Schwarzschild Perturbations in the Lorenz Gauge

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Talk Plan

- Why Lorenz Gauge?
- Schwarzschild perturbations reformulated in the Lorenz gauge:
  Now both odd- and even- parity modes.
- Code for time evolution of Lorenz-gauge perturbations from a point particle:
  Circular orbits in Schwarzschild.
- Future work: Self-force calculations, generic orbits, Kerr black hole.

Gauge freedom in Perturbation Theory

- ☐ $g_{\alpha\beta} = g_{\alpha\beta}^{(\mathrm{bckgrnd})} + \boxed{h_{\alpha\beta}} \leftarrow$ 10 functions, but only 2 "physical" dof

  In particular, $h_{\alpha\beta}$ and $h_{\alpha\beta} + 2\xi_{(\alpha;\beta)}$ represent the same physical perturbation, for any ("small", differentiable) displacement field ξ_α

- ☐ Useful gauges in black hole perturbation theory

  - Regge-Wheeler gauge: $h_{\alpha\beta} = \sum_{lmi} h^{(i)lm}(r,t)\, Y_{\alpha\beta}^{(i)lm}(\theta,\varphi) \quad \rightarrow \quad h^{(i1,i2,i3,i4)} = 0$
  - Radiation gauge: $h_{\alpha\beta}\, l^{\beta} = 0 \quad \text{or} \quad h_{\alpha\beta}\, n^{\beta} = 0$
  - Lorenz ("harmonic") gauge: $\bar{h}_{\alpha\beta}{}^{;\beta} = 0$

Grav. self-force and gauge freedom

- $$F^{\mu}_{\rm self} = \lim_{x\to prtcl} \nabla^{\mu\alpha\beta}\left(h_{\alpha\beta} - h^{\rm dir}_{\alpha\beta}\right) \qquad \text{(MSTQW 97)}$$

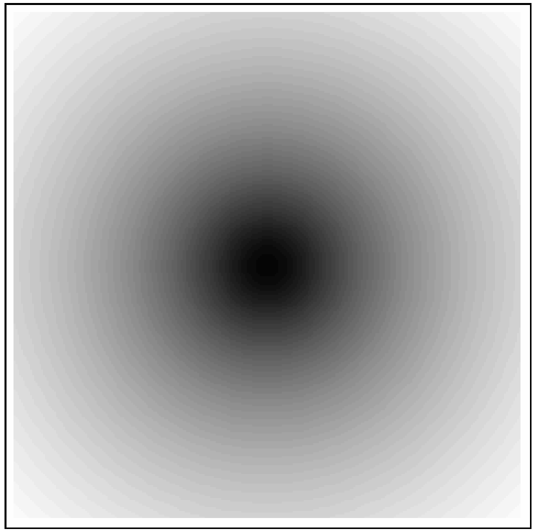
  $$= \lim_{x\to prtcl} \nabla^{\mu\alpha\beta}\left(h_{\alpha\beta} - h^{\rm S}_{\alpha\beta}\right) \qquad \text{(Detweiler \& Whiting 03)}$$

  Lorenz gauge

  Lorenz gauge

- Problem: Standard BH perturbation techniques give $h_{\alpha\beta}$ in other gauges.
- Solutions attempted:
  - Transform h (RW/Radiation) $\rightarrow$ h (Lorenz), and evaluate contribution of difference to SF
  - Transform h^S (Lorenz) $\rightarrow$ h^S (RW/Radiation), and try make sense of SF in RW/Rad gauges
- Our approach: Step back; reformulate BH perturbation theory in Lorenz gauge.

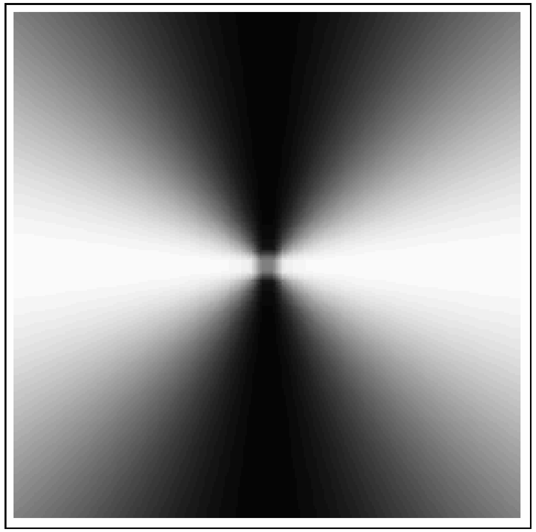
Why Lorenz gauge? (I)



Lorenz gauge:

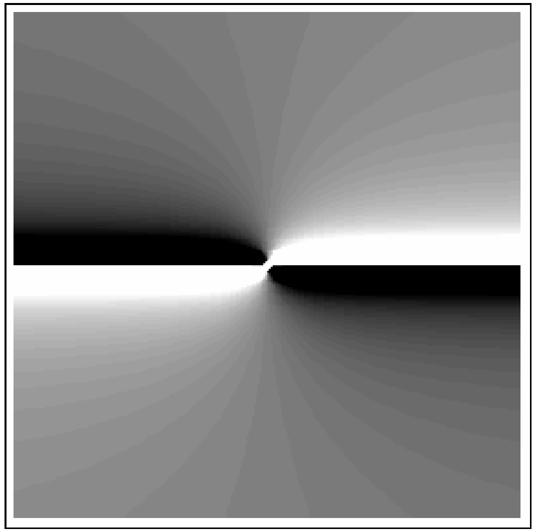
Particle singularities look like particle singularities:

Pointlike and isotropic



Regge-Wheeler gauge:

Particle singularities are not isotropic. Field depends on direction of approach to singularity



Radiation gauge:

Typically, particle singularities are not even isolated!

Why Lorenz gauge? (II)

- RW/Zerilli/Moncrief/Teukolsky variables are discontinuous at the particle. The (mode decomposed) Lorenz-gauge MP is continuous. No δ-function derivatives in the source term of the field equations.
- No need to resort to complicated MP reconstruction procedures, as when working with Moncrief/Teukolsky variables.
- In particular, no need to take derivatives of numerical integration variables – advantage in numerical implementation.
- Field equations manifestly hyperbolic.

- Price to pay: Field equations remain coupled.

Schwarzschild perturbations in the Lorenz gauge: formulation

1. Linearized Einstein equations in the Lorenz gauge

Linearize Einstein's equations in perturbation $h_{\alpha\beta}(x)$ about BH background $g_{\alpha\beta}$. Take source to be a point particle moving on a geodesic $x = x_p(\tau)$ of $g_{\alpha\beta}$. Get

$$\begin{aligned}\Box\bar{h}_{\alpha\beta} + 2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\bar{h}_{\mu\nu} + g_{\alpha\beta}\bar{h}^{\mu\nu}{}_{;\mu\nu} - 2g^{\mu\nu}\bar{h}_{\mu(\alpha;\nu\beta)} \\ = -16\pi\mu\int_{-\infty}^{\infty}(-g)^{-1/2}\,\delta^4[x^\mu - x_p^\mu(\tau)]u_\alpha u_\beta\, d\tau \equiv S_{\alpha\beta},\end{aligned}$$

where

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}h.$$

Impose Lorenz gauge condition,

$$g^{\beta\gamma}\bar{h}_{\alpha\beta;\gamma} = 0.$$

Get

$$\boxed{\Box\bar{h}_{\alpha\beta} + 2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\bar{h}_{\mu\nu} = S_{\alpha\beta}}$$

2. Tensor-harmonic decomposition

$$\bar{h}_{\alpha\beta} = \sum_{l,m} \sum_{i=1}^{10} \bar{h}^{(i)lm}(r,t)\, Y^{(i)lm}_{\alpha\beta}(r;\theta,\varphi)$$

$$S_{\alpha\beta} = \sum_{l,m} \sum_{i=1}^{10} S^{(i)lm}(r,t)\, Y^{(i)lm}_{\alpha\beta}(r;\theta,\varphi)$$

3. Make sure variables are suitable for numerical time-evolution

Redefine

$$\bar{h}_{\alpha\beta} = \sum_{l,m} \sum_{i=1}^{10} \frac{R^{(i)}(r)}{r}\, \bar{h}^{(i)lm}(r,t)\, Y^{(i)lm}_{\alpha\beta}(r;\theta,\varphi),$$

with

$$R^{(2,5,9)} = f^{-1}, \quad R^{(3)} = f^{-2}, \quad R^{(i)} = 1 \text{ for rest.}$$

Then all $\bar{h}^{(i)lm}$ are dimensionless and $\propto$ const at both $r \to 2M$ and $r \to \infty$.

4. Get separated equations for the $h^{(i)}$'s

$$\boxed{\Box^{2d}_{\mathrm{sc}}\bar{h}^{(i)lm} + \mathcal{M}^{(i)}_{\;(j)}\bar{h}^{(j)lm} = \tilde{S}^{(i)lm},}$$

where

$$\Box^{2d}_{\mathrm{sc}} \equiv \partial_{uv} + \frac{f}{4}\left[\frac{f'}{r} + \frac{l(l+1)}{r^2}\right],$$

$$\tilde{S}^{(i)lm} = 4\pi r f R^{-1}\int_{\infty}^{\infty} d\tau\, r_p^{-2}\delta(t-t_p)\delta(r-r_p)u_\alpha u_\beta \eta^{\alpha\mu}\eta^{\beta\nu}[Y^{(i)}_{\mu\nu}(\Omega_p)]^*,$$

$$\mathcal{M}^{(1)}_{(j)}\bar{h}^{(j)} = \frac{1}{2} f f' \bar{h}^{(1)}_{,r} - \frac{1}{2} f' i \bar{h}^{(2)}_{,t} + \frac{f^2}{2r^2}\left(\bar{h}^{(1)} - \bar{h}^{(3)} - \bar{h}^{(5)} - f\bar{h}^{(6)}\right),$$

$$\mathcal{M}^{(2)}_{(j)}\bar{h}^{(j)} = \frac{1}{2} f f' \bar{h}^{(2)}_{,r} + \frac{1}{2} i f' \bar{h}^{(1)}_{,t} + \frac{f^2}{2r^2}\left(\bar{h}^{(2)} - \bar{h}^{(4)}\right),$$

$$\mathcal{M}^{(3)}_{(j)}\bar{h}^{(j)} = \frac{1}{2} f f' \bar{h}^{(3)}_{,r} + \frac{1}{2r^2}\left[1 - 8M/r + 10(M/r)^2\right]\bar{h}^{(3)} - \frac{f^2}{2r^2}\left[\bar{h}^{(1)} - \bar{h}^{(5)} - (1 - 4M/r)\bar{h}^{(6)}\right],$$

$$\mathcal{M}^{(4)}_{(j)}\bar{h}^{(j)} = \frac{1}{4} f' f \bar{h}^{(4)}_{,r} + \frac{1}{4} f' i \bar{h}^{(5)}_{,t} - \frac{3}{4} f'(f/r)\bar{h}^{(4)} - \frac{1}{2} l(l+1)\,(f/r^2)\bar{h}^{(2)},$$

$$\mathcal{M}^{(5)}_{(j)}\bar{h}^{(j)} = \frac{1}{4} f f' \bar{h}^{(5)}_{,r} - \frac{1}{4} f' i \bar{h}^{(4)}_{,t} + \frac{f}{r^2}(1 - 3.5M/r)\bar{h}^{(5)} - \frac{f}{2r^2}\, l(l+1)\left(\bar{h}^{(1)} - \bar{h}^{(3)} - f\bar{h}^{(6)}\right) - \frac{f^2}{2r^2}\bar{h}^{(7)},$$

$$\mathcal{M}^{(6)}_{(j)}\bar{h}^{(j)} = -\frac{f}{2r^2}\left[\bar{h}^{(1)} - \bar{h}^{(5)} - (1 - 4M/r)\left(f^{-1}\bar{h}^{(3)} + \bar{h}^{(6)}\right)\right],$$

$$\mathcal{M}^{(7)}_{(j)}\bar{h}^{(j)} = -\frac{f}{2r^2}\left(\bar{h}^{(7)} + \lambda\,\bar{h}^{(5)}\right),$$

EVEN PARITY

ODD PARITY

$$\mathcal{M}^{(8)}_{(j)}\bar{h}^{(j)} = \frac{1}{4} f f' \left(\bar{h}^{(8)}_{,r} - \frac{3}{r}\bar{h}^{(8)} - i f^{-1}\bar{h}^{(9)}_{,t}\right),$$

$$\mathcal{M}^{(9)}_{(j)}\bar{h}^{(j)} = \frac{1}{4} f' \left(f\bar{h}^{(9)}_{,r} + i\bar{h}^{(8)}_{,t}\right) + \frac{f}{r^2}\left[(1 - 3.5M/r)\bar{h}^{(9)} - (f/2)\bar{h}^{(10)}\right],$$

$$\mathcal{M}^{(10)}_{(j)}\bar{h}^{(j)} = -\frac{f}{2r^2}\left(\bar{h}^{(10)} + \lambda\,\bar{h}^{(9)}\right).$$

