Quasi-local contribution to the gravitational self-force

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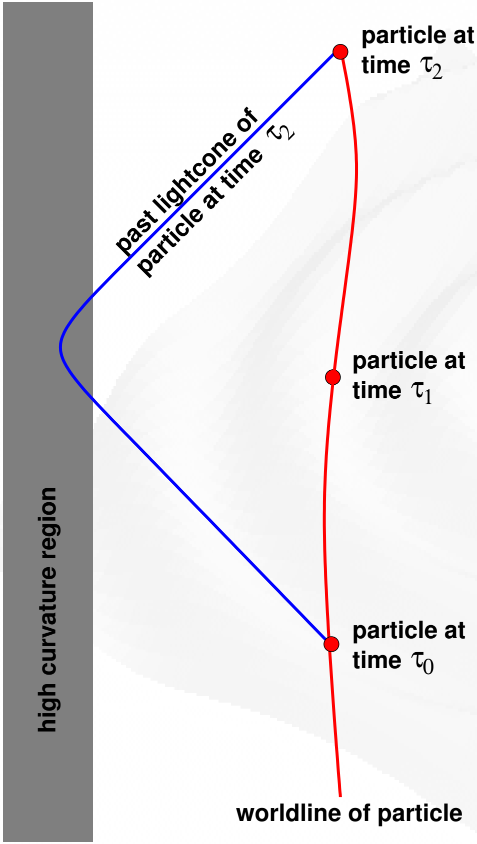
Outline

- The “trouble” with self-force
- The Poisson-Wiseman approach
- Calculating Hadamard’s V
- Convergence of expansions
- Summary

The “trouble” with self-force

- Consider a point particle carrying charge q coupled to a massless field ϕ and moving freely in a curved background geometry.
- In the absence of curvature and external matter, the particle does not interact with itself.
- However, in a curved background, even in vacuum, the charge can exert a force on itself because:
  - light-cones can be deformed sufficiently that the particle can be in it’s own null past.
  - Huygen’s principle fails in curved spacetimes, so the field propagates in time-like as well as null directions.
- The interaction of the particle at the present with its own field from the past gives rise to a self-force.

The “trouble” with self-force (cont.)



- The worldline of a particle and its past lightcone at τ_2 .
- The field generated by the particle at τ_1 can affect the particle at τ_2 because massless fields have time-like propagating components in curved space.
- At time τ_0 , the particle is light-like separated from its future self (at τ_2), so the light-like propagating components of the field from τ_0 can also interact with the particle at τ_2 .

The "trouble" with self-force (cont.)

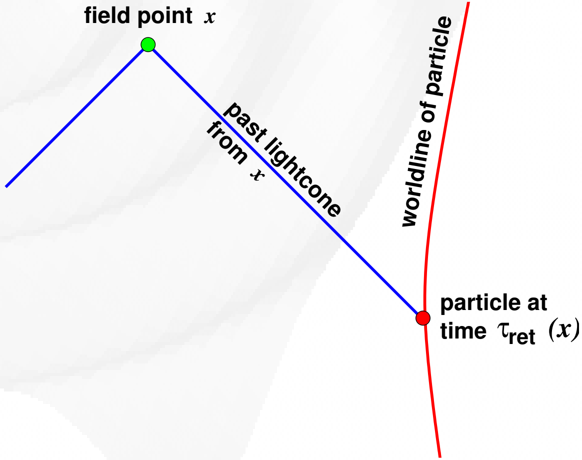
- Schematically, the self-force is given by the gradient of the tail field,

$$f^\alpha(x(\tau)) \sim q\,\nabla^\alpha \phi_{\text{tail}}(x(\tau)). \tag{1}$$

- The tail field at x from the particle on worldline $x'(\tau')$ is

$$\phi_{\text{tail}}(x) \sim q \lim_{\varepsilon\to 0^+} \int_{-\infty}^{\tau_{\text{ret}}(x)-\varepsilon} d\tau'\, G_{\text{ret}}[x, x'(\tau')], \tag{2}$$

  - G_{ret} is the retarded Green's function for the field ϕ
  - $\tau_{\text{ret}}(x)$ is the proper time at which the past light-cone from x intersects the world-line $x'(\tau')$.



