# Flow of mass and angular momentum into a black hole

Time-domain formalisms and small-hole/slow-motion approximation

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# 1. Goals and motivations

**To obtain practical time-domain formulae for the fluxes of mass and angular momentum into a black hole.**

For Schwarzschild holes, time-domain integration of the Regge-Wheeler and Zerilli equations is very efficient; prior to Martel (2004), there were no formulae linking the standard gauge-invariant variables to horizon fluxes.

For Kerr holes, time-domain integration of the Teukolsky equation is now feasible; the Teukolsky-Press flux formulae are formulated in the frequency domain.

(The membrane-paradigm flux formulae are formulated in the time domain, but this formalism does not involve the standard gauge-invariant quantities and is less practical.)

**To calculate the fluxes in a SH/SM approximation, in which the black-hole mass $M$ and the background spacetime's radius of curvature $\mathcal{R}$ are such that $M/\mathcal{R} \ll 1$.**

If the black hole is in on a circular orbit of radius $b$ in the field of another body of mass $M_{\rm ext}$, then

$$\frac{M}{\mathcal{R}} \sim \frac{M}{M + M_{\rm ext}} V^3, \qquad V = \sqrt{\frac{M + M_{\rm ext}}{b}}$$

This is small whenever $M/M_{\rm ext} \ll 1$, in which case $V$ can be arbitrary **(small-hole approximation)**.

This is also small whenever $V \ll 1$, in which case $M/M_{\rm ext}$ can be arbitrary **(slow-motion approximation)**.

# 2. Main results

For a Schwarzschild hole, the flux formulae are expressed in terms of gauge-invariant Regge-Wheeler and Zerilli-Moncrief functions,

$$\langle \dot{M} \rangle = \frac{1}{64\pi} \sum_{lm} (l-1)l(l+1)(l+2) \Big\langle 4 \big| \Psi^{lm}_{\rm RW}(v) \big|^2 + \big| \dot{\Psi}^{lm}_{\rm ZM}(v) \big|^2 \Big\rangle$$

For a Kerr hole, the flux formulae are expressed in terms of the gauge-invariant Teukolsky function

$$C_{\alpha\gamma\beta\delta} k^\alpha m^\gamma k^\beta m^\delta \equiv \psi_0(v, r_+, \theta, \psi) = \sum_m \psi^m(v,\theta) e^{im\psi}$$

For example,

$$\begin{aligned} \langle \dot{M} \rangle &= \frac{r_+^2 + a^2}{4\kappa} \sum_m \Big[ 2\kappa \int \big\langle |\Phi^m_+|^2 \big\rangle \sin\theta \, d\theta \\ &\quad - im\Omega_{\rm H} \int \big\langle \bar{\Phi}^m_+ \Phi^m_- - \Phi^m_+ \bar{\Phi}^m_- \big\rangle \sin\theta \, d\theta \Big] \end{aligned}$$

For a small Schwarzschild hole on a circular orbit (radius $b$) in the field of another Schwarzschild black hole (mass $M_{\rm ext} \gg M$),

$$\langle \dot{M} \rangle = \frac{32}{5}\left(\frac{M}{M_{\rm ext}}\right)^6 V^{18}\frac{(1-V^2)(1-2V^2)}{(1-3V^2)^2}, \qquad V = \sqrt{\frac{M_{\rm ext}}{b}}$$

For a small Kerr hole on the same (equatorial) orbit,

$$\langle \dot{M} \rangle = -\epsilon\frac{8}{5}\left(\frac{M}{M_{\rm ext}}\right)^5 \chi(1+3\chi^2)V^{15}\frac{(1-2V^2)\left(1-\frac{4+27\chi^2}{4+12\chi^2}V^2\right)}{(1-3V^2)^2}$$

where $\chi \equiv a/M \equiv J/M^2$ and $\epsilon \equiv \hat{\boldsymbol{L}}\cdot\hat{\boldsymbol{s}} = \pm 1$.

# 3. Generator dynamics

[The techniques described here are adapted from Hawking & Hartle (1972), Price & Thorne (1986), and the Membrane Paradigm book.]

