(\frac{r_<}{r_>}\right) Y_{\ell m}^*(\Omega_0) Y_{\ell m}(\Omega)$

Full relativistic treatment

Mino, Nakano and Sasaki ('03)

Tensor harmonics expansion of local quantities

Some ambiguity in spherical extension.

→ Gauge transformation to RW gauge

Standard form recovers under H gauge.

$$
\begin{aligned}
F^{r(\mathrm{even})}_{S,H}|_\ell = \sum_m \Big[& -\frac{1}{2}\,(2\,\mathcal{E}^2\,r_0^3 - i\,u_r\,\mathcal{L}\,m\,r_0^2 + 2\,i\,r_0\,u_r\,\mathcal{L}\,M\,m \\
& \quad -2\,r_0\,\mathcal{L}^2 + 4\,\mathcal{L}^2\,M)\,\mathcal{L}^2\,m^2\;G^{\mathrm{S,H}}_{\ell m}(t_0,r_0)/r_0^6 \\
& +\frac{1}{2}\,(2\,\mathcal{E}^2\,r_0^3 - i\,u_r\,\mathcal{L}\,m\,r_0^2 + 2\,i\,r_0\,u_r\,\mathcal{L}\,M\,m \\
& \quad -2\,r_0\,\mathcal{L}^2 + 4\,\mathcal{L}^2\,M)\,\mathcal{L}^2\;K^{\mathrm{S,H}}_{\ell m}(t_0,r_0)/r_0^6 \\
& +\frac{1}{2}\,\frac{\mathcal{E}\,\mathcal{L}^2\,u_r\,m^2\;\partial_t G^{\mathrm{S,H}}_{\ell m}(t_0,r_0)}{r_0^2} \\
& -\frac{1}{2}\,\frac{\mathcal{E}^3\,r_0\,u_r\;\partial_t H^{\mathrm{S,H}}_{0\ell m}(t_0,r_0)}{r_0-2\,M} \\
& -\frac{i\,(\mathcal{E}^2\,r_0^3 - r_0\,\mathcal{L}^2 + 2\,\mathcal{L}^2\,M)\,\mathcal{E}\,\mathcal{L}\,m\;\partial_t h^{(\mathrm{e})\mathrm{S,H}}_{1\ell m}(t_0,r_0)}{r_0^4\,(r_0-2\,M)} \\
& -\frac{i\,\mathcal{E}^2\,u_r\,\mathcal{L}\,m\;\partial_t h^{(\mathrm{e})\mathrm{S,H}}_{0\ell m}(t_0,r_0)}{(r_0-2\,M)\,r_0} + \cdots\cdots\Big]\,Y_{\ell m}(\Omega_0)\,.
\end{aligned}
$$

$$
\mathcal{E} = -g_{tt}\frac{dt}{d\tau}\,, \qquad \mathcal{L} = g_{\phi\phi}\frac{d\phi}{d\tau}\,, \qquad u_r = g_{rr}\frac{dr}{d\tau}\,.
$$

Interruption factor of standard form:

$$
\begin{aligned}
h^{(\mathrm{e})\mathrm{S,H}}_{0\ell m}(t,r) &\sim \frac{1}{\ell(\ell+1)}\,, \\
h^{(\mathrm{e})\mathrm{S,H}}_{1\ell m}(t,r) &\sim \frac{1}{\ell(\ell+1)}\,, \\
G^{\mathrm{S,H}}_{\ell m}(t,r) &\sim \frac{1}{\ell(\ell+1)(\ell-1)(\ell+2)}\,.
\end{aligned}
$$

$$h_{0\ell m}^{(e)S,H}(t,r) \sim \frac{1}{\ell(\ell+1)}\,,$$
$$h_{1\ell m}^{(e)S,H}(t,r) \sim \frac{1}{\ell(\ell+1)}\,,$$
$$G_{\ell m}^{S,H}(t,r) \sim \frac{1}{\ell(\ell+1)(\ell-1)(\ell+2)}\,.$$

<Standard form>

Regularization parameter:

$$\boxed{F^{\mu(\pm)}|_{\ell} = \pm A^{\mu} L + B^{\mu} + \frac{C^{\mu}}{L} + D_{\ell}^{\mu}}$$