The "trouble" with self-force (cont.)

- The ε limit in

$$f^\alpha \sim q^2 \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\tau-\varepsilon} d\tau' \, \nabla^\alpha G_{\mathrm{ret}}[x(\tau), x'(\tau')], \tag{3}$$

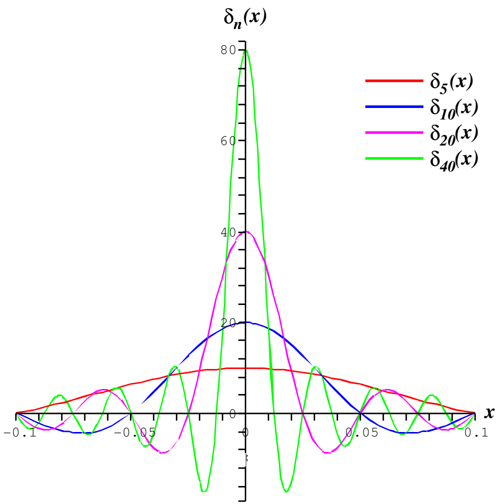
  allows us to formally avoid a singular distributional contribution to $\nabla^\alpha G_{\mathrm{ret}}[x(\tau), x'(\tau')]$ when $x(\tau') = x(\tau)$

- Usually, however, one must approximate G_{ret} by a finite sum over the mode (eigenfunction) solutions of the wave function

$$G_{\mathrm{ret}}(x, x') \sim \sum_k u(k; x)\bar{u}(k, x'). \tag{4}$$

- Although the modes are all finite, they are localized in frequency space rather than coördinate space, and are "aware" that they must sum to an infinite value at $x' = x$. This causes the number of modes needed to achieve a finite accuracy diverge as $\varepsilon \to 0$.

The “trouble” with self-force (cont.)



- To illustrate, consider

$$\delta(x) = \int_{-\infty}^{\infty} dk\ e^{2\pi i k x}.$$

- We approximate by truncating at a finite mode number n ,

$$\delta_n(x) \equiv \int_{-n}^{n} dk\ e^{2\pi i k x} = \frac{\sin(2\pi n x)}{\pi x}.$$

- We need $n \gg 1/x$ to have $\delta_n(x)$ settle down at x . In the limit $x \rightarrow 0$, we need $n \rightarrow \infty$ to achieve a given accuracy.

The Poisson-Wiseman Approach

- One method for solving this problem was proposed by Poisson and Wiseman, First Capra Ranch Radiation Reaction Meeting, Frank Capra's Ranch, (1998) .
- They noted that within a normal neighbourhood of x the retarded Green's function can be expressed in the Hadamard form

$$G_{\text{ret}}(x,x') \sim \Theta(t-t')\Big[U(x,x')\,\delta[\sigma(x,x')] - V(x,x')\,\Theta[-\sigma(x,x')]\Big]. \tag{5}$$

Here:

- $\Theta(t-t')$ is 1 for x in the causal future of x' and 0 otherwise,
- $\sigma(x,x')$ is half the square of the geodesic distance from x to x' , and
- $U(x,x')$ and $V(x,x')$ are everywhere smooth.

The Poisson-Wiseman Approach (cont.)

- Let us break the self-force integral into two parts

$$f^\alpha \sim q^2 \left[\int_{-\infty}^{\tau-\Delta\tau} d\tau' \, \nabla^\alpha G_{\text{ret}}[x, x'(\tau')] + \lim_{\varepsilon\to 0^+} \int_{\tau-\Delta\tau}^{\tau-\varepsilon} d\tau' \, \nabla^\alpha G_{\text{ret}}[x, x'(\tau')] \right] \tag{6}$$

- Provided that we choose $\Delta\tau$ such that $x(\tau')$ is within the normal neighbourhood of $x(\tau)$ for all $\tau - \Delta\tau < \tau' < \tau$, we can use for the second integral above

$$G_{\text{ret}}(x, x') \sim \Theta(t - t') \Big[U \; \delta[\sigma] - V \; \Theta[-\sigma] \Big] . \tag{7}$$

- Notice, however, that the particle moves on a timelike world-line, so $\sigma[x(\tau), x'(\tau')] = 0$ only for $\tau' = \tau$, which is not integrated over. Thus, the U term can never contribute to the self-force.

