Adiabatic radiation reaction

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Collaborators

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Hughes, Drasco, Flanagan, Franklin, PRL **94**, 221101 (2005)

Outline

- Adiabatic?
- Generic orbits
- Snapshots
- Future

Adiabatic?

Types of waveforms

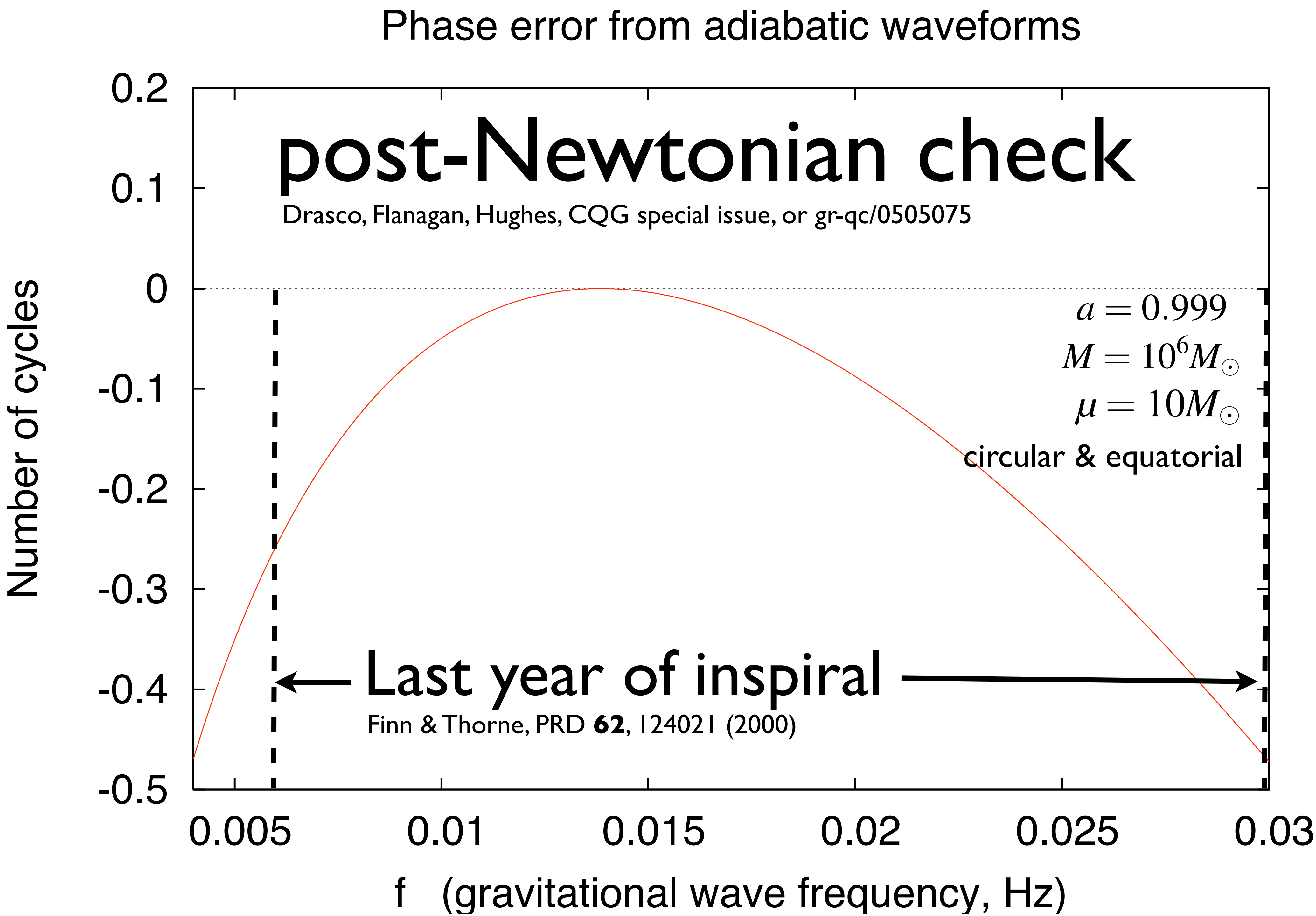
- **Kludge** (available)
  - Gair et al.
  - explore data analysis & event rates
- **Adiabatic** (our goal)
  - Inspiral is sequence of geodesics
  - detection waveforms
- **Self-force** (Capra)
  - measurement waveforms