5. Write gauge conditions in mode-decomposed form,

$$\bar{h}_{t\beta}{}^{;\beta} = \frac{i}{2fr} Y^{lm}(\theta,\varphi) \times \left[i\bar{h}^{(1)}_{,t} + i\bar{h}^{(3)}_{,t} + f\left(\bar{h}^{(2)}_{,r} + \bar{h}^{(2)}/r - \bar{h}^{(4)}/r\right)\right] = 0,$$

$$\bar{h}_{r\beta}{}^{;\beta} = -\frac{1}{2f^2 r} Y^{lm}(\theta,\varphi) \times \left[i\bar{h}^{(2)}_{,t} - f\left(\bar{h}^{(1)}_{,r} - \bar{h}^{(3)}_{,r}\right) + (1-4M/r)\bar{h}^{(3)}/r - (f/r)\left(\bar{h}^{(1)} - \bar{h}^{(5)} - 2f\bar{h}^{(6)}\right)\right] = 0$$

$$\begin{aligned}(\sin\theta\,\bar{h}_{\theta\beta}{}^{;\beta})_{,\theta} + (\bar{h}_{\varphi\beta}{}^{;\beta}/\sin\theta)_{,\varphi} &= \frac{1}{2f}\,\sin\theta\,Y^{lm}(\theta,\varphi) \\ &\times \left[i\bar{h}^{(4)}_{,t} - f\left(\bar{h}^{(5)}_{,r} + 2\bar{h}^{(5)}/r + l(l+1)\,\bar{h}^{(6)}/r - \bar{h}^{(7)}/r\right)\right] = 0,\end{aligned}$$

$$\begin{aligned}(\bar{h}_{\theta\beta}{}^{;\beta})_{,\varphi} - (\bar{h}_{\varphi\beta}{}^{;\beta})_{,\theta} &= \frac{-i}{2f}\,\sin\theta\,Y^{lm}(\theta,\varphi) \\ &\times \left[i\bar{h}^{(8)}_{,t} + f\left(\bar{h}^{(9)}_{,r} + 2\bar{h}^{(9)}/r - \bar{h}^{(10)}/r\right)\right] = 0\end{aligned}$$

… and use them to reduce the system of field equations:

$$\mathcal{M}^{(1)}_{(j)}\bar{h}^{(j)} = \frac{1}{2}ff'\bar{h}^{(3)}_{,r} + \frac{f}{2r^2}(1-4M/r)(\bar{h}^{(1)}-\bar{h}^{(5)}) - \frac{1}{2r^2}(1-6M/r+12M/r^2)\bar{h}^{(3)} + \frac{f^2}{2r^2}(6M/r-1)\bar{h}^{(}$$

$$\mathcal{M}^{(3)}_{(j)}\bar{h}^{(j)} = \frac{1}{2}ff'\bar{h}^{(3)}_{,r} + \frac{1}{2r^2}\left[1-8M/r+10(M/r)^2\right]\bar{h}^{(3)} - \frac{f^2}{2r^2}\left[\bar{h}^{(1)}-\bar{h}^{(5)}-(1-4M/r)\bar{h}^{(6)}\right],$$

$$\mathcal{M}^{(5)}_{(j)}\bar{h}^{(j)} = \frac{f}{r^2}\left[(1-4.5M/r)\bar{h}^{(5)} - \frac{1}{2}l(l+1)(\bar{h}^{(1)}-\bar{h}^{(3)}) + \frac{1}{2}(1-3M/r)\left(l(l+1)\bar{h}^{(6)}-\bar{h}^{(7)}\right)\right],$$

$$\mathcal{M}^{(6)}_{(j)}\bar{h}^{(j)} = -\frac{f}{2r^2}\left[\bar{h}^{(1)}-\bar{h}^{(5)}-(1-4M/r)\left(f^{-1}\bar{h}^{(3)}+\bar{h}^{(6)}\right)\right],$$

$$\mathcal{M}^{(7)}_{(j)}\bar{h}^{(j)} = -\frac{f}{2r^2}\left(\bar{h}^{(7)}+\lambda\,\bar{h}^{(5)}\right),$$

$$\xcancel{\mathcal{M}^{(2)}_{(j)}\bar{h}^{(j)} = \frac{1}{2}ff'\bar{h}^{(2)}_{,r} + \frac{1}{2}\,if'\bar{h}^{(1)}_{,t} + \frac{f^2}{2r^2}\left(\bar{h}^{(2)}-\bar{h}^{(4)}\right),}$$

$$\xcancel{\mathcal{M}^{(4)}_{(j)}\bar{h}^{(j)} = \frac{1}{4}f'f\bar{h}^{(4)}_{,r} + \frac{1}{4}\,f'i\bar{h}^{(5)}_{,t} - \frac{3}{4}f'(f/r)\bar{h}^{(4)} - \frac{1}{2}l(l+1)\,(f/r^2)\bar{h}^{(2)},}$$

EVEN PARITY

ODD PARITY

$$\xcancel{\mathcal{M}^{(9)}_{(j)}\bar{h}^{(j)} = \frac{f}{r^2}\,(1-4.5M/r)\,\bar{h}^{(9)} - \frac{f}{2r^2}\,(1-3M/r)\,\bar{h}^{(10)}.}$$

$$\xcancel{\mathcal{M}^{(10)}_{(j)}\bar{h}^{(j)} = -\frac{f}{2r^2}\left(\bar{h}^{(10)}+\lambda\,\bar{h}^{(9)}\right).}$$

$$\mathcal{M}^{(8)}_{(j)}\bar{h}^{(j)} = \frac{1}{4}ff'\left(\bar{h}^{(8)}_{,r} - \frac{3}{r}\bar{h}^{(8)} - if^{-1}\bar{h}^{(9)}_{,t}\right),$$

Circular equatorial orbit, Axially-symmetric (m=0) modes

Analytic solution for the axially-symmetric, odd-parity modes in the case of a circular orbit

Denote (for given l mode)

$$h^{(8)}_{m=0} \equiv \phi(r). \tag{1}$$

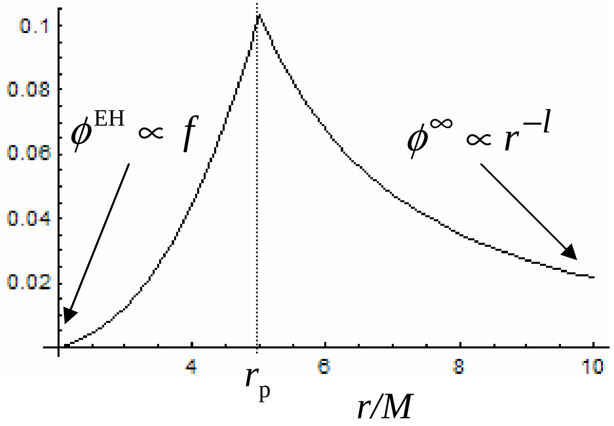
Then

$$\phi'' + V(r)\phi = \text{const} \times \delta(r - r_0). \tag{2}$$

Unique continuous solution with physical boundary conditions:

$$\phi(r) = \text{const} \times \begin{cases} \phi^{\rm EH}(r)\phi^{\infty}(r_p), & r \le r_p \\ \phi^{\infty}(r)\phi^{\rm EH}(r_p), & r \ge r_p, \end{cases} \tag{3}$$

$$\begin{aligned} \phi^{\rm EH}(r) &= \frac{x}{1+x}\sum_{n=0}^{l+1} \alpha_n^l x^n \qquad (x \equiv r/2M - 1) \\ \phi^{\infty}(r) &= \phi^{\rm EH} \ln f + \frac{1}{1+x}\sum_{n=0}^{l+1} \beta_n^l x^n. \end{aligned} \tag{4}$$



MP “reconstruction”

$$\boxed{h_{\alpha\beta} = h^{l=0,1}_{\alpha\beta} + \frac{1}{2r}\sum_{l=2}^{\infty}\sum_{m=-l}^{l} h^{lm}_{\alpha\beta},} \tag{1}$$

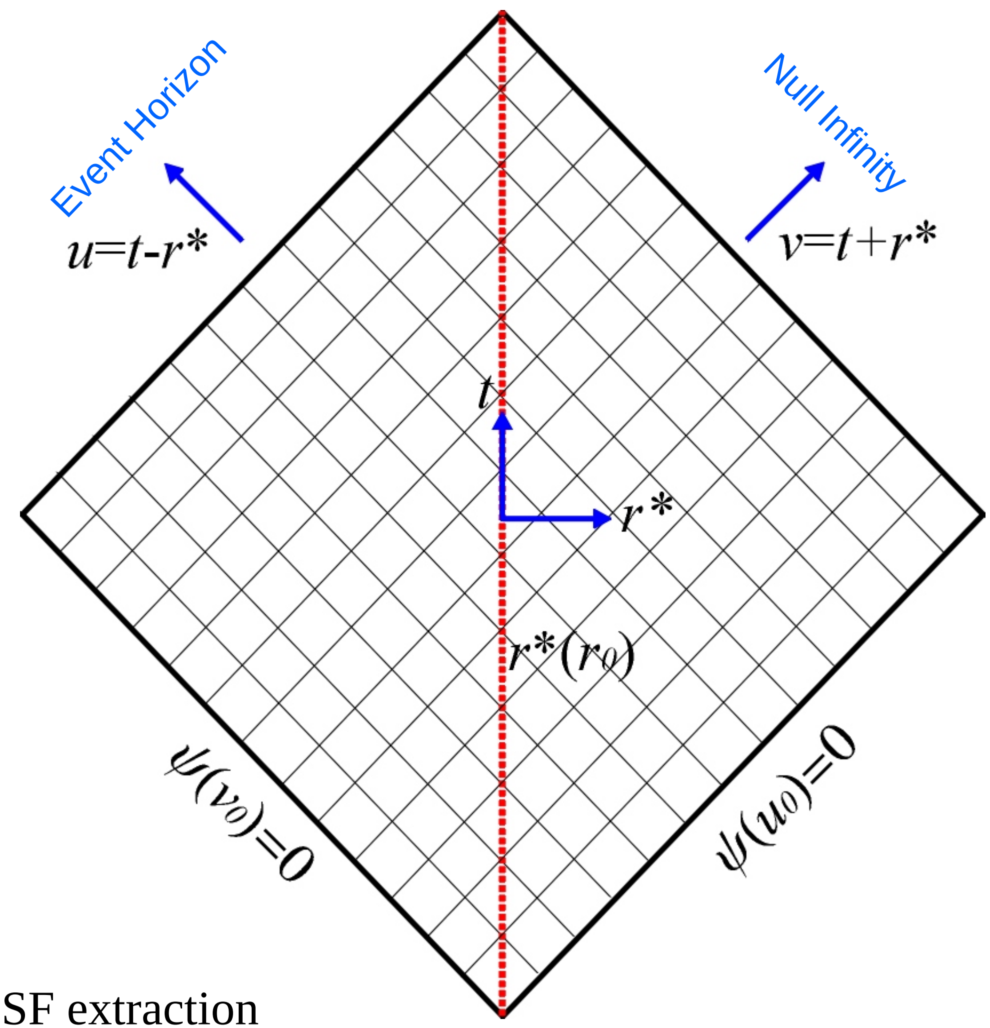
$$\begin{aligned}
h^{lm}_{tt} &= \left(\bar{h}^{(1)} + f\bar{h}^{(6)}\right) Y^{lm}, \\
h^{lm}_{tr} &= f^{-1}\bar{h}^{(2)} Y^{lm}, \\
h^{lm}_{rr} &= f^{-2}\left(\bar{h}^{(1)} - f\bar{h}^{(6)}\right) Y^{lm}, \\
h^{lm}_{t\theta} &= ir\left(\bar{h}^{(4)} Y^{lm}_{\mathrm{V1}} - \bar{h}^{(8)} Y^{lm}_{\mathrm{V2}}\right), \\
h^{lm}_{t\varphi} &= r\sin\theta\left(\bar{h}^{(4)} Y^{lm}_{\mathrm{V2}} - \bar{h}^{(8)} Y^{lm}_{\mathrm{V1}}\right), \\
h^{lm}_{r\theta} &= rf^{-1}\left(\bar{h}^{(5)} Y^{lm}_{\mathrm{V1}} + \bar{h}^{(9)} Y^{lm}_{\mathrm{V2}}\right) \\
h^{lm}_{r\varphi} &= -irf^{-1}\sin\theta\left(\bar{h}^{(5)} Y^{lm}_{\mathrm{V2}} + \bar{h}^{(9)} Y^{lm}_{\mathrm{V1}}\right) \\
h^{lm}_{\theta\theta} &= r^2\left[\left(f^{-1}\bar{h}^{(3)} - \bar{h}^{(6)}\right) Y^{lm} + \bar{h}^{(7)} Y^{lm}_{\mathrm{T1}} + \bar{h}^{(10)} Y^{lm}_{\mathrm{T2}}\right], \\
h^{lm}_{\theta\varphi} &= -ir^2\sin\theta\left(\bar{h}^{(7)} Y^{lm}_{\mathrm{T2}} + \bar{h}^{(10)} Y^{lm}_{\mathrm{T1}}\right), \\
h^{lm}_{\varphi\varphi} &= r^2\sin^2\theta\left[\left(f^{-1}\bar{h}^{(3)} - \bar{h}^{(6)}\right) Y^{lm} - \bar{h}^{(7)} Y^{lm}_{\mathrm{T1}} - \bar{h}^{(10)} Y^{lm}_{\mathrm{T2}}\right],
\end{aligned} \tag{2}$$

$$\begin{aligned}
Y^{lm}_{\mathrm{V1}} &\equiv \frac{1}{l(l+1)}\, Y^{lm}_{,\theta}, \\
Y^{lm}_{\mathrm{V2}} &\equiv \frac{i}{l(l+1)}\, \sin^{-1}\theta\, Y^{lm}_{,\varphi}, \\
Y^{lm}_{\mathrm{T1}} &\equiv \frac{1}{\lambda l(l+1)}\left[\sin\theta\left(\sin^{-1}\theta\, Y^{lm}_{,\theta}\right)_{,\theta} - \sin^{-2}\theta\, Y_{,\varphi\varphi}\right], \\
Y^{lm}_{\mathrm{T2}} &\equiv \frac{2i}{\lambda l(l+1)}\left(\sin^{-1}\theta\, Y^{lm}_{,\varphi}\right)_{,\theta}.
\end{aligned} \tag{3}$$

Code for time evolution of the MP Eqs

Circular geodesics in Schwarzschild

Grid for 1+1d numerical evolution



Grid size:

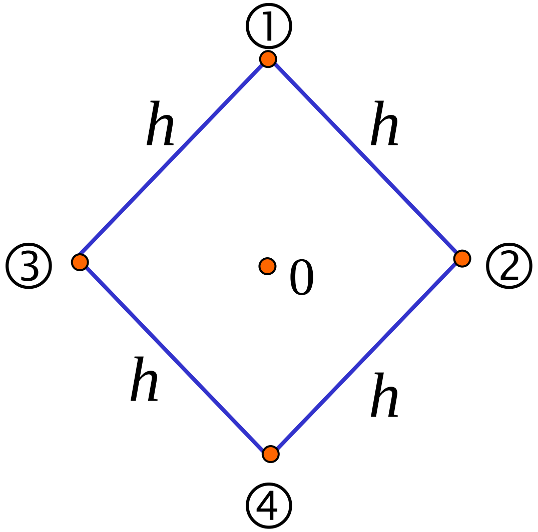
- At least 3-4 T_{orb} (>300M for $r_0=6M$)

Resolution:

- ~1 grid pt/M^2 for fluxes extraction
- ~10^6 grid pts/M^2 (near particle) for SF extraction

Finite difference scheme (2nd order convergent)

[Demonstrated here for a scalar field –
works similarly for our coupled MP Eqs]

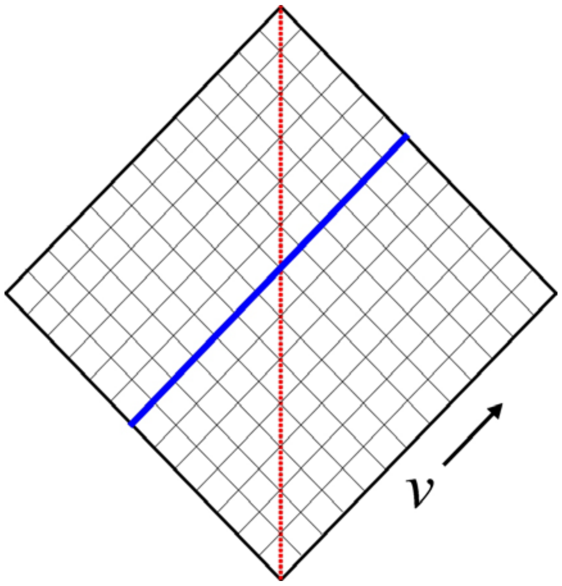


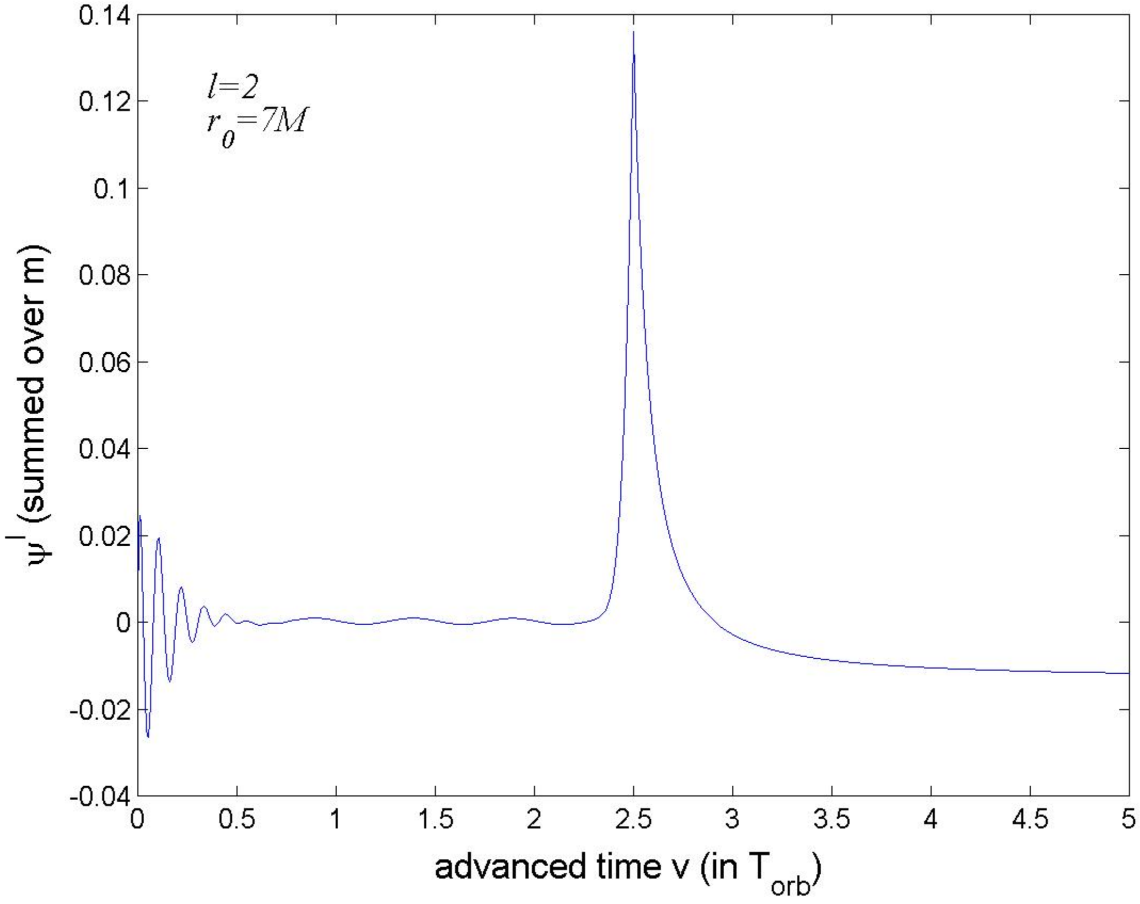
$$\int_{\text{cell}} du dv\ \left[\psi^{lm}_{,uv} + V_{\text{sc}}\psi^{lm} = S^{lm}\right]$$

$$\psi(1)+\psi(4)-\psi(3)-\psi(2)+\frac{1}{2}h^2V_{\text{sc}}(0)[\psi(3)+\psi(2)]+O(h^3)=\int_{\text{cell}} S\, dudv \equiv h\cdot Z$$

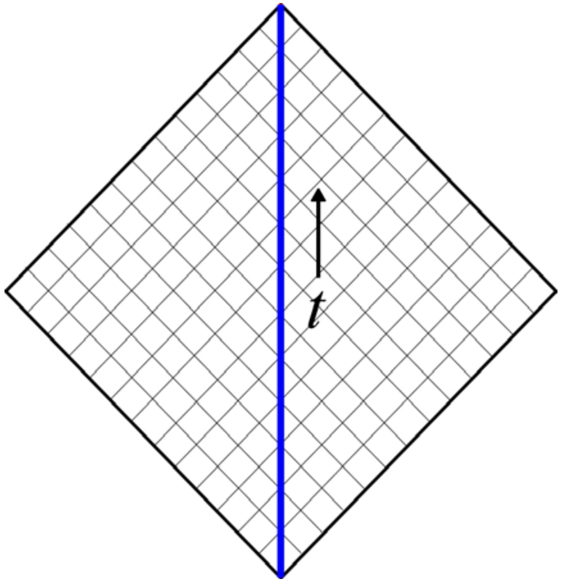
$$Z=\frac{f\alpha_{lm}}{r_0E}\times\begin{cases}0, & \text{no particle in cell}\\ 1, & \text{particle in cell, } m=0\\ \frac{\sin(m\omega h/2)}{m\omega h/2}\exp[-im\omega t(0)], & \text{particle in cell, } m\neq 0\end{cases}$$