A^{μ}-term: Quadratic divergence

B^{μ}-term: Linear divergence

A^{μ} and B^{μ} are independent of ℓ ($L = \ell + 1/2$)

C^{μ}-term: Logarithmic divergence

$$C^{\mu} = 0\,.$$

D_{ℓ}^{μ}-term: Remaining finite contribution

$$\begin{aligned} D_{\ell}^{\mu} &= \frac{d^{\mu}}{(2\ell-1)(2\ell+3)} \\ &\quad + \frac{e^{\mu}}{(2\ell-1)(2\ell+3)(2\ell-3)(2\ell+5)} + \cdots\,. \\ &= \frac{d^{\mu}}{4(L^2-1)} + \frac{e^{\mu}}{16(L^2-1)(L^2-4)} + \cdots\,. \end{aligned}$$

$$
\begin{aligned}
h^{(e)S,H}_{0\ell m}(t,r) &\sim \frac{1}{\ell(\ell+1)}\,,\\
h^{(e)S,H}_{1\ell m}(t,r) &\sim \frac{1}{\ell(\ell+1)}\,,\\
G^{S,H}_{\ell m}(t,r) &\sim \frac{1}{\ell(\ell+1)(\ell-1)(\ell+2)}\,.
\end{aligned}
$$

Summation over m modes:

$$
\sum_m \left[\partial_\phi^{2k} Y^*_{\ell m}(\Omega_0)\right] Y_{\ell m}(\Omega_0) = \sum_m \left[(-1)^k\, m^{2k}\, Y^*_{\ell m}(\Omega_0)\right] Y_{\ell m}(\Omega_0)
$$

$$
\sim \ell(\ell+1) \times (\text{function of } \ell)\,.
$$

$$
\begin{aligned}
&\sum_m \left[\left(\partial_\theta^2 - \partial_\phi^2\right) \partial_\phi^{2k} Y^*_{\ell m}(\Omega_0)\right] Y_{\ell m}(\Omega_0)\\
&\quad = \sum_m \left[\left(-\ell(\ell+1) + 2\,m^2\right)(-1)^k\, m^{2k}\, Y^*_{\ell m}(\Omega_0)\right] Y_{\ell m}(\Omega_0)
\end{aligned}
$$

$$
\sim \ell(\ell+1)(\ell-1)(\ell+2) \times (\text{function of } \ell)\,.
$$

$$
\sum_m \left|\partial_\theta Y_{\ell m}(\Omega_0)\right|^2 \sim \ell(\ell+1)\,.
$$

$$
\sum_m m^{2k} \left|\partial_\theta Y_{\ell m}(\Omega_0)\right|^2 \sim \ell(\ell+1)(\ell-1)(\ell+2) \times (\text{function of } \ell)\,.
$$

*** We are trying to prove "standard form"!**

Generators of gauge transformation:

$$\xi_\mu^{(\rm odd)} = \sum_{\ell m} \Lambda_{\ell m}^{\rm S,H\to RW}(t,\,r) \times \left\{0, 0, \frac{-1}{\sin\theta}\partial_\phi Y_{\ell m}(\theta,\phi), \sin\theta\partial_\theta Y_{\ell m}(\theta,\phi)\right\}\,,$$

$$\xi_\mu^{(\rm even)} = \sum_{\ell m}\{M_{0\ell m}^{\rm S,H\to RW}(t,\,r)Y_{\ell m}(\theta,\phi), M_{1\ell m}^{\rm S,H\to RW}(t,\,r)Y_{\ell m}(\theta,\phi), M_{2\ell m}^{\rm S,H\to RW}(t,\,r)\partial_\theta Y_{\ell m}(\theta,\phi), M_{2\ell m}^{\rm S,H\to RW}(t,\,r)\partial_\phi Y_{\ell m}(\theta,\phi)\}\,.$$