Accuracy

$$\vec{a} = \frac{\mu}{M}\left[\vec{a}_{1,\text{diss}} + \vec{a}_{1,\text{cons}} + \frac{\mu}{M}(\vec{a}_{2,\text{diss}} + \vec{a}_{2,\text{cons}}) + O\left(\frac{\mu^2}{M^2}\right)\right]$$

$$\phi(t) = \frac{M}{\mu}\left[\phi_1(t, t_1) + \frac{\mu}{M}\phi_2(t, t_1) + O\left(\frac{\mu^2}{M^2}\right)\right]$$

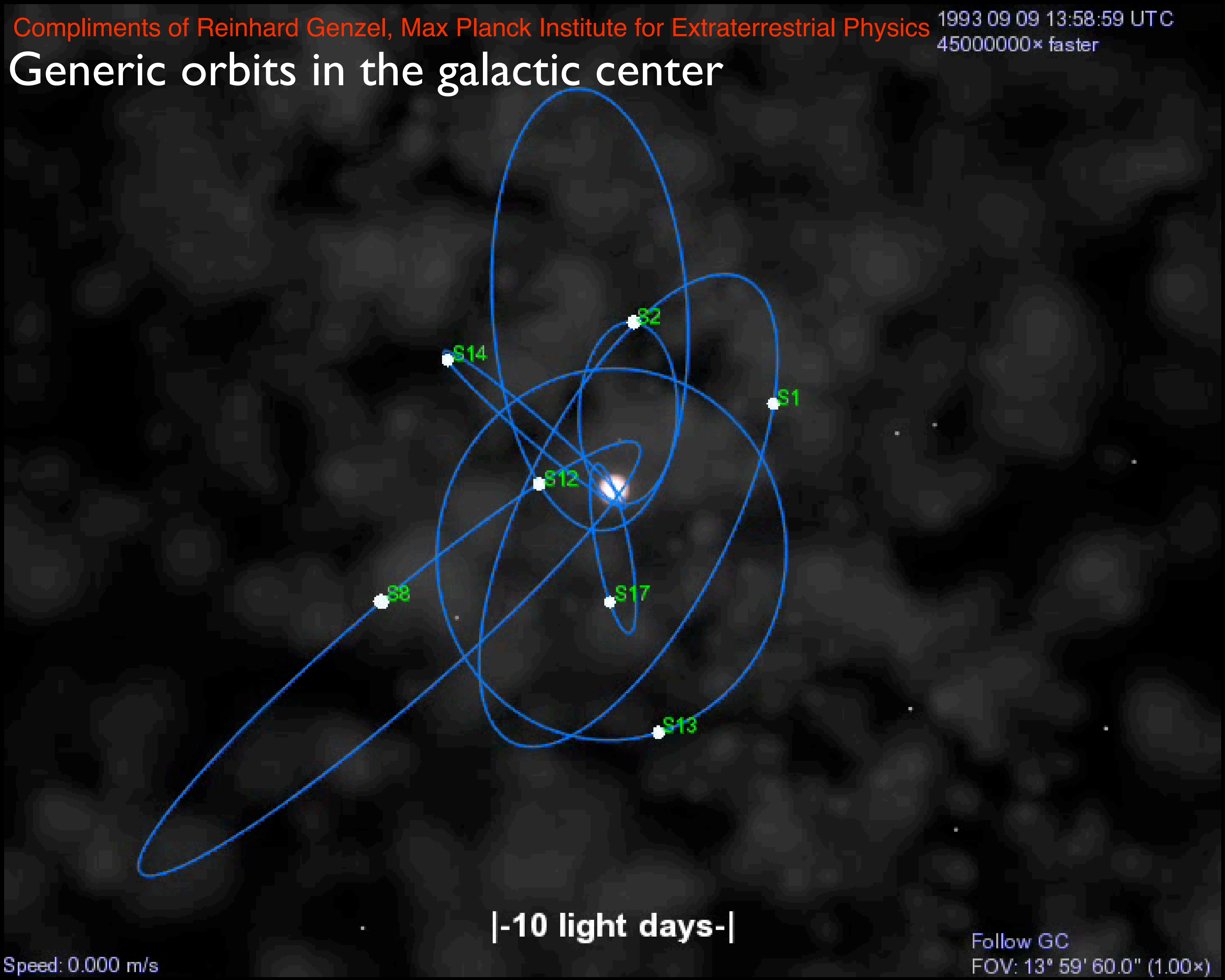
Dissipative effects accumulate secularly.
Conservative effects oscillate, and are **reduced in order**