The Poisson-Wiseman Approach

- We can therefore write the self-force as

$$f^\alpha \sim q^2 \left[\int_{-\infty}^{\tau - \Delta\tau} d\tau' \, \nabla^\alpha G_{\text{ret}}[x, x'(\tau')] - \int_{\tau-\Delta\tau}^{\tau} d\tau' \, \nabla^\alpha V[x, x'(\tau')] \right] \tag{8}$$

- We have dropped the Θ s from the second term because they are always unity in the domain of integration. We have dropped the ε limit because $\nabla^\alpha V$ is everywhere finite, so the end point does not change the value of the integral.
- $G_{\text{ret}}(x, x')$ can now be approximated throughout the domain of the first integral to any given (non-zero) accuracy with a finite (but possibly large) number of terms, because we are no longer approaching arbitrarily close to the singularity at $x' = x$.
- How can we approximate $\nabla^\alpha V$?

Calculating Hadamard's V

- $\nabla^\alpha V$ is a bitensor, meaning it acts as a two-index tensor at both x and at x' .
- There is an extensive literature on series expansions of bitensors. One begins with a formal power series in $\sigma(x, x')$,

$$V(x, x') = \sum_{n=0}^{\infty} V^n(x, x')\sigma^n. \tag{9}$$

- Substituting this expansion into the Hadamard form for G_{ret} which in turn is substituted into the Green's function differential equation and equating powers of σ , one gets a set of differential equations for the coefficients $V^n(x, x')\sigma^n$.

Calculating Hadamard's V (cont.)

$$\mathfrak{d}U = 0, \tag{10}$$

$$V^0 + \mathfrak{d}V^0 = -\frac{1}{2}\mathfrak{D}U, \tag{11}$$

$$(n+1)V^n + \mathfrak{d}V^n = -\frac{1}{2n}\mathfrak{D}V^{n-1}, \tag{12}$$

where

- $\mathfrak{d} \equiv \sigma^{;\mu}\left[\nabla_\mu - \frac{1}{2}(\ln\Delta)_{;\mu}\right]$,
- $\mathfrak{D}$ is the wave operator for the field ϕ ,
- $\Delta \equiv \det\left[-g_\alpha{}^{\alpha'}(x,x')\sigma_{;\alpha'\beta}\right]$ is the Van Vleck - Morrette determinant, and
- $g^\alpha{}_{\alpha'}(x,x')$ is the bivector of parallel displacement, which is defined such that $g^\alpha{}_{\alpha'}u^{\alpha'}(x') = u^\alpha(x)$ where $u^\alpha \equiv dx^\alpha/d\tau$.

Calculating Hadamard's V (cont.)

- The solution to the U is well known to be

$$U = \Delta^{1/2}. \tag{13}$$

- We look for power series solutions to the equations for the V^n s, i.e. assume

$$V^n(x, x') = v^n + v^n_\lambda \sigma^{;\lambda} + \frac{1}{2} v^n_{\lambda\rho} \sigma^{;\lambda} \sigma^{;\rho} + \dots . \tag{14}$$

- Following e.g. Allen, Follaci and Ottewill, PRD 38, (1988) , we can use known expansions of bitensors, such as

$$\Box \Delta^{1/2} = \frac{1}{240} C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} g_{\alpha\beta} \sigma^{;\alpha} \sigma^{;\beta} + O\left[(\sigma^{;\alpha})^3 \right], \tag{15}$$

to solve the differential equations for V^n s order by order for the v^n coefficients.

Calculating Hadamard's V (cont.)

- Results to date:
  - Roberts, Class. Quantum Grav. 6, (1989) stopped just short calculating the leading term in the expansion of the normal neighbourhood integral. A trivial extension of his result gives:

$$\begin{aligned} f^{\alpha}_{\mathrm{QL}} &\equiv -q^2 \int_{\tau-\Delta\tau}^{\tau} \nabla^{\alpha} V\left(z(\tau), z(\tau')\right) d\tau' \\ &= -\frac{q^2}{4} C^{\alpha}{}_{\beta\gamma}{}^{\delta}{}_{;\delta} u^{\beta} u^{\gamma} \Delta\tau^2 + O(\Delta\tau^3) \end{aligned} \tag{16}$$

  - P. R. Anderson and Hu, Phys. Rev. D. 69 (2004) used a Hadamard-WKB expansion for the Euclidean Green's function to calculate V to sixth order for a minimally-coupled scalar charge in a Schwarzschild background. They did not explicitly calculate the self-force in that paper, however.