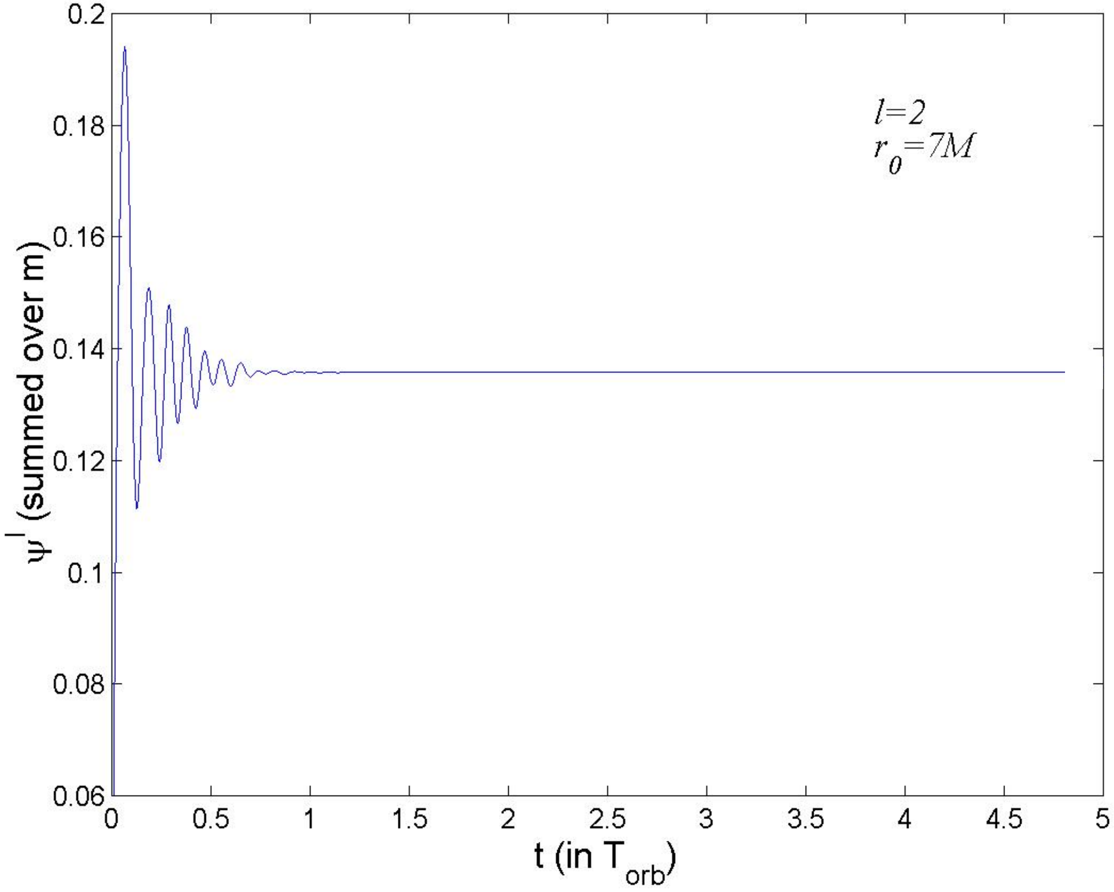
Results: Scalar field



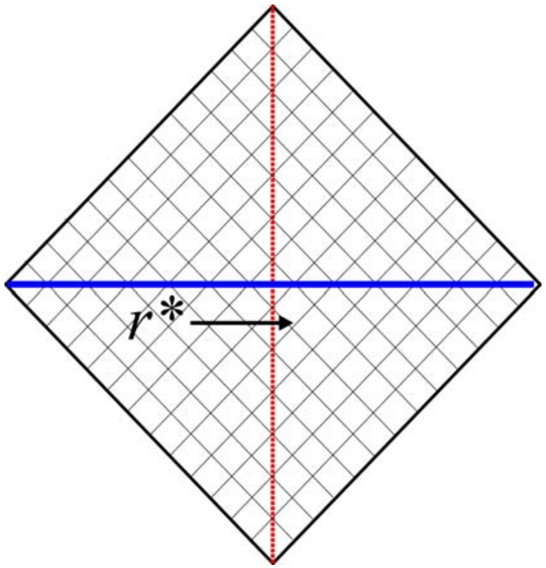


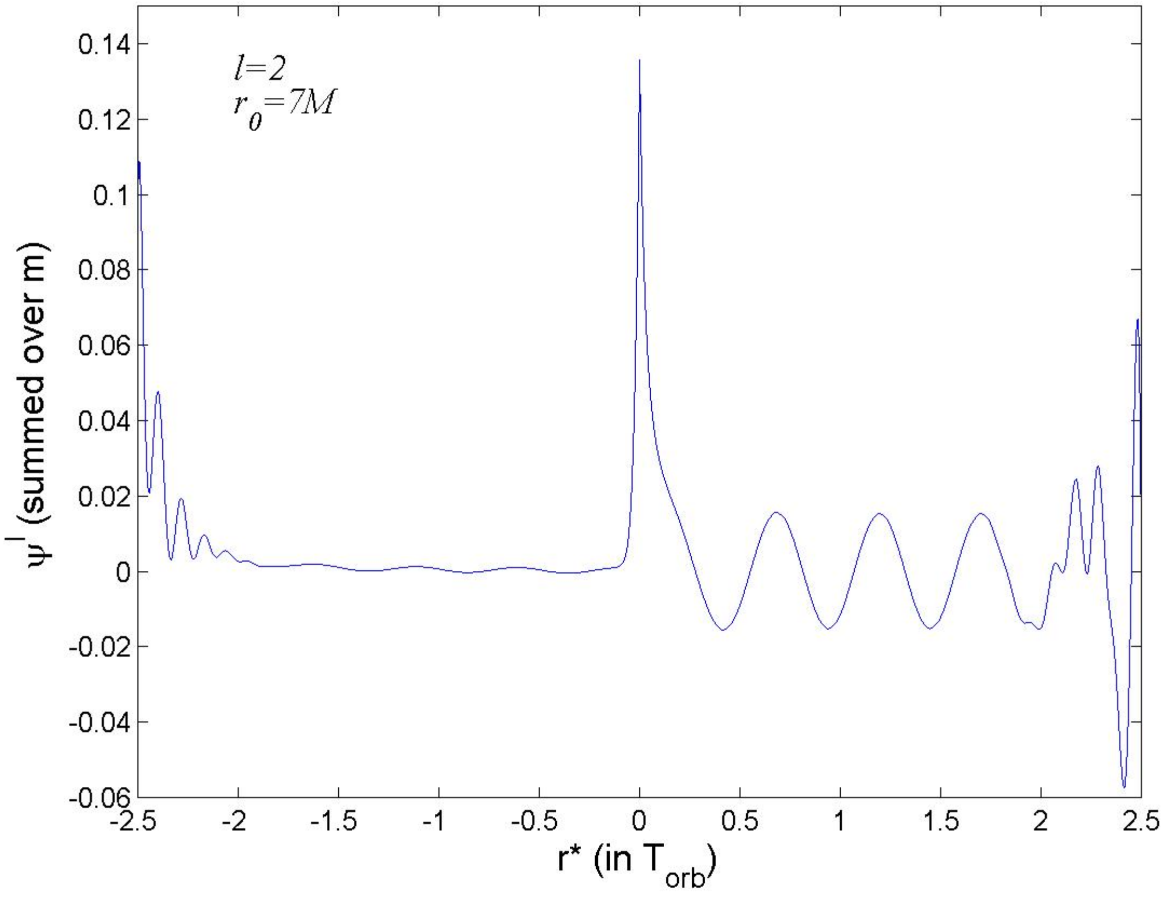
Results: Scalar field



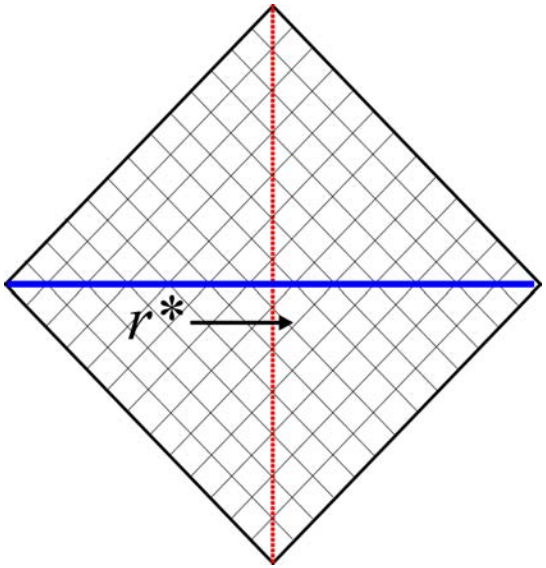


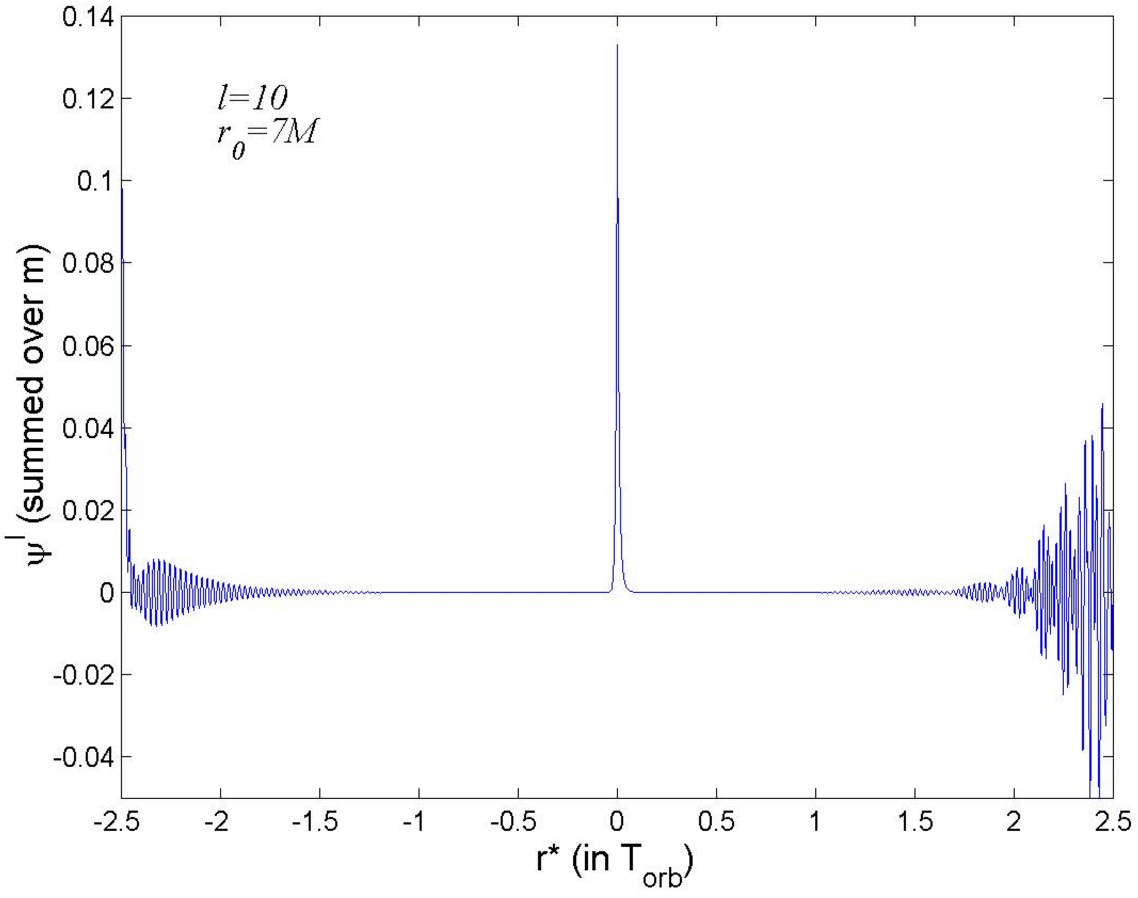
Results: Scalar field



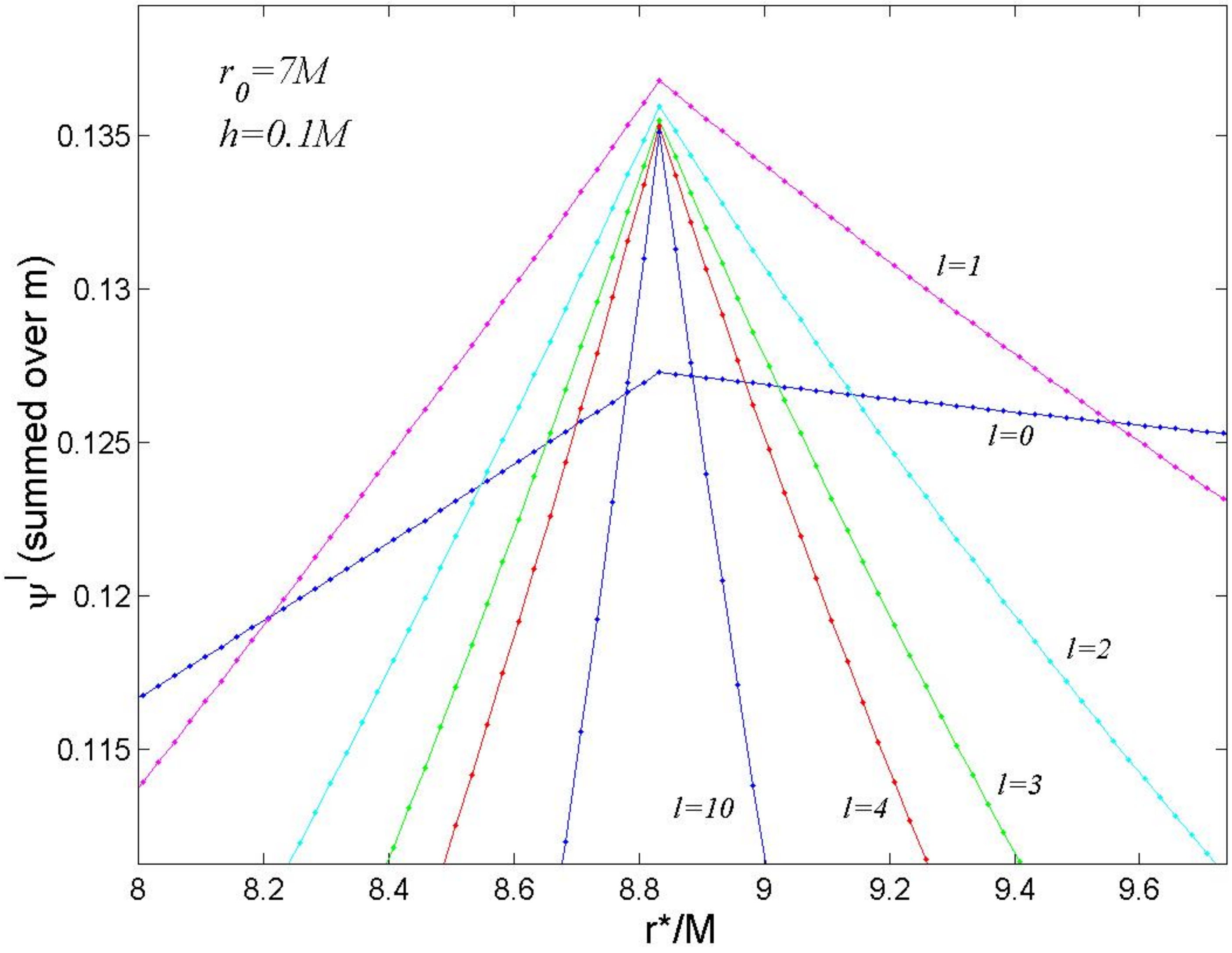


Results: Scalar field

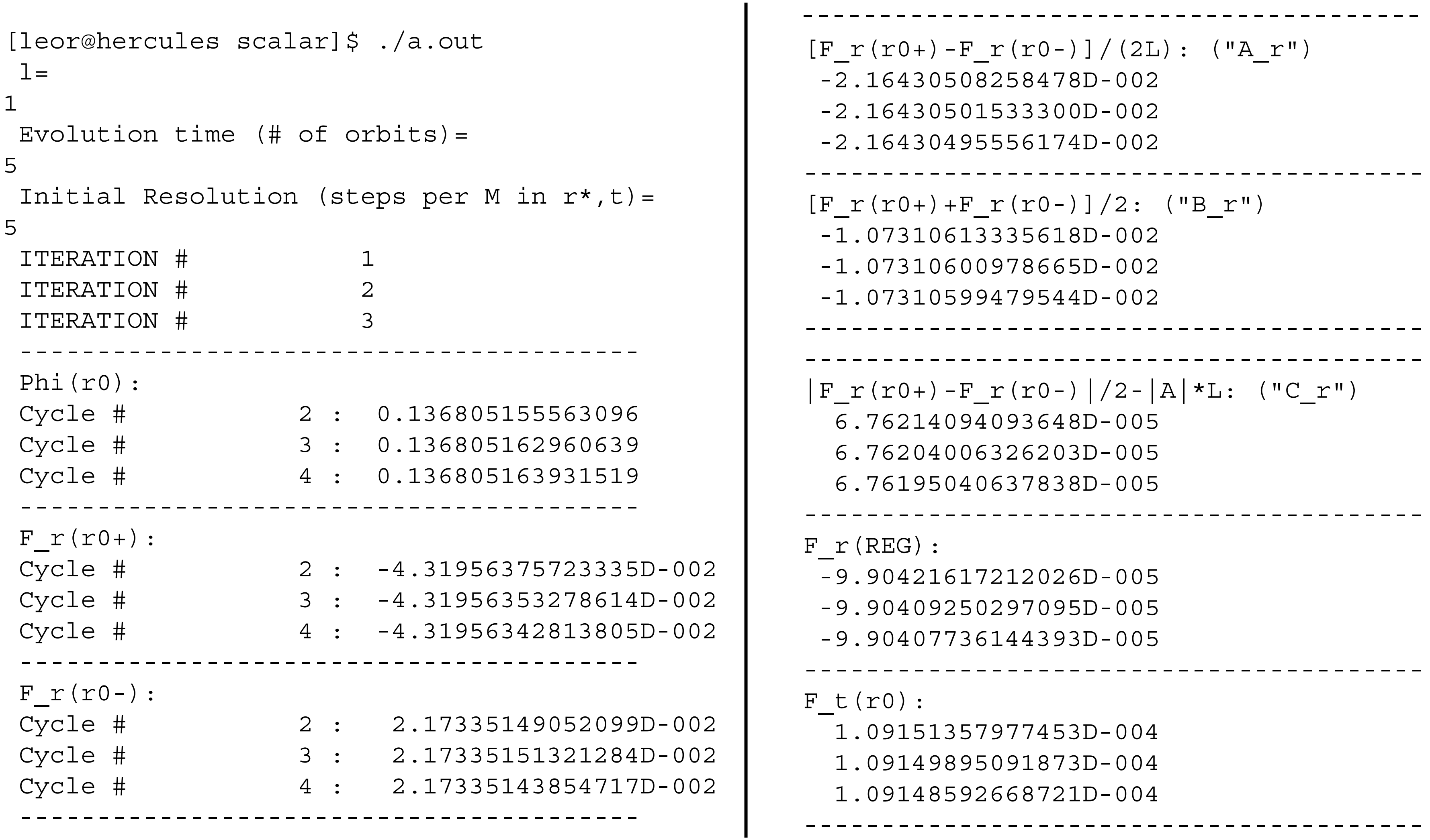




Results: Scalar field



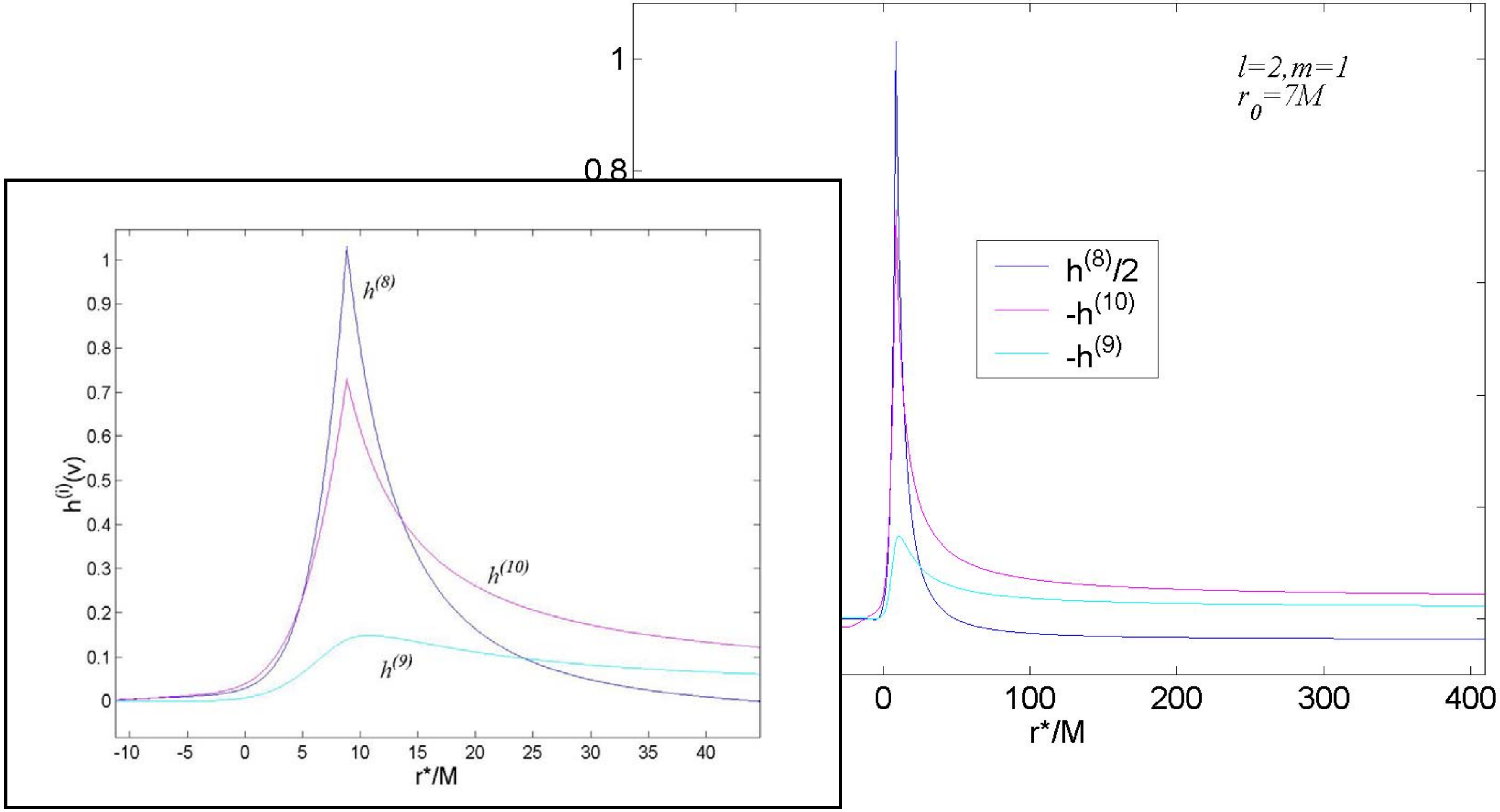
Results: Scalar force (sample output, screen capture)



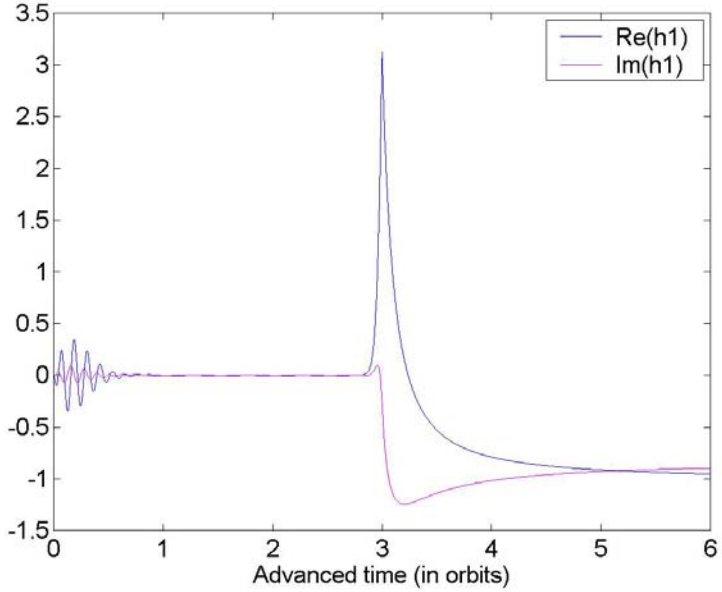
```
[leor@hercules scalar]$ ./a.out
 l=
1
 Evolution time (# of orbits)=
5
 Initial Resolution (steps per M in r*,t)=
5
 ITERATION #           1
 ITERATION #           2
 ITERATION #           3
 ----------------------------------------
 Phi(r0):
 Cycle #          2 :  0.136805155563096
 Cycle #          3 :  0.136805162960639
 Cycle #          4 :  0.136805163931519
 ----------------------------------------
 F_r(r0+):
 Cycle #          2 :  -4.31956375723335D-002
 Cycle #          3 :  -4.31956353278614D-002
 Cycle #          4 :  -4.31956342813805D-002
 ----------------------------------------
 F_r(r0-):
 Cycle #          2 :   2.17335149052099D-002
 Cycle #          3 :   2.17335151321284D-002
 Cycle #          4 :   2.17335143854717D-002