Regge-Wheeler gauge condition:

$$\boxed{h_{0\ell m\omega}^{(\rm e)S,RW} = h_{1\ell m\omega}^{(\rm e)S,RW} = G_{\ell m\omega}^{\rm S,RW} = h_{2\ell m\omega}^{\rm S,RW} = 0}$$

$$h_{2\ell m}^{\rm S,H}(t,\,r) = -2\,i\,\Lambda_{\ell m}^{\rm S,H\to RW}(t,\,r)\,,$$

$$h_{0\ell m}^{(\rm e)S,H}(t,\,r) = -M_{0\ell m}^{\rm S,H\to RW}(t,\,r) - \partial_t M_{2\ell m}^{\rm S,H\to RW}(t,\,r)\,,$$

$$h_{1\ell m}^{(\rm e)S,H}(t,\,r) = -M_{1\ell m}^{\rm S,H\to RW}(t,\,r) - r^2\,\partial_r\left(\frac{M_{2\ell m}^{\rm S,H\to RW}(t,\,r)}{r^2}\right)\,,$$

$$G_{\ell m}^{\rm S,H}(t,\,r) = -\frac{2}{r^2}M_{2\ell m}^{\rm S,H\to RW}(t,\,r)\,.$$

Gauge deterministic on sphere:

$$\Lambda_{\ell m}^{\rm S,H\to RW}(t,\,r) = \frac{i}{2}h_{2\ell m}^{\rm S,H}(t,\,r)\,,$$

$$M_{2\ell m}^{\rm S,H\to RW}(t,\,r) = -\frac{r^2}{2}G_{\ell m}^{\rm S,H}(t,\,r)\,,$$

$$M_{0\ell m}^{\rm S,H\to RW}(t,\,r) = -h_{0\ell m}^{(\rm e)S,H}(t,\,r) - \partial_t M_{2\ell m}^{\rm S,H\to RW}(t,\,r)\,,$$

$$M_{1\ell m}^{\rm S,H\to RW}(t,\,r) = -h_{1\ell m}^{(\rm e)S,H}(t,\,r) - r^2\,\partial_r\left(\frac{M_{2\ell m}^{\rm S,H\to RW}(t,\,r)}{r^2}\right)\,.$$

NOT necessary to integrate w.r.t. t or r.

Gauge functions are determined uniquely.

(for $\ell \geq 2$)

The odd parity components:

$$h_{0\ell m}^{\rm S,RW}(t,\,r) = h_{0\ell m}^{\rm S,H}(t,\,r) + \partial_t\,\Lambda_{\ell m}^{\rm S,H\to RW}(t,\,r)\,,$$
$$h_{1\ell m}^{\rm S,RW}(t,\,r) = h_{1\ell m}^{\rm S,H}(t,\,r) + r^2\,\partial_r\left(\frac{\Lambda_{\ell m}^{\rm S,H\to RW}(t,\,r)}{r^2}\right)\,,$$

The even parity components:

$$H_{0\ell m}^{\rm S,RW}(t,\,r) = H_{0\ell m}^{\rm S,H}(t,\,r) + \frac{2\,r}{r-2\,M}\Big[\partial_t\,M_{0\ell m}^{\rm S,H\to RW}(t,\,r) - \frac{M(r-2\,M)}{r^3}M_{1\ell m}^{\rm S,H\to RW}(t,\,r)\Big]\,,$$
$$H_{1\ell m}^{\rm S,RW}(t,\,r) = H_{1\ell m}^{\rm S,H}(t,\,r) + \Big[\partial_t\,M_{1\ell m}^{\rm S,H\to RW}(t,\,r) + \partial_r\,M_{0\ell m}^{\rm S,H\to RW}(t,\,r) - \frac{2\,M}{r(r-2\,M)}M_{0\ell m}^{\rm S,H\to RW}(t,\,r)\Big]\,,$$
$$H_{2\ell m}^{\rm S,RW}(t,\,r) = H_{2\ell m}^{\rm S,H}(t,\,r) + \frac{2(r-2\,M)}{r}\Big[\partial_r\,M_{1\ell m}^{\rm S,H\to RW}(t,\,r) + \frac{M}{r(r-2\,M)}M_{1\ell m}^{\rm S,H\to RW}(t,\,r)\Big]\,,$$
$$K_{\ell m}^{\rm S,RW}(t,\,r) = K_{\ell m}^{\rm S,H}(t,\,r) + \frac{2(r-2\,M)}{r^2}M_{1\ell m}^{\rm S,H\to RW}(t,\,r)\,,$$

$$\boxed{h_{0\ell m\omega}^{\rm (e)S,RW} = h_{1\ell m\omega}^{\rm (e)S,RW} = G_{\ell m\omega}^{\rm S,RW} = h_{2\ell m\omega}^{\rm S,RW} = 0}$$