Calculating Hadamard's V (cont.)

- Anderson, Flanagan and Ottewill, PRD 71, (2005) have recently found the first two non-vanishing orders for this part of the gravitational self-force,

$$\begin{aligned} f^{\alpha}_{QL}(\tau,\Delta\tau) \;=\; & -\mu^2(\delta^{\alpha\beta}+u^\alpha u^\beta)\Big\{ C_{\beta\gamma\delta\varepsilon}\, C_{\sigma}{}^{\gamma}{}_{\rho}{}^{\varepsilon}\, u^\delta u^\sigma u^\rho\, \Delta\tau^2 \\ & + u^\gamma u^\delta \Big[\frac{1}{6} C_{\gamma\mu\delta\nu} C_{\varepsilon}{}^{\mu}{}_{\sigma}{}^{\nu}{}_{;\beta}\, u^\varepsilon u^\sigma - \frac{3}{20} C_{\beta\gamma\mu\delta;\nu} C_{\varepsilon}{}^{\mu}{}_{\sigma}{}^{\nu} u^\varepsilon u^\sigma \\ & + \frac{1}{3}\left(\frac{1}{2} C_{\mu\nu\gamma\lambda} C^{\mu\nu}{}_{\delta}{}^{\lambda}{}_{;\beta} + C_{\mu\varepsilon\gamma\lambda} C^{\mu}{}_{\sigma\delta}{}^{\lambda}{}_{;\beta} u^\varepsilon u^\sigma \right) \\ & - \frac{19}{60}\left(\frac{1}{2} C_{\mu\nu\gamma\lambda} C^{\mu\nu}{}_{\delta\beta}{}^{;\lambda} + C_{\mu\varepsilon\gamma\lambda} C^{\mu}{}_{\sigma\delta\beta}{}^{;\lambda} u^\varepsilon u^\sigma \right) \Big] \Delta\tau^3 \\ & + O(\Delta\tau^4) \Big\}. \end{aligned} \tag{17}$$

Convergence of expansions

- We now know how to expand both integrals in

$$f^\alpha \sim q^2 \Bigg[\int_{-\infty}^{\tau-\Delta\tau} d\tau' \, \nabla^\alpha G_{\mathsf{ret}}[x, x'(\tau')] + \lim_{\varepsilon \to 0^+} \int_{\tau-\Delta\tau}^{\tau-\varepsilon} d\tau' \, \nabla^\alpha G_{\mathsf{ret}}[x, x'(\tau')] \Bigg] \tag{18}$$

- The larger $\Delta\tau$ is chosen, the better the expansion of the first integral converges. The smaller $\Delta\tau$ is chosen, the better the expansion of the second integral converges.
- Recall that $\tau - \Delta\tau$ must be within the normal neighbourhood of τ for a valid expansion of the second integral.
- Is there any $\Delta\tau$ such that both expnasions converge sufficiently well?

Convergence of expansions (cont.)

- Recently, Alan Wiseman and I have investigated this question by examining
  - QL convergence for:
    - minimally coupled scalar field
    - Schwarzschild background (mass M)
    - static particle and circular geodesic (Schwarzschild radius R).
  - mode-sum convergence for:
    - minimally coupled scalar field
    - Schwarzschild background
    - static particle at $6M$
    - order M Green's function
- See gr-qc/0506136 for details (Warning: there are typos we discovered in proof that are not corrected on archived version yet.)

Convergence of expansions (cont.)