 ----------------------------------------
```

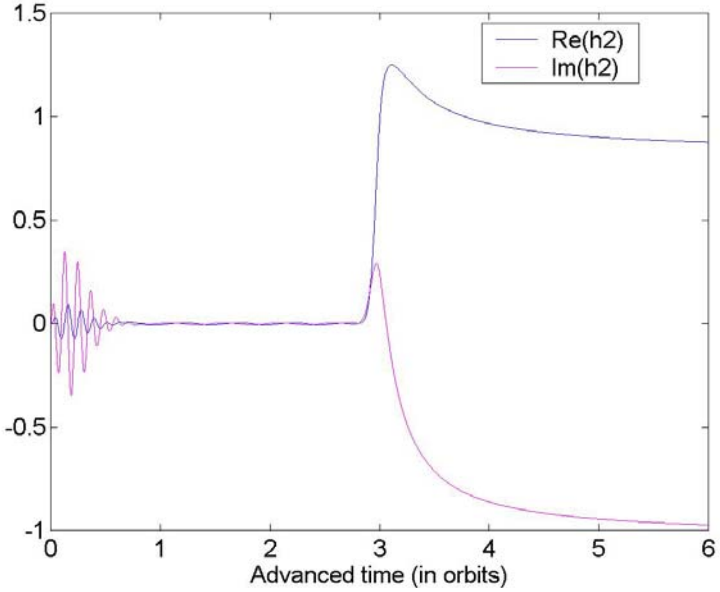
```
 ----------------------------------------
[F_r(r0+)-F_r(r0-)]/(2L): ("A_r")
 -2.16430508258478D-002
 -2.16430501533300D-002
 -2.16430495556174D-002
 ----------------------------------------
[F_r(r0+)+F_r(r0-)]/2: ("B_r")
 -1.07310613335618D-002
 -1.07310600978665D-002
 -1.07310599479544D-002
 ----------------------------------------
 ----------------------------------------
|F_r(r0+)-F_r(r0-)|/2-|A|*L: ("C_r")
  6.76214094093648D-005
  6.76204006326203D-005
  6.76195040637838D-005
 ----------------------------------------
F_r(REG):
 -9.90421617212026D-005
 -9.90409250297095D-005
 -9.90407736144393D-005
 ----------------------------------------
F_t(r0):
  1.09151357977453D-004
  1.09149895091873D-004
  1.09148592668721D-004
 ----------------------------------------
```

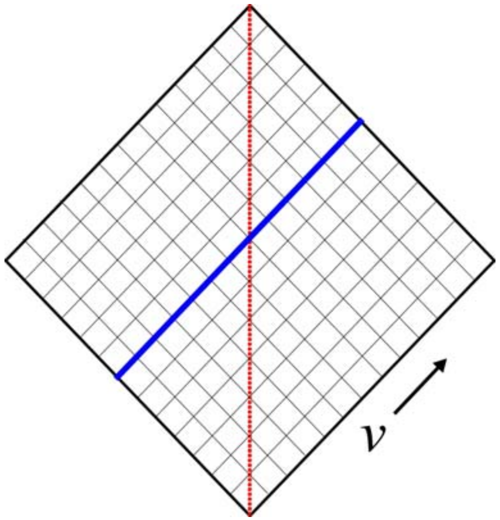
Sample results: Metric Perturbation (odd parity)

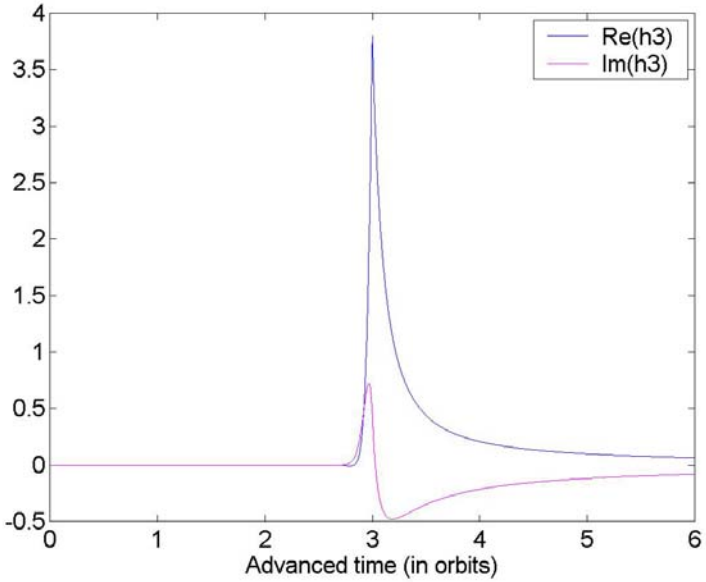


Sample results: Metric Perturbation (Even parity,l=m=2, r_{p}=7M)

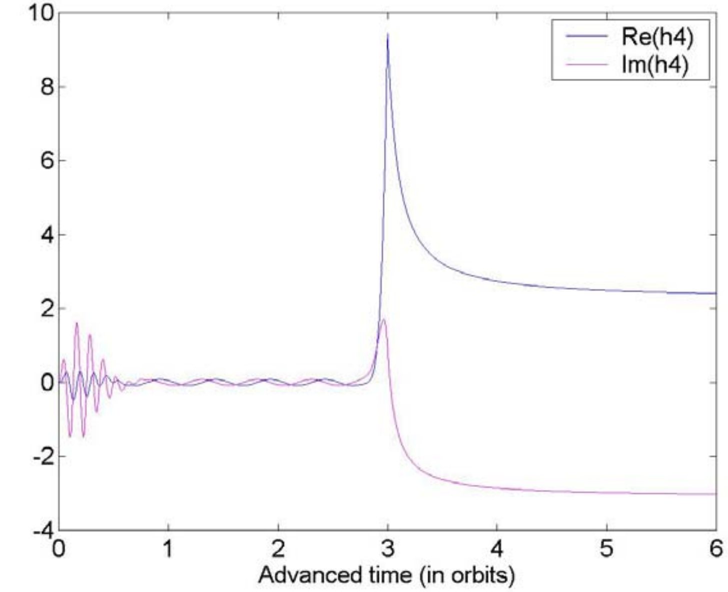


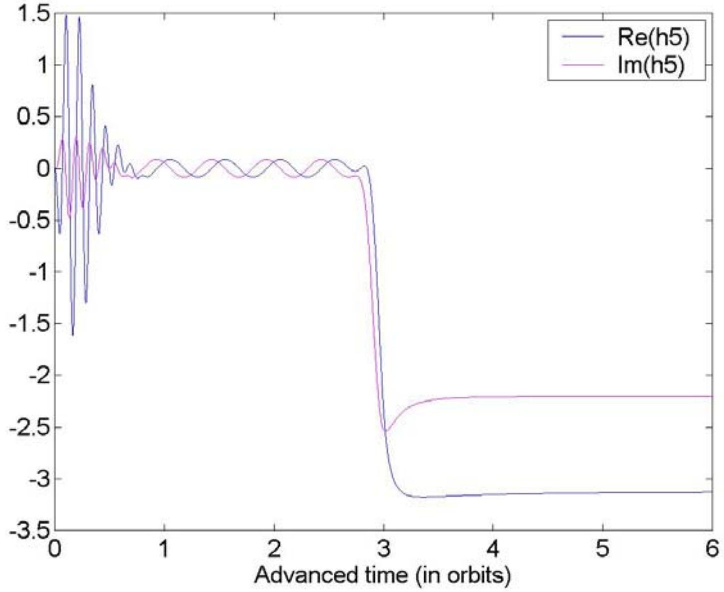


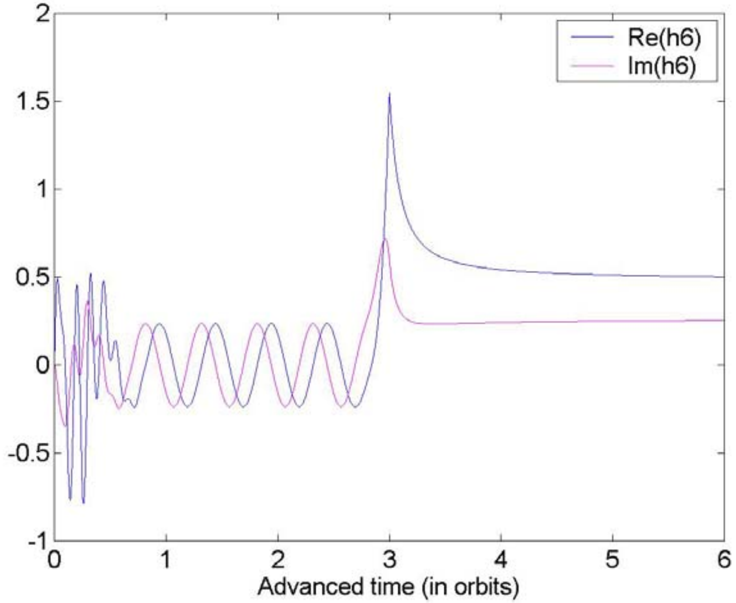


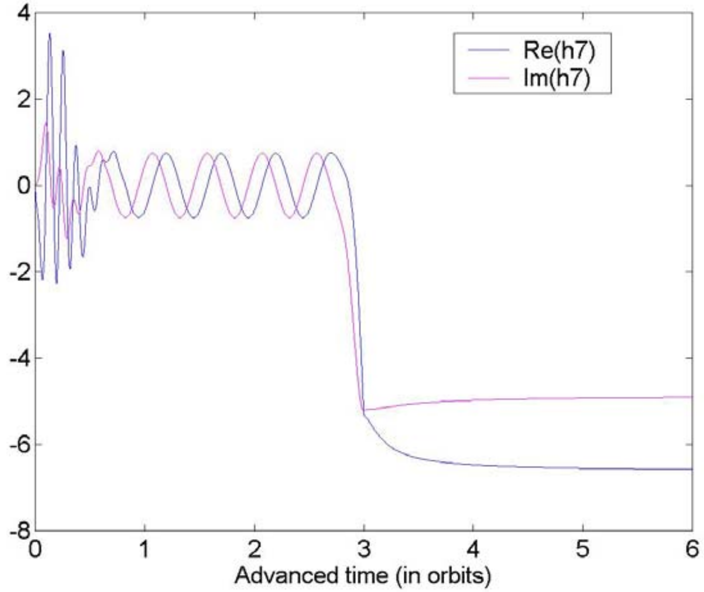


Sample results: MP (even parity,*l*=*m*=2, r_p=7*M*)