S-force difference between H and RW for general orbit

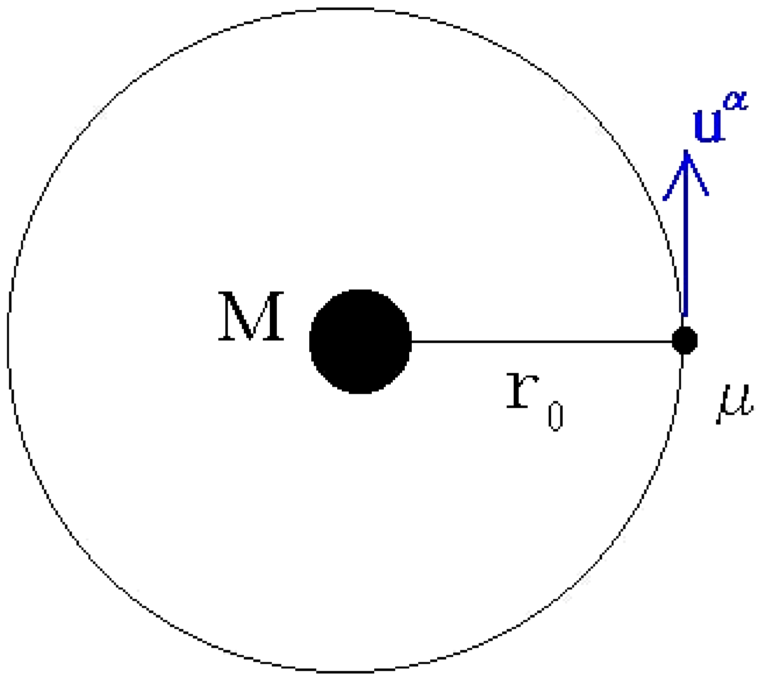
(In terms of the M.P. under H gauge)

$$
\begin{aligned}
\delta F^{r(\mathrm{even})}_{\mathrm{S,H}\to\mathrm{RW}}|_\ell = \sum_m \Big[& \frac{1}{2}(-i\, r_0\, u_r\, \mathcal{L}\,(r_0 - 2\,M)\, m + 4\,\mathcal{L}^2\, M \\
& +2\,\mathcal{E}^2\, r_0^3 - 2\, r_0\, \mathcal{L}^2)\, \mathcal{L}^2\, m^2\; G^{\mathrm{S,H}}_{\ell m}(t_0, r_0)/r_0^6 \\
& -\frac{1}{2}(\mathcal{L}^2\, r_0\,(r_0^3\, \mathcal{E}^2 - 2\, r_0^3 + 4\, r_0^2\, M \\
& -\mathcal{L}^2\, r_0 + 2\,\mathcal{L}^2\, M)\, m^2 \\
& +i\, r_0\, \mathcal{L}\, u_r\,(2\, r_0^5 + 2\, r_0^4\, \mathcal{E}^2 - 7\, r_0^3\, M \\
& -4\, r_0^3\, M\, \mathcal{E}^2 + 6\, r_0^2\, M^2 - r_0^2\, \mathcal{L}^2 \\
& +3\, r_0^2\, \mathcal{L}^2\, M - 2\,\mathcal{L}^2\, M^2)\, m + 4\, r_0\, \mathcal{L}^4\, M \\
& +2\,\mathcal{E}^2\, r_0^3\, \mathcal{L}^2\, M - \mathcal{E}^2\, r_0^4\, \mathcal{L}^2 - 4\,\mathcal{L}^4\, M^2 \\
& -2\, r_0^6\, \mathcal{E}^2 + 2\,\mathcal{E}^4\, r_0^6 - r_0^2\, \mathcal{L}^4) \\
& \times \partial_r G^{\mathrm{S,H}}_{\ell m}(t_0, r_0)/r_0^6 \\
& + \frac{r_0^2\, \mathcal{E}^3\, u_r\;\; \partial_t^2 h^{(\mathrm{e})\mathrm{S,H}}_{0\ell m}(t_0, r_0)}{(r_0 - 2\,M)^2} \\
& -\frac{1}{2}(-2\, r_0^3 + 3\,\mathcal{E}^2\, r_0^3 + 4\, M\, r_0^2 - 3\, r_0\, \mathcal{L}^2 \\
& +6\,\mathcal{L}^2\, M)\, r_0\, \mathcal{E}^2 \\
& \times \partial_t^2 \partial_r G^{\mathrm{S,H}}_{\ell m}(t_0, r_0)/(r_0 - 2\,M)^2 \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \Big]\, Y_{\ell m}(\Omega_0)\,.
\end{aligned}
$$