The null generators of an (evolving) event horizon are described by parametric relations $z^\alpha(v, \theta^A)$, with **$\boldsymbol{v}$ the parameter** ("advanced time") and **$\boldsymbol{\theta}^A$ generator labels** ("comoving coordinates").

The vectors

$$k^\alpha = \frac{\partial z^\alpha}{\partial v} : \text{tangent to generators}$$

$$e^\alpha_A = \frac{\partial z^\alpha}{\partial \theta^A} : \text{transverse to generators}$$

are tangent to the horizon; $k^\alpha$ is null and is orthogonal to $e^\alpha_A$.

The tangent vector satisfies the geodesic equation, $k^{\alpha}{}_{;\beta} k^{\beta} = \kappa k^{\alpha}$, which defines the surface gravity $\kappa$.

The **expansion scalar $\boldsymbol{\theta}$** and **shear tensor $\boldsymbol{\sigma}_{AB}$** are defined by

$$k_{\alpha;\beta} e^{\alpha}_{A} e^{\beta}_{B} = \frac{1}{2} \frac{\partial \gamma_{AB}}{\partial v} = \frac{1}{2} \theta \, \gamma_{AB} + \sigma_{AB}$$

where

$$\gamma_{AB} = g_{\alpha\beta} e^{\alpha}_{A} e^{\beta}_{B}$$

is the **horizon metric**.

In the absence of matter, the expansion and shear evolve according to

$$\left( \frac{\partial}{\partial v} - \kappa \right) \theta = -\frac{1}{2} \theta^2 - \sigma_{AB} \sigma^{AB}$$

$$\left( \frac{\partial}{\partial v} - \kappa \right) \sigma^{A}{}_{B} = -\theta \sigma^{A}{}_{B} - C^{A}{}_{B}$$

Here,

$$C_{AB} = C_{\alpha\gamma\beta\delta} e^{\alpha}_{A} k^{\gamma} e^{\beta}_{B} k^{\delta}$$

are tangential components of the **Weyl tensor**.

In the absence of caustics, the **horizon area** grows according to

$$\dot{A} = \frac{d}{dv} \oint \sqrt{\gamma}\, d^2\theta = \oint \frac{\partial\sqrt{\gamma}}{\partial v}\, d^2\theta = \oint \theta\, dS$$

where $dS = \sqrt{\gamma}\, d^2\theta$ is an element of surface area.

# 4. Perturbation equations

The perturbation of the horizon is driven by the Weyl curvature, $C_{AB} = O(\varepsilon)$, which implies that $\sigma_{AB} = O(\varepsilon)$ and $\theta = O(\varepsilon^2)$.

To leading order in $\varepsilon$, the perturbation equations reduce to

$$\left(\frac{\partial}{\partial v} - \kappa\right)\sigma_{AB} = -C_{AB} + O(\varepsilon^2)$$

$$\left(\frac{\partial}{\partial v} - \kappa\right)\theta = -\sigma_{AB}\sigma^{AB} + O(\varepsilon^3)$$

where $\kappa$ now stands for the unperturbed (Kerr) surface gravity.

The Weyl curvature varies over a **short time scale** — the generators rotate with a fast angular velocity $\Omega_{\rm H}$ and the hole might be in a tidal field that varies rapidly.

The shear equation must be integrated exactly, with teleological boundary conditions:

$$\sigma_{AB}(v,\theta^A) = \int_v^\infty e^{\kappa(v-v')} C_{AB}(v',\theta^A)\, dv'$$

The shear also varies over a short time scale, but its square contains a piece that varies slowly, and this will drive a **slow, secular change in the expansion**.

This is described by $(\partial_v \ll \kappa)$

$$\langle\theta\rangle = \frac{1}{\kappa}\bigl\langle \sigma_{AB}\sigma^{AB} \bigr\rangle$$

Integrating this over $dS$ gives the averaged rate of growth of the horizon area.