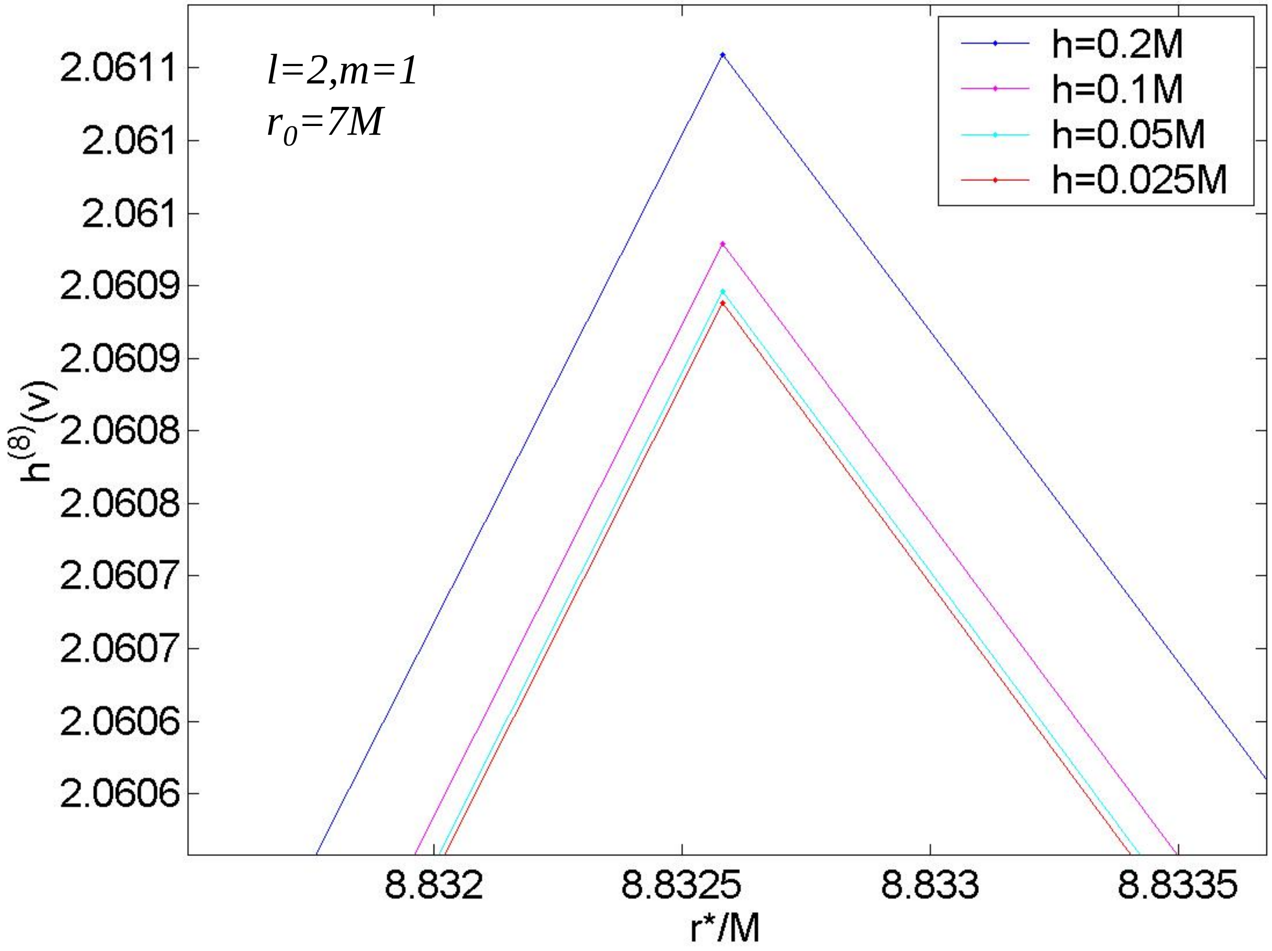


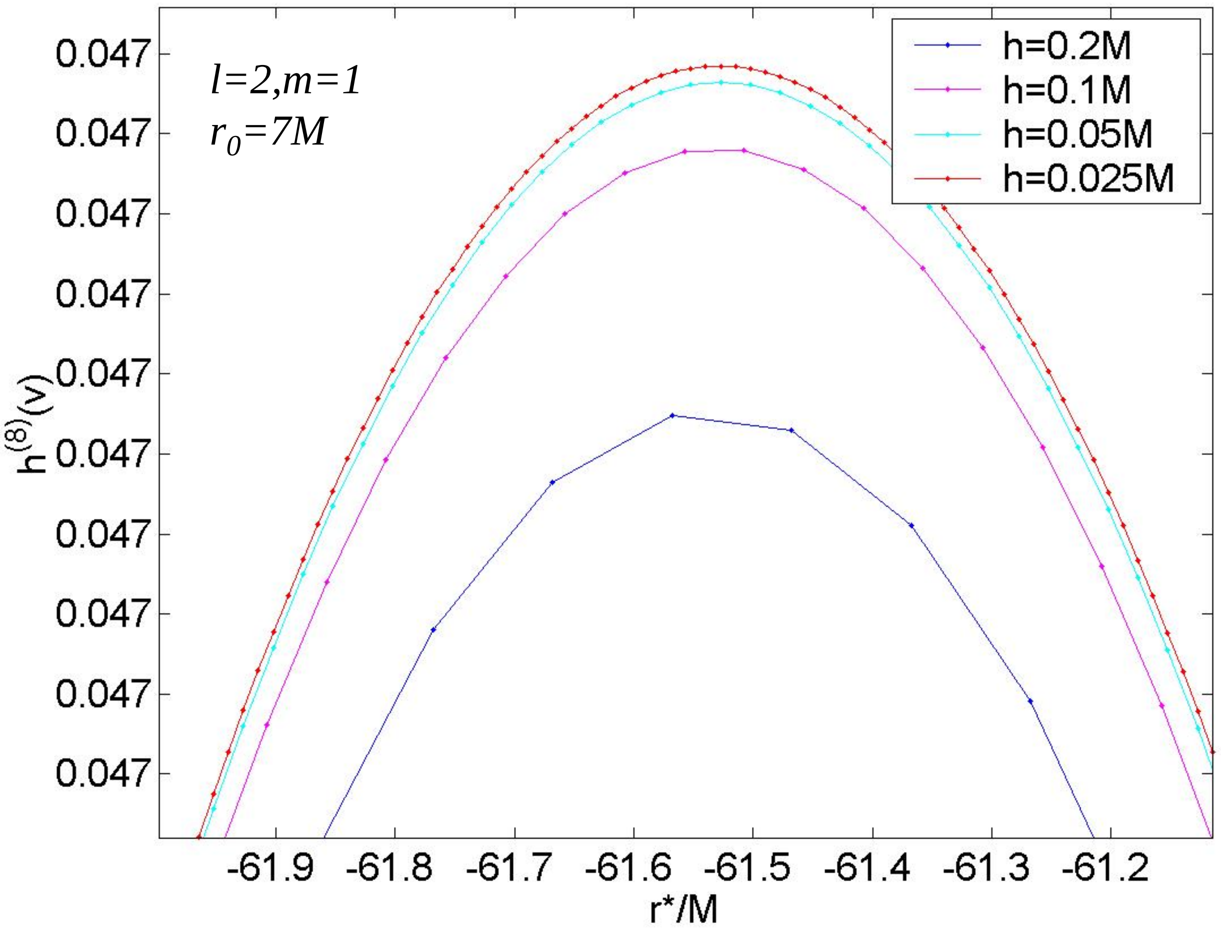




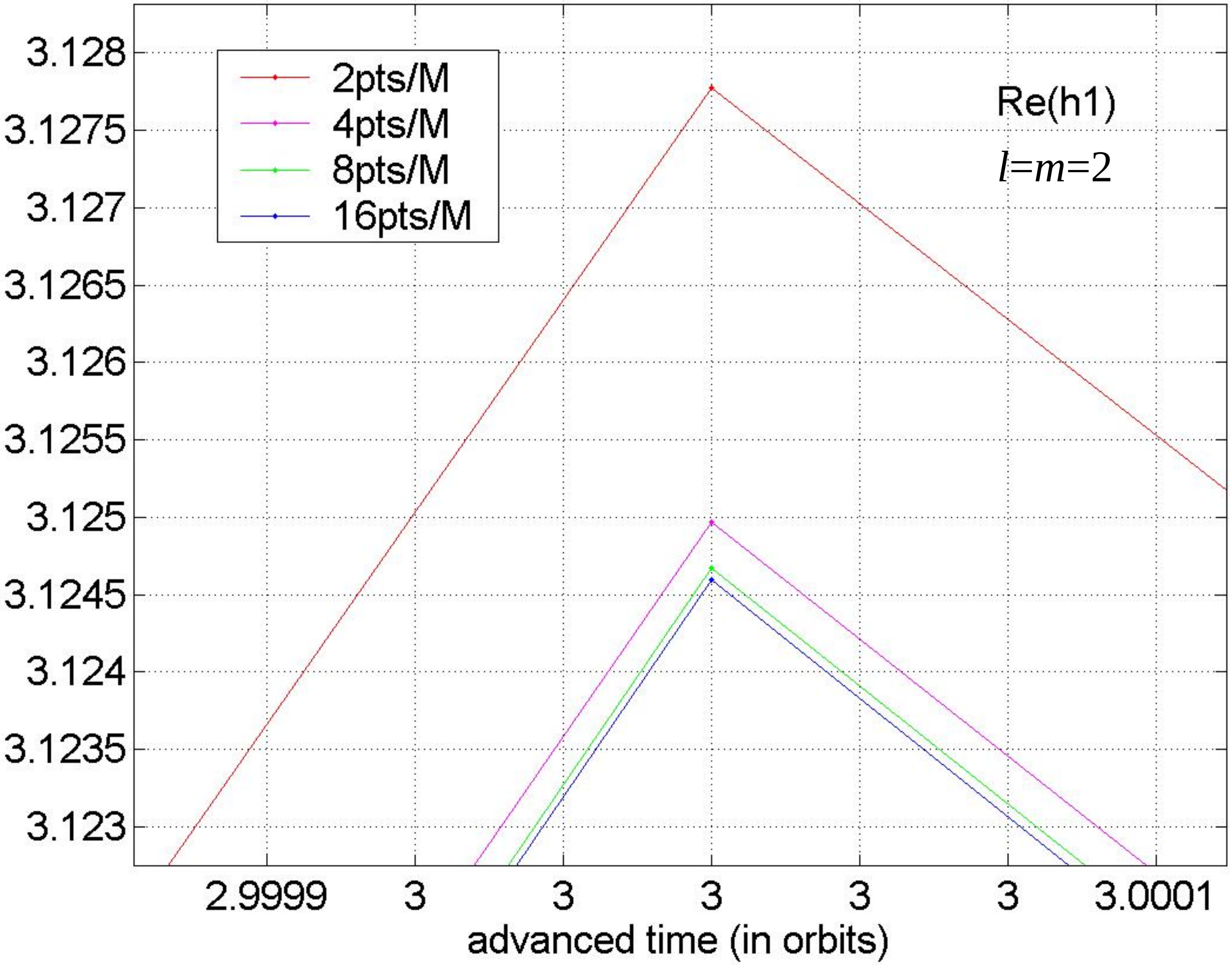
Tests of Code

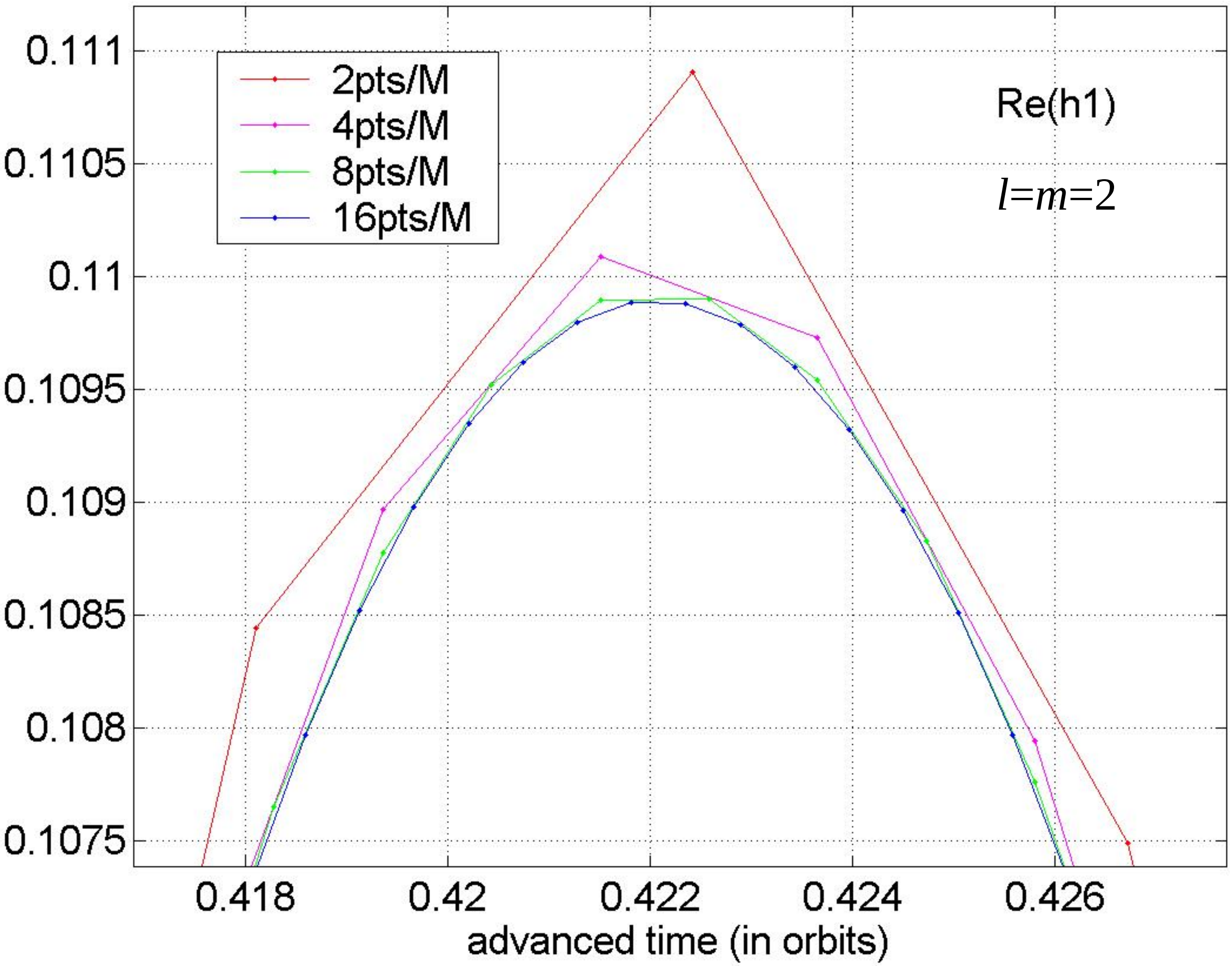
Test #1: 2nd-order Numerical Convergence





l=2,m=1
r_0=7M
h=0.2M
h=0.1M
h=0.05M
h=0.025M
0.047
0.047
0.047
0.047
0.047
0.047
0.047
0.047
0.047
0.047
$h^{(8)}(v)$
-61.9
-61.8
-61.7
-61.6
-61.5
-61.4
-61.3
-61.2
r*/M


3.128
3.1275
3.127
3.1265
3.126
3.1255
3.125
3.1245
3.124
3.1235
3.123
2pts/M
4pts/M
8pts/M
16pts/M
Re(h1)
$l=m=2$
2.9999
3
3
3
3
3
3.0001
advanced time (in orbits)


0.111
0.1105
0.11
0.1095
0.109
0.1085
0.108
0.1075
0.418
0.42
0.422
0.424
0.426
advanced time (in orbits)
2pts/M
4pts/M
8pts/M
16pts/M
Re(h1)
$l=m=2$

Test #2: conservation of the gauge condition

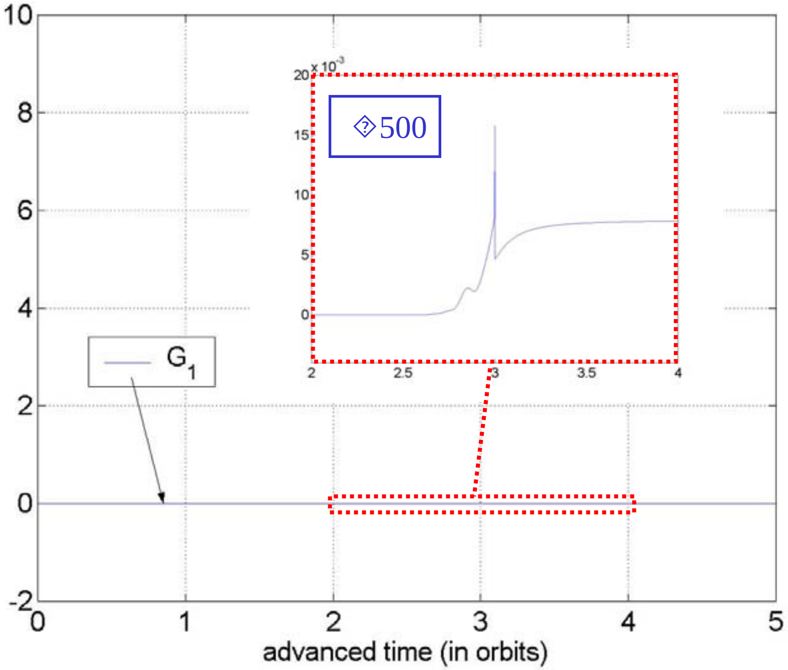
For each l, m, define

$$G_1 \equiv M \left| i\bar{h}^{(1)}_{,t} + i\bar{h}^{(3)}_{,t} + f\left(\bar{h}^{(2)}_{,r} + \bar{h}^{(2)}/r - \bar{h}^{(4)}/r\right) \right|, \tag{1a}$$