Circular orbit:

$$\{z_0^\alpha\} = \left\{t_0,\ r_0,\ \frac{\pi}{2},\ \phi_0\right\}$$

$$\{u^\alpha\} = \left\{\frac{1}{\sqrt{1-3M/r_0}},\ 0,\ 0,\ \sqrt{\frac{M}{r_0^2(r_0-3M)}}\right\}$$



$$F^\phi = \frac{r_0 - 2\,M}{r_0^3\,\Omega} F^t\,,$$

$$\Omega = \frac{u^\phi}{u^t}\,.$$

S-force difference between H and RW for circular orbit

$$\delta F^{t(\mathrm{odd})}_{\mathrm{S,H}\to\mathrm{RW}}|_\ell = 0\,,$$
$$\delta F^{r(\mathrm{odd})}_{\mathrm{S,H}\to\mathrm{RW}}|_\ell = 0\,,$$
$$\delta F^{t(\mathrm{even})}_{\mathrm{S,H}\to\mathrm{RW}}|_\ell = 0\,,$$
$$\delta F^{r(\mathrm{even})}_{\mathrm{S,H}\to\mathrm{RW}}|_\ell = \sum_m \Big[-3\,\frac{M\,(r_0-2\,M)^2\,h^{(\mathrm{e})\mathrm{S,H}}_{1\ell m}(t_0,r_0)}{r_0^4\,(r_0-3\,M)} + \frac{3}{2}\,\frac{M\,(r_0-2\,M)^2\,\partial_r G^{\mathrm{S,H}}_{\ell m}(t_0,r_0)}{r_0^2\,(r_0-3\,M)}\Big]\,Y_{\ell m}(\Omega_0)\,.$$

Because

Source (frequency domain): $\delta(\omega - m\,\Omega)$

$\rightarrow\ \partial_t = -i\,m\,\Omega$

Gauge force: 2PN $O(v^4)$

Because

$$h^{(\mathrm{e})\mathrm{S,H}}_{1\ell m}(t,r) \sim \frac{M}{\ell(\ell+1)}\,,$$
$$G^{\mathrm{S,H}}_{\ell m}(t,r) \sim \frac{M}{\ell(\ell+1)(\ell-1)(\ell+2)}\,.$$

1PN Circular Example:

We consider only r-component. ($\ell \geq 2$)

$$F^r_{\mathrm{RW}}\big|_\ell = F^r_{\mathrm{full,RW}}\big|_\ell - F^r_{\mathrm{S,RW}}\big|_\ell = -\mu^2 \frac{45\,M}{8(2\ell-1)(2\ell+3)\,r_0^3}$$

After summing over ℓ mode

$$F^r_{\mathrm{RW}}(\ell \geq 2) = -\frac{3\mu^2 M}{4r_0^3}$$

For $\ell = 0$ and 1 modes with appropriate boundary [Detweiler and Poisson ('04)]

$$F^r_{\mathrm{RW}}(\ell = 0,\, 1) = \frac{2\,\mu^2}{r_0^2} - \frac{57\,\mu^2 M}{4r_0^3}$$

Regularized gravitational self-force:

$$F^r_{\mathrm{RW}} = \frac{2\,\mu^2}{r_0^2} - \frac{15\,\mu^2 M}{r_0^3}\,.$$

→ Correction to the radius of the orbit that deviates from the geodesic in the unperturbed background.

§3. Discussion

Schwarzschild background / Regge-Wheeler gauge

S-part of the self-force

Standard form?

$$F^{\mu(\pm)}|_\ell = \pm A^\mu L + B^\mu + \frac{C^\mu}{L} + D^\mu_\ell$$

*** Plunging into a B.H.**

Standard form recovers.

Barack and Ori ('01), Barack and Lousto ('02)

*** 1PN calculation for circular orbit**

Standard form recovers.

Higher PN order calculation?

Black Hole Perturbation Club

http://www2.yukawa.kyoto-u.ac.jp/~misao/BHPC/