Finally, integration of the metric equation $\partial_v \gamma_{AB} = 2\sigma_{AB}$ gives

$$\gamma_{AB}(v, \theta^A) = \gamma^0_{AB}(\theta) + \delta\gamma_{AB}(v, \theta^A)$$

where $\gamma^0_{AB}$ is the Kerr horizon metric, and

$$\begin{aligned}
\delta\gamma_{AB}(v, \theta^A) &= 2\int_{-\infty}^{v} \sigma_{AB}(v', \theta^A)\, dv' \\
&= \frac{2}{\kappa}\int_{-\infty}^{v} C_{AB}(v', \theta^A)\, dv' \\
&\quad + \frac{2}{\kappa}\int_{v}^{\infty} e^{\kappa(v-v')} C_{AB}(v', \theta^A)\, dv'
\end{aligned}$$

is the perturbation.

# 5. From area growth to fluxes

For matter perturbations it can be shown [Carter (1979)] that the rates of change of area, mass, and angular momentum are related by

$$\frac{\kappa}{8\pi}\dot{A} = \dot{M} - \Omega_{\rm H}\dot{J} \qquad \text{(first law)}$$

We assume that this holds also for gravitational perturbations, at least on average.

For matter perturbations it can be shown [Carter (1979)] that if the matter field is decomposed into modes $e^{-i\omega v}e^{im\psi}$, then the mode contributions to the averaged fluxes are related by

$$\langle \dot{M} \rangle_{m,\omega} = \frac{\omega}{m} \langle \dot{J} \rangle_{m,\omega}$$

We assume that this holds also for gravitational perturbations [Teukolsky & Press (1974)].

These relations imply

$$\langle \dot{M} \rangle_{m,\omega} = \frac{\omega}{k} \frac{\kappa}{8\pi} \langle \dot{A} \rangle_{m,\omega}$$

$$\langle \dot{J} \rangle_{m,\omega} = \frac{m}{k} \frac{\kappa}{8\pi} \langle \dot{A} \rangle_{m,\omega}$$

where $k = \omega - m\Omega_{\rm H}$.

In spacetime coordinates $(v, r, \theta, \psi)$ the mode decomposition of the metric perturbation is

$$\delta\gamma_{AB} = \sum_m \int d\omega \, \gamma_{AB}^{m,\omega}(r,\theta) e^{-i\omega v} e^{im\psi}$$

In horizon coordinates $(v, r = r_+, \theta, \phi = \psi - \Omega_H v)$ we have

$$\delta\gamma_{AB} = \sum_m \int d\omega \, \gamma_{AB}^{m,\omega}(\theta) e^{-ikv} e^{im\phi}$$

The shear tensor is

$$\sigma_{AB} = \frac{1}{2}\sum_m \int d\omega\,(-ik)\gamma^{m,\omega}_{AB}(\theta)e^{-ikv}e^{im\phi}$$

Then

$$\begin{aligned}\frac{\kappa}{8\pi}\langle\dot{A}\rangle &= \frac{1}{8\pi}\oint\left\langle\sigma^{AB}\sigma_{AB}\right\rangle dS \\ &= \sum_m \int d\omega\,\frac{-ik}{16\pi}\oint\left\langle\sigma^{AB}\gamma^{m,\omega}_{AB}(\theta)e^{-ikv}e^{im\phi}\right\rangle dS\end{aligned}$$

It follows that

$$\langle\dot{M}\rangle = \sum_m \int d\omega\,\frac{-i\omega}{16\pi}\oint\left\langle\sigma^{AB}\gamma^{m,\omega}_{AB}(\theta)e^{-ikv}e^{im\phi}\right\rangle dS$$

$$\langle\dot{J}\rangle = \sum_m \int d\omega\,\frac{-im}{16\pi}\oint\left\langle\sigma^{AB}\gamma^{m,\omega}_{AB}(\theta)e^{-ikv}e^{im\phi}\right\rangle dS$$

Substituting $-i\omega \to \pounds_t$ and $im \to \pounds_\phi$ and reconstructing the metric perturbation, we arrive at

$$\langle \dot{M} \rangle = \frac{1}{16\pi} \oint \langle \sigma^{AB} \pounds_t \gamma_{AB} \rangle \, dS$$

$$\langle \dot{J} \rangle = -\frac{1}{16\pi} \oint \langle \sigma^{AB} \pounds_\phi \gamma_{AB} \rangle \, dS$$

The Lie derivatives are in the directions of $t^\alpha$ and $\phi^\alpha$, the timelike and rotational Killing vectors of the Kerr spacetime; $k^\alpha = t^\alpha + \Omega_{\rm H} \phi^\alpha$.