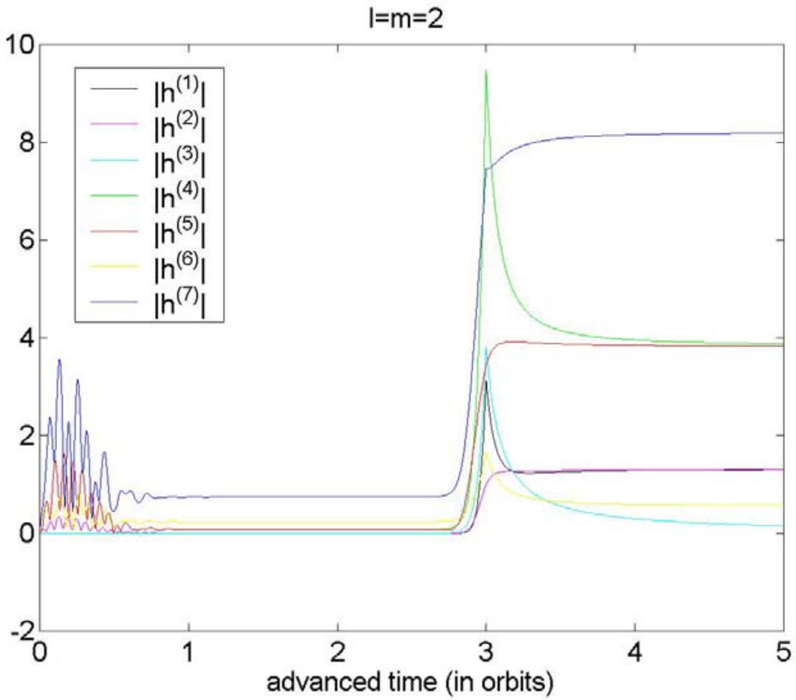
$$G_2 \equiv M \left| i\bar{h}^{(2)}_{,t} - f\left(\bar{h}^{(1)}_{,r} - \bar{h}^{(3)}_{,r}\right) + (1 - 4M/r)\bar{h}^{(3)}/r - (f/r)\left(\bar{h}^{(1)} - \bar{h}^{(5)} - 2f\bar{h}^{(6)}\right) \right|, \tag{1b}$$

$$G_3 \equiv M \left| i\bar{h}^{(4)}_{,t} - f\left(\bar{h}^{(5)}_{,r} + 2\bar{h}^{(5)}/r + l(l+1)\,\bar{h}^{(6)}/r - \bar{h}^{(7)}/r\right) \right|, \tag{1c}$$

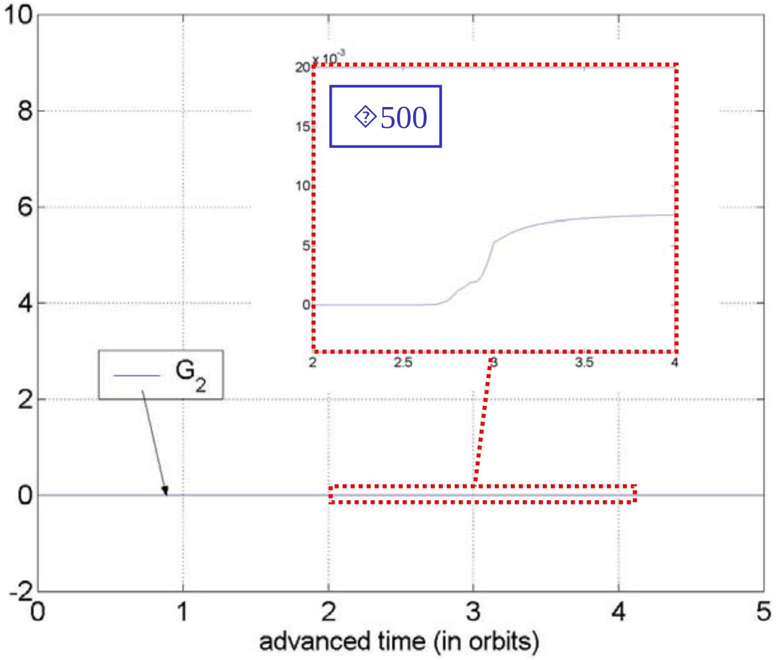
$$G_4 \equiv M \left| i\bar{h}^{(8)}_{,t} + f\left(\bar{h}^{(9)}_{,r} + 2\bar{h}^{(9)}/r - \bar{h}^{(10)}/r\right) \right| \tag{1d}$$



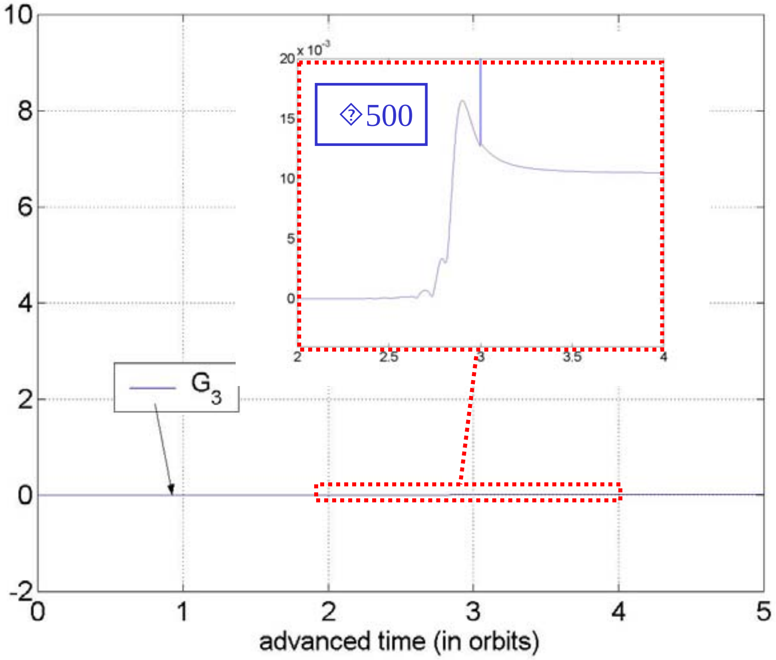
10
8
6
4
2
0
-2
0
1
2
3
4
5
advanced time (in orbits)
G1
20 x 10^-3
15
10
5
0
2
2.5
3
3.5
4
500



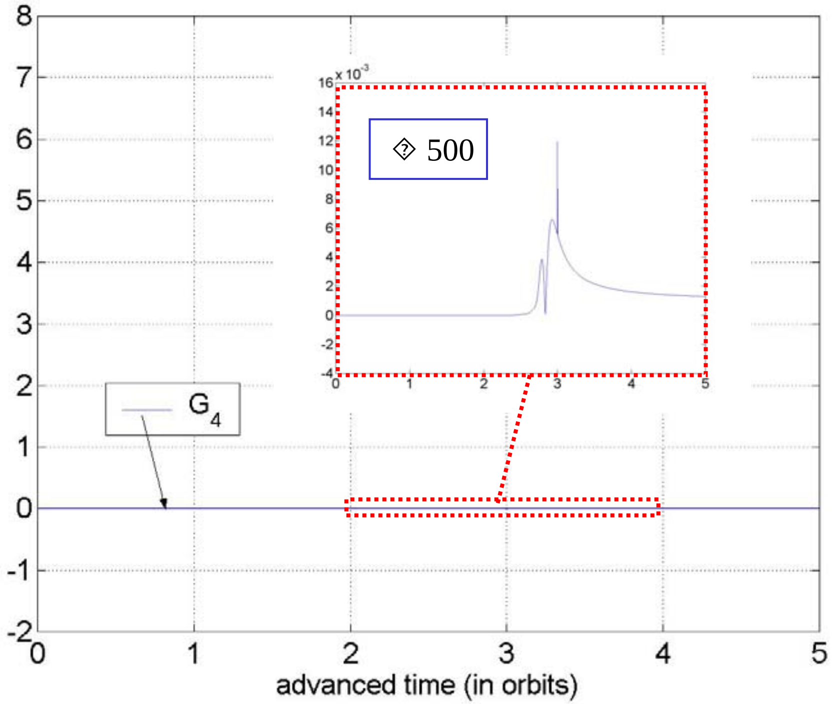
l=m=2
|h(1)|
|h(2)|
|h(3)|
|h(4)|
|h(5)|
|h(6)|
|h(7)|
10
8
6
4
2
0
-2
0
1
2
3
4
5
advanced time (in orbits)