- For QL expansion, we use the Anderson and Hu result. Although expansion is to seventh order, only two terms remain for both static particle and circular geodesic orbits.
- E.g. for the static particle, the only non-vanishing parts are the fifth and seventh order terms of the radial component

$$f^r_{\mathrm{QL}}[5] \equiv \frac{9}{2240}\frac{q^2M^2}{R^{15}}(4R-11M)(R-2M)^5\,\Delta\tau^5, \tag{19}$$

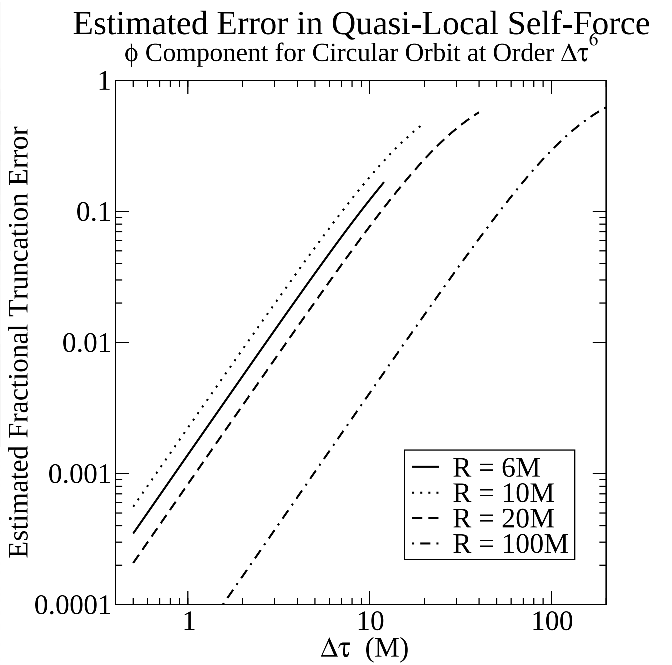
$$f^r_{\mathrm{QL}}[7] \equiv \frac{1}{3360}\frac{q^2M^2}{R^{20}}\left(20R^3-195MR^2+598M^2R-585M^3\right)(R-2M)^6\,\Delta\tau^7. \tag{20}$$

- To estimate truncation error, we use

$$\varepsilon \equiv \frac{f^r_{\mathrm{QL}}[7]}{f^r_{\mathrm{QL}}[5]+f^r_{\mathrm{QL}}[7]}. \tag{21}$$

Convergence of expansions (cont.)

- This is an estimate of the local truncation error, i.e. an estimated bound on the fractional error of not including the next term in the series. Unfortunately, there is no way to estimate global truncation error.



Convergence of expansions (cont.)

- For the mode sum expansion, we use the multipole expansion of order M Green's function for a static scalar charge.
- The sum up to the N^{th} multipole for the mode-sum integral gives

$$f_r(\Delta\tau, N) = -\frac{q^2}{R^2 u^t}\frac{(N+1)}{4\sqrt{2}}\int_{-1}^{\cos\beta_{\text{max}}}\frac{P_{N+1}(\xi)-P_N(\xi)}{(1-\xi)^{3/2}}d\xi \,, \tag{22}$$

where $u^t = 1 - M/R + O[M^2]$ and

$$\cos\beta_{\text{max}} \equiv 1 - \frac{1}{2}\left(\frac{\Delta\tau}{u^t\, R}\right)^2 . \tag{23}$$

Convergence of expansions (cont.)

- On dimensional grounds, one can argue that the radial component of the self force should be of the form

$$f_r = \lambda \frac{q^2 M}{R^3}\,, \tag{24}$$

- We know in this case that $\lambda = 0$.
- Taking $\Delta\tau = R$, we get

Modes	$(M^2 u^t/q^2) f_r(\Delta\tau, N)$	bound on λ
10	0.0017	0.36
50	0.00078	0.17
100	0.00056	0.12
200	0.00041	0.088
300	0.00036	0.078

Summary

- The Poisson-Wiseman approach breaks the self-force into two integrals, the near-past (QL) integral and the far-past integral.
- The near-past integral admits a power series expansion within a normal neighbourhood of the particle. This expansion seems to converge well out to near the edge of the normal neighbourhood.
- The far-past integral admits a mode sum expansion. This did not converge very well if a substantial part of the normal neighbourhood was within the domain of the integral.
- There are many open issues:
  - Can the mode-sum convergence be accelerated?
  - To what order is the quasi-local expansion feasible?
  - Are convergence calculations for the simple cases we looked at indicative of more interesting cases?
  - Gauge problems for the gravitational case?