These flux formulae first appeared in the Membrane Paradigm book, but the derivation given here is different.

They can now be re-expressed in terms of standard **gauge-invariant quantities**.

# 6. Curvature formalism

Let $\psi_0$ be a complex component of the perturbed Weyl tensor,

$$\psi_0 = \delta\big(C_{\alpha\gamma\beta\delta} k^\alpha m^\gamma k^\beta m^\delta\big) = \big(\delta C_{\alpha\gamma\beta\delta}\big) k^\alpha m^\gamma k^\beta m^\delta$$

Then we have

$$C_{AB} = \bar{m}_A \bar{m}_B \psi_0 + m_A m_B \bar{\psi}_0, \qquad m^\alpha = m^A e^\alpha_A$$

The shear tensor is obtained by integration

$$\sigma_{AB} = \bar{m}_A \bar{m}_B \Phi_+ + m_A m_B \bar{\Phi}_+$$

where

$$\Phi_+(v, \theta^A) \equiv \int_v^\infty e^{\kappa(v-v')} \psi_0(v', \theta^A)\, dv'$$

Finally, the metric perturbation is

$$\delta\gamma_{AB} = \frac{2}{\kappa}\Big[\bar{m}_A \bar{m}_B (\Phi_- + \Phi_+) + m_A m_B (\bar{\Phi}_- + \bar{\Phi}_+)\Big]$$

where

$$\Phi_-(v,\theta^A) \equiv \int_{-\infty}^{v} \psi_0(v',\theta^A)\,dv'$$

After decomposition into modes $e^{im\psi}$, substitution of these expressions into the flux formulae returns

$$\begin{aligned}
\langle \dot{M} \rangle &= \frac{r_+^2+a^2}{4\kappa} \sum_m \Big[ 2\kappa \int \big\langle |\Phi_+^m|^2 \big\rangle \sin\theta\,d\theta \\
&\quad - im\Omega_{\rm H} \int \big\langle \bar{\Phi}_+^m \Phi_-^m - \Phi_+^m \bar{\Phi}_-^m \big\rangle \sin\theta\,d\theta \Big] \\
\langle \dot{J} \rangle &= -\frac{r_+^2+a^2}{4\kappa} \sum_m (im) \int \big\langle \bar{\Phi}_+^m \Phi_-^m - \Phi_+^m \bar{\Phi}_-^m \big\rangle \sin\theta\,d\theta
\end{aligned}$$

The gauge-invariant function $\psi_0 = \sum_m \psi^m(v,\theta) e^{im\psi}$ can be obtained by solving the **Teukolsky equation**.

In the frequency domain these formulae are the same as in Teukolsky & Press (1974).

# 7. Metric formalism

For a perturbed Schwarzschild black hole, the metric perturbation (in the adopted comoving gauge) can be expressed in terms the **Regge-Wheeler** and **Zerilli-Moncrief** functions, which are gauge invariant.

We have

$$\begin{aligned}\delta\gamma_{AB}(v,\theta^A) &= 2M\sum_{lm}\biggl[2X^{lm}_{AB}(\theta^A)\int^v \Psi^{lm}_{\mathrm{RW}}(v')\,dv' \\ &\qquad + Z^{lm}_{AB}(\theta^A)\Psi^{lm}_{\mathrm{ZM}}(v)\biggr]\end{aligned}$$

where $X^{lm}_{AB}(\theta^A)$ and $Z^{lm}_{AB}(\theta^A)$ are respectively odd-parity and even-parity tensorial spherical harmonics.