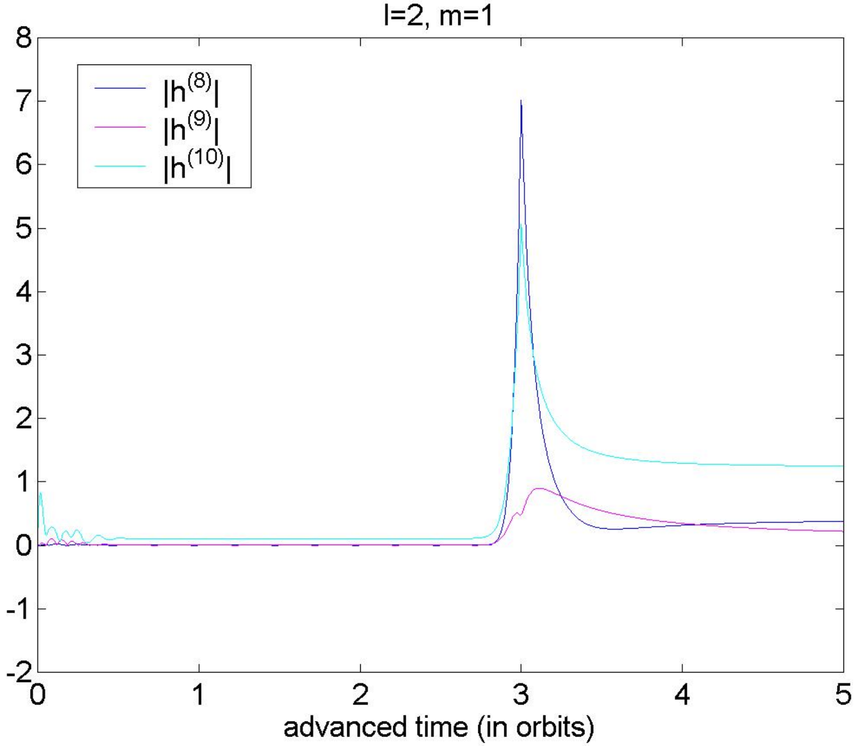


10
8
6
4
2
0
-2
0
1
2
3
4
5
advanced time (in orbits)
G2
20 x 10^-3
15
10
5
0
2
2.5
3
3.5
4
500



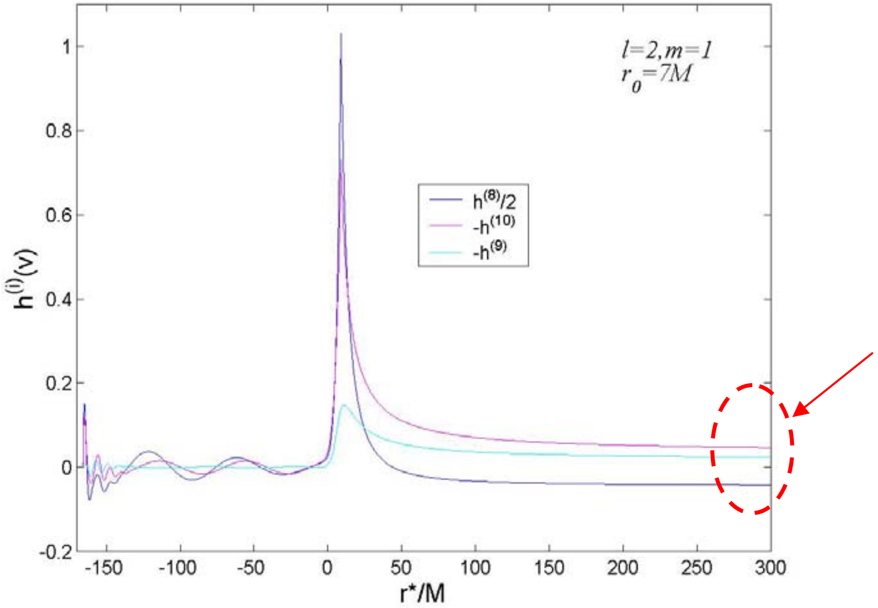
10
8
6
4
2
0
-2
0
1
2
3
4
5
advanced time (in orbits)
G3
20 x 10^-3
15
10
5
0
2
2.5
3
3.5
4
500


8
7
6
5
4
3
2
1
0
-1
-2
0
1
2
3
4
5
advanced time (in orbits)
G_4
x 10^{-3}
16
14
12
10
8
6
4
2
0
-2
-4
0
1
2
3
4
5
◇ 500


l=2, m=1
$|h^{(8)}|$
$|h^{(9)}|$
$|h^{(10)}|$
8
7
6
5
4
3
2
1
0
-1
-2
0
1
2
3
4
5
advanced time (in orbits)

Test #3: Flux of energy radiated to infinity

$$\begin{aligned}\mu^{-1}\dot{E}^{\infty}_{lm} &= \int \frac{d\Omega}{16\pi r^2}\left(\left|(\sin\theta)^{-1}\dot{h}_{\theta\varphi}\right|^2 + \frac{1}{4}\left|\dot{h}_{\theta\theta} - (\sin\theta)^{-2}\dot{h}_{\varphi\varphi}\right|^2\right)\Bigg|_{\text{wave zone}} \\ &= \frac{m^2\omega^2}{32\pi}\left(\left|\bar{h}^{(7)}_{lm}\right|^2 + \left|\bar{h}^{(10)}_{lm}\right|^2\right)\Bigg|_{\text{wave zone}}\end{aligned}$$



l	m	**Poisson's:**	**Martel's:**		**Ours:**	
		f-domain, from ψ	t-domain, from ψ		t-domain, from $h_{\alpha\beta}$	
2	1	$8.1633e{-}07$	$8.1623e{-}07$	[0.01%]	$8.1654e{-}07$	[0.03%]
	2	$1.7063e{-}04$	$1.7051e{-}04$	[0.07%]	$1.7061e{-}04$	[0.01%]
3	1	$2.1731e{-}09$	$2.1741e{-}09$	[0.05%]	$2.1735e{-}09$	[0.03%]
	2	$2.5199e{-}07$	$2.5164e{-}07$	[0.14%]	$2.5221e{-}07$	[0.09%]
	3	$2.5471e{-}05$	$2.5432e{-}05$	[0.15%]	$2.5485e{-}05$	[0.05%]
4	1	$8.3956e{-}13$	$8.3507e{-}13$	[0.53%]	$8.4160e{-}13$	[0.24%]
	2	$2.5091e{-}09$	$2.4986e{-}09$	[0.42%]	$2.5111e{-}09$	[0.08%]
	3	$5.7751e{-}08$	$5.7464e{-}08$	[0.50%]	$5.7796e{-}08$	[0.08%]
	4	$4.7256e{-}06$	$4.7080e{-}06$	[0.37%]	$4.7312e{-}06$	[0.12%]
5	1	$1.2594e{-}15$	$1.2544e{-}15$	[0.40%]	$1.2601e{-}15$	[0.06%]
	2	$2.7896e{-}12$	$2.7587e{-}12$	[1.11%]	$2.7911e{-}12$	[0.05%]
	3	$1.0933e{-}09$	$1.0830e{-}09$	[0.94%]	$1.0942e{-}09$	[0.08%]
	4	$1.2324e{-}08$	$1.2193e{-}08$	[1.06%]	$1.2328e{-}08$	[0.03%]
	5	$9.4563e{-}07$	$9.3835e{-}07$	[0.77%]	$9.4701e{-}07$	[0.15%]
	total	$2.0317e{-}04$	$2.0273e{-}04$	[0.22%]	$2.0292e{-}04$	[0.12%]

TABLE I. $\dot{E}^{\infty}_{lm}$ [in units of $(\mu/M)^2$], for $r_0 = 7.9456M$.

What’s next?

Gravitational Self Force

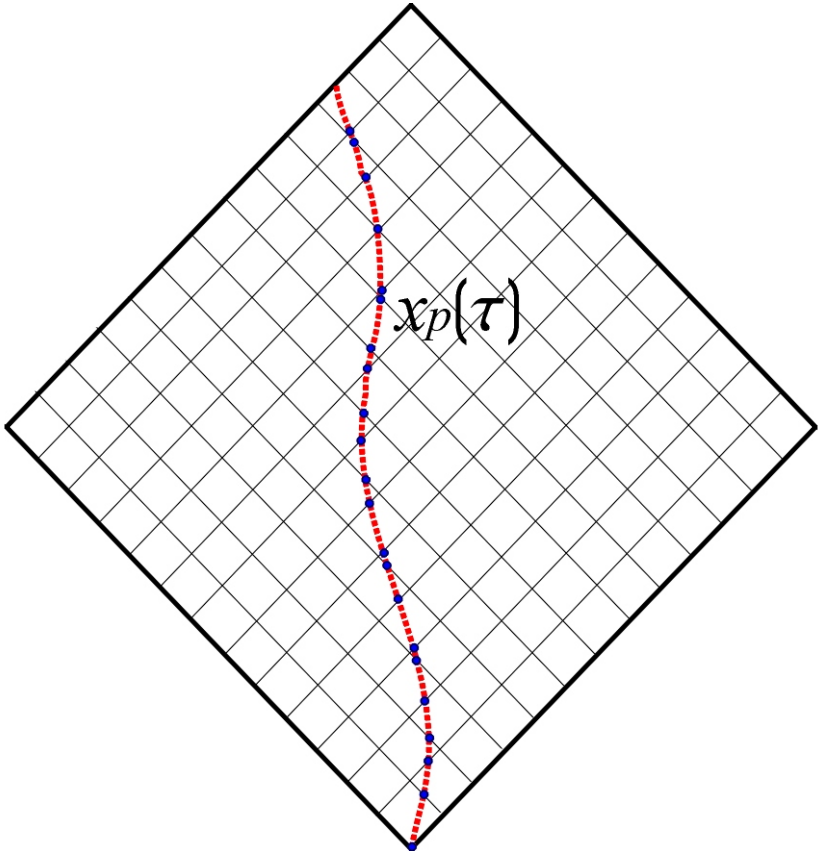
- Getting the self force is now straightforward:
  - No need to take 3 derivatives, just one – as in scalar case
  - Modes l=0,1 given (almost) analytically, in the harmonic gauge
  - Most crucial: no gauge problem; mode-sum implemented as is:

$$\boxed{F_{\text{self}}(x_p) = \sum_{l=0}^{\infty} \left[F_{\text{full}}(h_{\alpha\beta})\big|_{x \to x_p} - A(l+1/2) - B - C/(l+1/2) \right] - D}$$

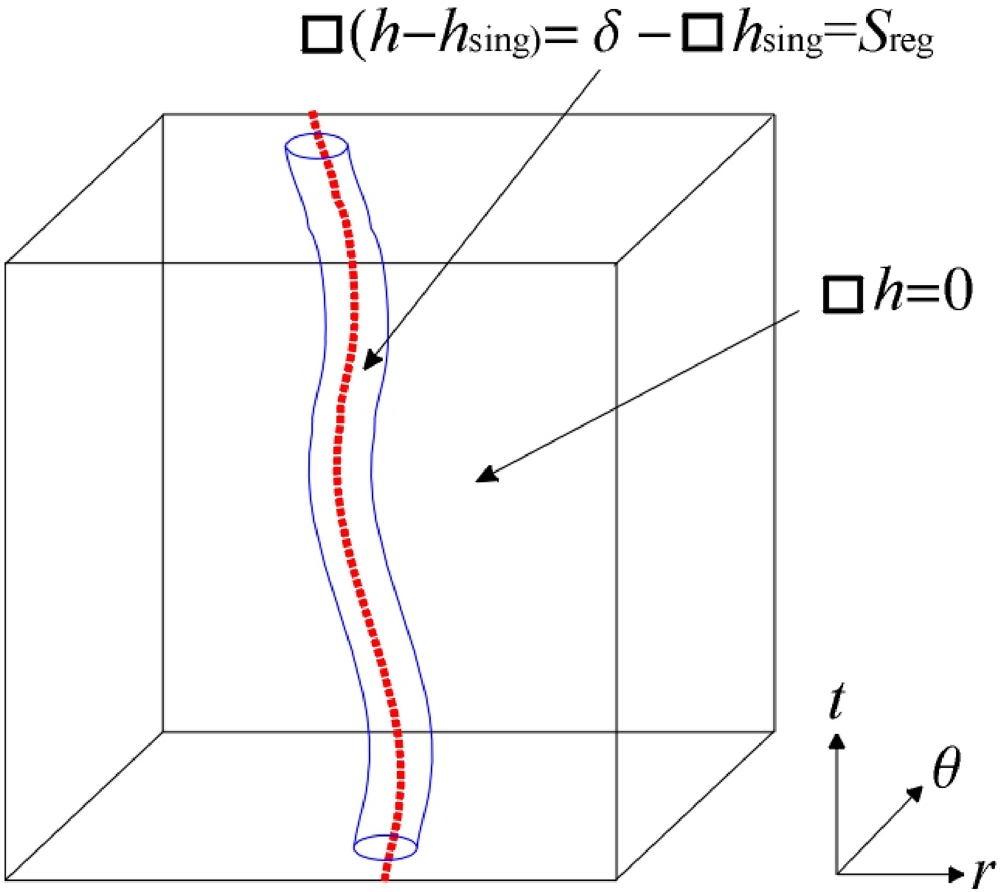
- Two options:
  - Either decompose F_{full} in Scalar harmonics & use the “standard” RP;
  - Or decompose F_{full} in Vector harmonics and re-derive the RP accordingly.

Eccentric orbits in Schwarzschild

- Generalization is easy, as code is time-domain



Orbits in Kerr



N. Hernandez (MSc Thesis, Brownsville 2005) shows promise of this approach