Phase error from adiabatic waveforms
post-Newtonian check
Drasco, Flanagan, Hughes, CQG special issue, or gr-qc/0505075
$a = 0.999$
$M = 10^6 M_\odot$
$\mu = 10 M_\odot$
circular & equatorial
Last year of inspiral
Finn & Thorne, PRD **62**, 124021 (2000)
Number of cycles
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
0.005
0.01
0.015
0.02
0.025
0.03
f (gravitational wave frequency, Hz)

Generic orbits

Compliments of Reinhard Genzel, Max Planck Institute for Extraterrestrial Physics

Generic orbits in the galactic center



Mino time

Mino, PRD **67**, 084027 (2003)

$$\frac{d\tau}{d\lambda} = r^2 + a^2\cos^2\theta = \Sigma$$

- Decouples radial and polar geodesic equations

$$\left(\frac{dr}{d\lambda}\right)^2 = V_r(r) \qquad \left(\frac{d\theta}{d\lambda}\right)^2 = V_\theta(\theta)$$

$$\frac{dt}{d\lambda} = V_t(r,\theta) \qquad \frac{d\phi}{d\lambda} = V_\phi(r,\theta)$$

Harmonic Structure

Schmidt, CQG **19**, 2743 (2002)
Drasco & Hughes, PRD **69**, 044015 (2004)

- Radial and polar motion are periodic

$$r(\lambda) = \sum_{n=-\infty}^{\infty} r_n e^{-in\Upsilon_r\lambda} \qquad \theta(\lambda) = \sum_{k=-\infty}^{\infty} \theta_k e^{-ik\Upsilon_\theta\lambda}$$

- Axial and time motion are linear & doubly-periodic

$$t(\lambda) = t_0 + \Gamma\lambda + \Delta t(r,\theta) \qquad \phi(\lambda) = \phi_0 + \Upsilon_\phi\lambda + \Delta\phi(r,\theta)$$

$$\Delta t = \sum_{k=1}^{\infty} \Delta t_k^\theta \sin(k\Upsilon_\theta\lambda) + \sum_{n=1}^{\infty} \Delta t_n^r \sin(n\Upsilon_r\lambda) \qquad \Delta\phi = \sum_{k=1}^{\infty} \Delta\phi_k^\theta \sin(k\Upsilon_\theta\lambda) + \sum_{n=1}^{\infty} \Delta\phi_n^r \sin(n\Upsilon_r\lambda)$$