This expression gives rise to the horizon flux formulae

$$\begin{aligned}
\langle \dot{M} \rangle &= \frac{1}{64\pi} \sum_{lm} (l-1)l(l+1)(l+2) \Big\langle 4 \big| \Psi^{lm}_{\rm RW}(v) \big|^2 + \big| \dot{\Psi}^{lm}_{\rm ZM}(v) \big|^2 \Big\rangle \\
\langle \dot{J} \rangle &= \frac{1}{64\pi} \sum_{lm} (l-1)l(l+1)(l+2)(im) \\
&\quad \times \Big\langle 4\Psi^{lm}_{\rm RW}(v) \int^v \bar{\Psi}^{lm}_{\rm RW}(v')\, dv' + \dot{\Psi}^{lm}_{\rm ZM}(v) \bar{\Psi}^{lm}_{\rm ZM}(v) \Big\rangle
\end{aligned}$$

These were first obtained and used by Martel (2004); flaws in his derivation (based on the Isaacson stress-energy tensor) have been eliminated.

With $v \to u$, the same formulae apply at future null infinity.

# 8. SH/SM approximation (S)

The metric of a nonrotating black hole of mass $M$ immersed in an external universe with radius of curvature $\mathcal{R}$ can be expressed as an **expansion in powers of $M/\mathcal{R} \ll 1$** [Detweiler (2001); Alvi (2000); Poisson (2004)].

If the black hole is on a circular orbit of radius $b$ in the field of another body of mass $M_{\rm ext}$, then

$$\frac{M}{\mathcal{R}} \sim \frac{M}{M + M_{\rm ext}} V^3, \qquad V = \sqrt{\frac{M + M_{\rm ext}}{b}}$$

When $M \ll \mathcal{R}$ the hole can be thought of as moving on a **world line $\gamma$** in the background spacetime of the external universe, with a 4-velocity $u^\alpha$.

The hole is distorted by the **tidal gravitational field** supplied by the external universe, which is described by

$$\mathcal{E}_{\alpha\beta}(v) \equiv C_{\mu\alpha\nu\beta}(\gamma) u^\mu u^\nu, \qquad \mathcal{B}_{\alpha\beta}(v) \equiv \frac{1}{2} u^\mu \varepsilon_{\mu\alpha}{}^{\gamma\delta} C_{\nu\beta\gamma\delta}(\gamma) u^\nu$$

where $C_{\mu\alpha\nu\beta}$ is the Weyl tensor of the background spacetime.

From the perturbed metric the Regge-Wheeler and Zerilli-Moncrief functions can be computed, and these can be substituted into the flux formulae.

This gives

$$\langle \dot{M} \rangle = \frac{16M^6}{45} \left\langle \dot{\mathcal{E}}_{\alpha\beta} \dot{\mathcal{E}}^{\alpha\beta} + \dot{\mathcal{B}}_{\alpha\beta} \dot{\mathcal{B}}^{\alpha\beta} \right\rangle = O(M^6/\mathcal{R}^6)$$

$$\langle \dot{J} \rangle = -\frac{32M^6}{45} u^\mu \varepsilon_{\mu\alpha\gamma\delta} \left\langle \dot{\mathcal{E}}^\alpha{}_\beta \mathcal{E}^{\beta\gamma} + \dot{\mathcal{B}}^\alpha{}_\beta \mathcal{B}^{\beta\gamma} \right\rangle s^\delta = O(M^6/\mathcal{R}^5)$$

where $s^\alpha$ is the spin direction and $\dot{\mathcal{E}}_{\alpha\beta} = \mathcal{E}_{\alpha\beta;\mu} u^\mu$.

For a small Schwarzschild hole on a circular orbit (radius $b$) in the field of another Schwarzschild black hole (mass $M_{\rm ext} \gg M$),

$$\langle \dot{M} \rangle = \frac{32}{5} \left( \frac{M}{M_{\rm ext}} \right)^6 V^{18} \frac{(1 - V^2)(1 - 2V^2)}{(1 - 3V^2)^2}$$

$$\langle \dot{J} \rangle = \frac{32}{5} \left( \frac{M}{M_{\rm ext}} \right)^6 M_{\rm ext} V^{15} \frac{(1 - V^2)(1 - 2V^2)}{(1 - 3V^2)^2}$$

where $V = \sqrt{M_{\rm ext}/b}$.

In the slow-motion limit ($V \ll 1$), this agrees with Poisson & Sasaki (1995) and Alvi (2001).

# 9. SH/SM approximation (K)

For a Kerr black hole one must rely on the curvature formalism and solve the Teukolsky equation in the regime $M/\mathcal{R} \ll \chi$, where $\chi \equiv a/M \equiv J/M^2$.