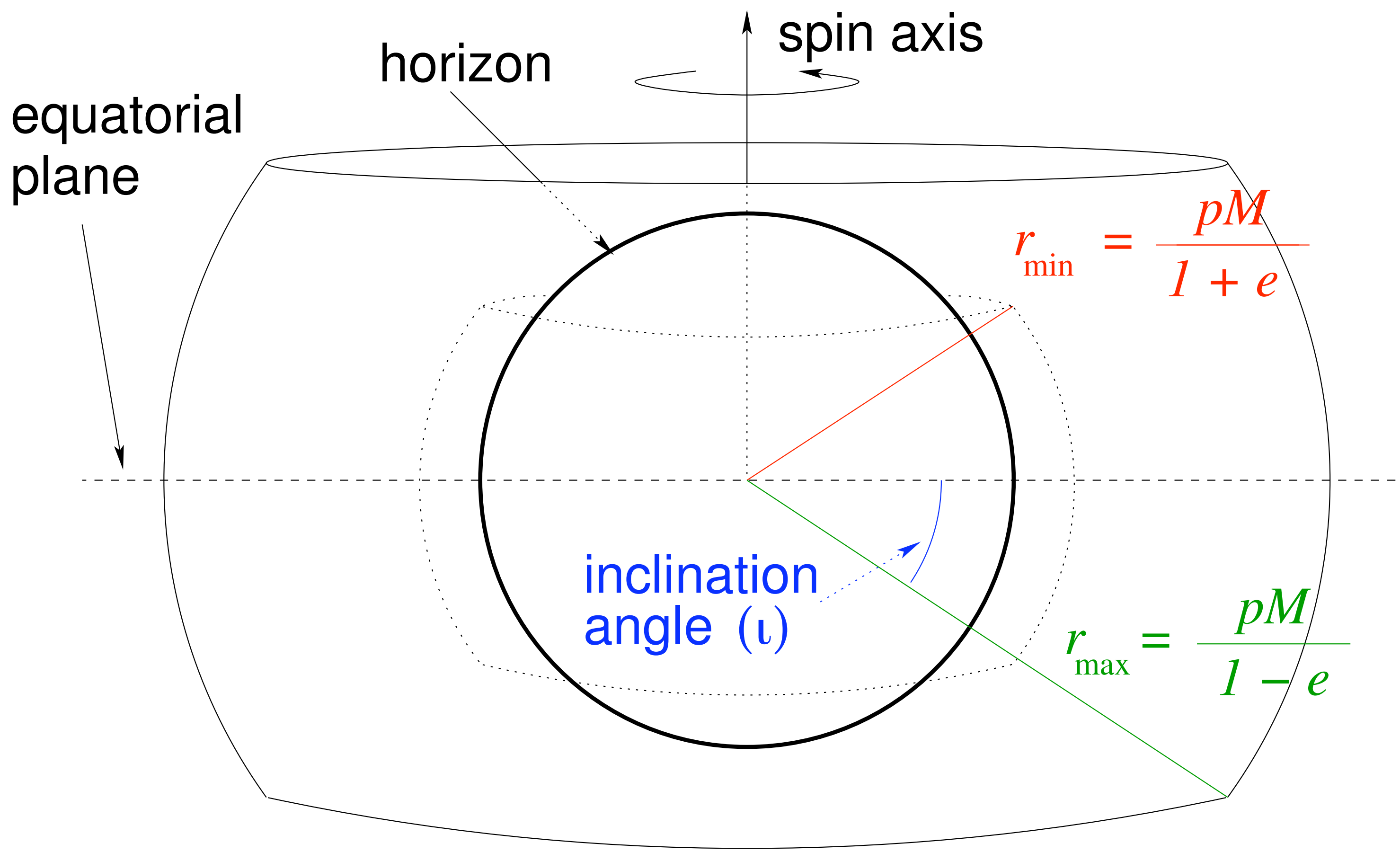
Coordinate frequencies

- Waveforms oscillate at integer-linear combinations of coordinate-time frequencies

$$\omega_r = \frac{\Upsilon_r}{\Gamma} \qquad \omega_\theta = \frac{\Upsilon_\theta}{\Gamma} \qquad \omega_\phi = \frac{\Upsilon_\phi}{\Gamma}$$