To leading order it suffices to solve the time-independent Teukolsky equation; the dependence of $\psi_0$ on $v$ is inherited from the asymptotic conditions in the external universe, which are encoded in $\mathcal{E}_{\alpha\beta}(v)$ and $\mathcal{B}_{\alpha\beta}(v)$.

The final result is

$$\begin{aligned}
\langle \dot{M} \rangle &= O(M^5/\mathcal{R}^5) \\
\langle \dot{J} \rangle &= -\frac{2}{45} M^5 \chi \Big[ 8(1+3\chi^2)\langle E_1 + B_1 \rangle - 3(4+17\chi^2)\langle E_2 + B_2 \rangle \\
&\qquad + 15\chi^2 \langle E_3 + B_3 \rangle \Big] \\
&= O(M^5/\mathcal{R}^4)
\end{aligned}$$

where

$$E_1 = \mathcal{E}_{\alpha\beta}\mathcal{E}^{\alpha\beta}, \qquad E_2 = \mathcal{E}_{\alpha\beta}s^\beta \mathcal{E}^\alpha{}_\gamma s^\gamma, \qquad E_3 = \left(\mathcal{E}_{\alpha\beta}s^\alpha s^\beta\right)^2$$

$$B_1 = \mathcal{B}_{\alpha\beta}\mathcal{B}^{\alpha\beta}, \qquad B_2 = \mathcal{B}_{\alpha\beta}s^\beta \mathcal{B}^\alpha{}_\gamma s^\gamma, \qquad B_3 = \left(\mathcal{B}_{\alpha\beta}s^\alpha s^\beta\right)^2$$

These results were previously obtained by D'Eath (1996).

For a small Kerr hole on an equatorial, circular orbit (radius $b$) in the field of a Schwarzschild black hole (mass $M_{\rm ext} \gg M$),

$$\langle \dot{M} \rangle = -\epsilon \frac{8}{5} \left( \frac{M}{M_{\rm ext}} \right)^5 \chi (1 + 3\chi^2) V^{15} \frac{(1 - 2V^2)\left(1 - \frac{4+27\chi^2}{4+12\chi^2} V^2\right)}{(1 - 3V^2)^2}$$

and $\langle \dot{J} \rangle = \langle \dot{M} \rangle / \Omega$, where $\epsilon \equiv \hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{s}} = \pm 1$ and $M_{\rm ext}\Omega = \epsilon V^3$.

In the slow-motion limit ($V \ll 1$), this agrees with Tagoshi et al (1997) and Alvi (2001).

# 10. SH/SM approx. (S vs K)

For a Kerr black hole we have obtained

$$\langle \dot{M} \rangle = O(M^5/\mathcal{R}^5), \qquad \langle \dot{J} \rangle = O(M^5/\mathcal{R}^4)$$

For a Schwarzschild black hole we have obtained

$$\langle \dot{M} \rangle = O(M^6/\mathcal{R}^6), \qquad \langle \dot{J} \rangle = O(M^6/\mathcal{R}^5)$$

These scalings can be understood by examining the special case of **rigid rotation** with an angular velocity $\Omega$, for which

$$\langle \dot{M} \rangle = \Omega(\Omega - \Omega_{\mathrm{H}})C, \qquad \langle \dot{J} \rangle = (\Omega - \Omega_{\mathrm{H}})C, \qquad C = O(M^6/\mathcal{R}^4)$$

For Kerr, the SH/SM approximation implies $\Omega \ll \Omega_H$, so that

$$\langle \dot{M} \rangle = -\Omega\Omega_{\rm H} C = O(M^5/\mathcal{R}^5), \qquad \langle \dot{J} \rangle = -\Omega_{\rm H} C = O(M^5/\mathcal{R}^4)$$

For Schwarzschild, $\Omega_{\rm H} = 0$ and

$$\langle \dot{M} \rangle = \Omega^2 C = O(M^6/\mathcal{R}^6), \qquad \langle \dot{J} \rangle = \Omega C = O(M^6/\mathcal{R}^5)$$

The scalings are thus naturally explained.

These observations were first made by Thorne and then elaborated on by Alvi (2001).