$$\omega_{mkn} = m\omega_\phi + k\omega_\theta + n\omega_r$$

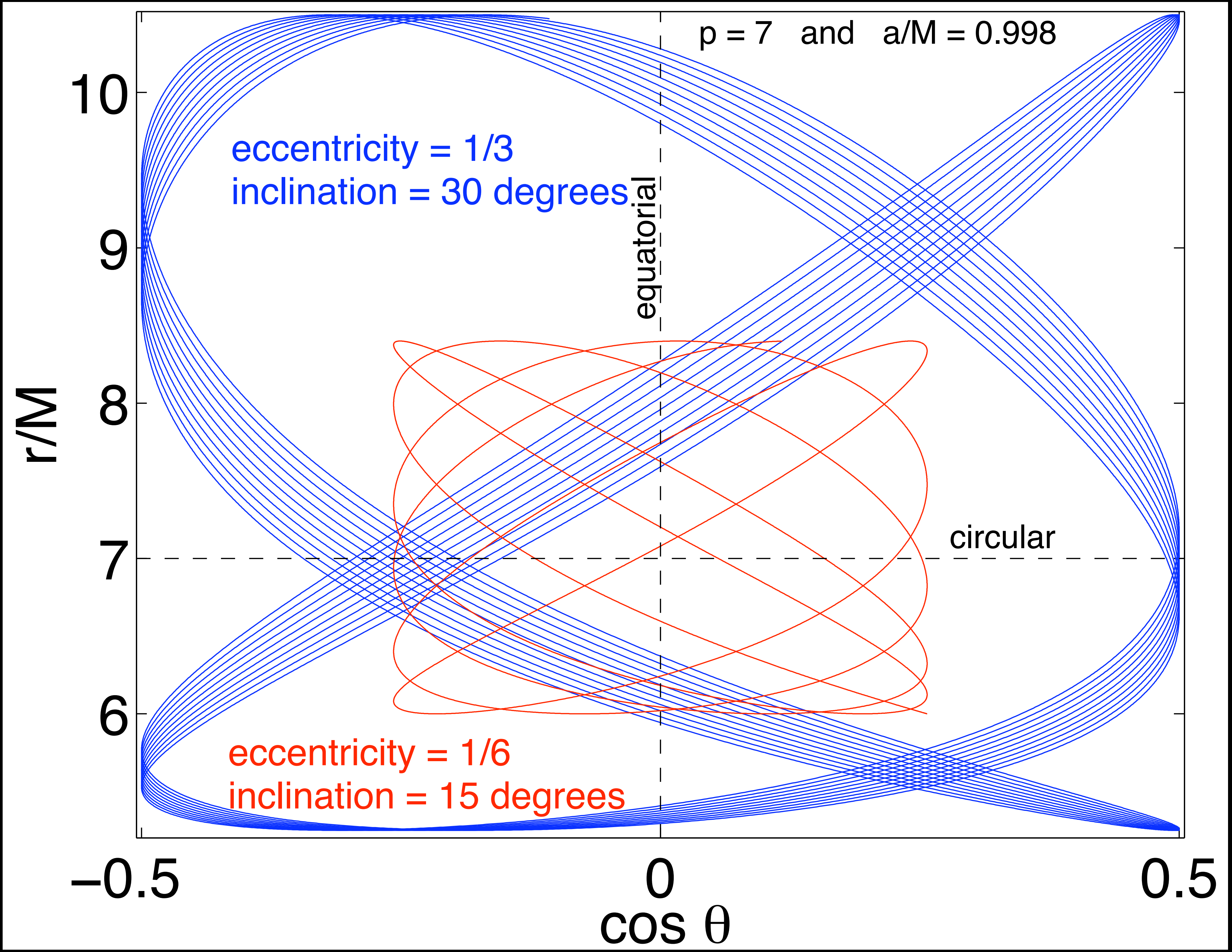
The orbital torus



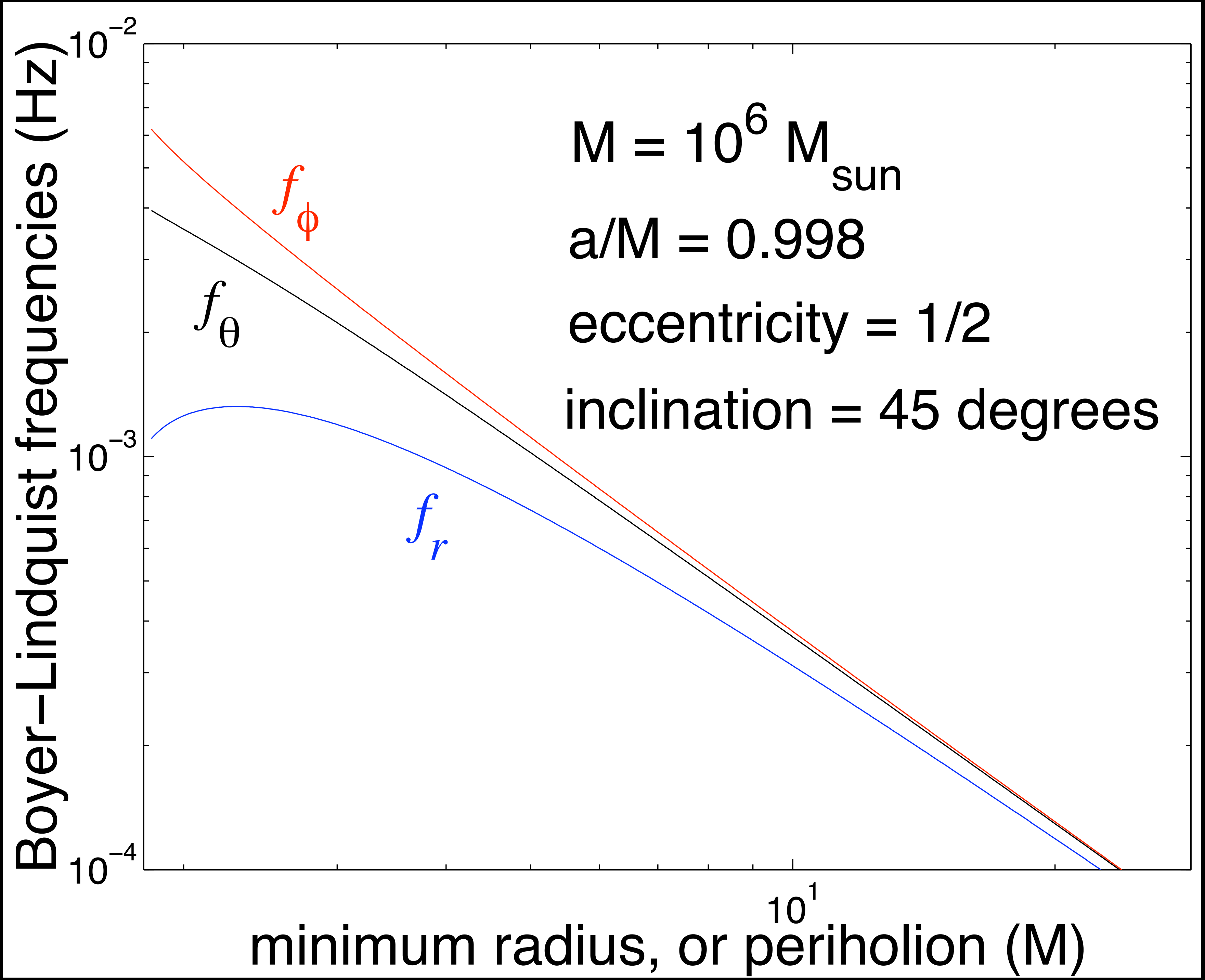
Orbit "shape" determined by three constants:

$(r_{\min}, r_{\max}, \iota)$ or (e, p, ι) or (E, L_z, Q) or...

Comparing a pair of generic orbits



Frequencies of generic orbits



Snapshots

Teukolsky-Sasaki-Nakamura formalism

- Source of perturbation: generic orbiting test mass
- Solution: discrete set of mode amplitudes

$$Z^{H,\infty}_{lmkn} = \frac{2\pi}{\Gamma}\left\langle J^{H,\infty}(r,\theta,\omega_{mkn})\right\rangle$$

$$\langle f(r,\theta)\rangle = \frac{1}{\Lambda_\theta\Lambda_r}\int_0^{\Lambda_\theta} d\lambda_\theta \int_0^{\Lambda_r} d\lambda_r\, f[r(\lambda_r),\theta(\lambda_\theta)]e^{ik\Upsilon_\theta\lambda_\theta + in\Upsilon_r\lambda_r}$$

$$\Lambda_{r,\theta} = \frac{2\pi}{\Upsilon_{r,\theta}}$$

Waveforms and fluxes

$$h_+ - ih_\times = -\frac{2}{r}\sum_{lmkn}\frac{Z^{\mathrm{H}}_{lmkn}}{\underline{\omega^2_{mkn}}}S_{lmkn}(\theta)e^{-i\omega_{mkn}(t-r)+im\phi}$$

$$\left\langle\frac{dE}{dt}\right\rangle = \sum_{lmkn}\frac{1}{4\pi\omega^2_{mkn}}\left(\underline{\left|Z^{\mathrm{H}}_{lmkn}\right|}^2 + \alpha_{lmkn}\underline{\left|Z^{\infty}_{lmkn}\right|}^2\right)$$

$$\left\langle\frac{dL_z}{dt}\right\rangle = \sum_{lmkn}\frac{m}{4\pi\omega^3_{mkn}}\left(\underline{\left|Z^{\mathrm{H}}_{lmkn}\right|}^2 + \alpha_{lmkn}\underline{\left|Z^{\infty}_{lmkn}\right|}^2\right)$$

Where α_{lmkn} is a simple quantity.

“Flux” of Carter

- Sago, Tanaka, Hikida, Nakano, to appear in PTP, or gr-qc/0506092

$$\left\langle \frac{dQ}{dt} \right\rangle^{\infty} = (\text{constants}) + 2 \sum_{lmkn} n\omega_r \left| Z^{\mathrm{H}}_{lmkn} \right|^2$$

due to gravitational perturbations.

- Drasco, Flanagan, Hughes, CQG special issue, or gr-qc/0505075

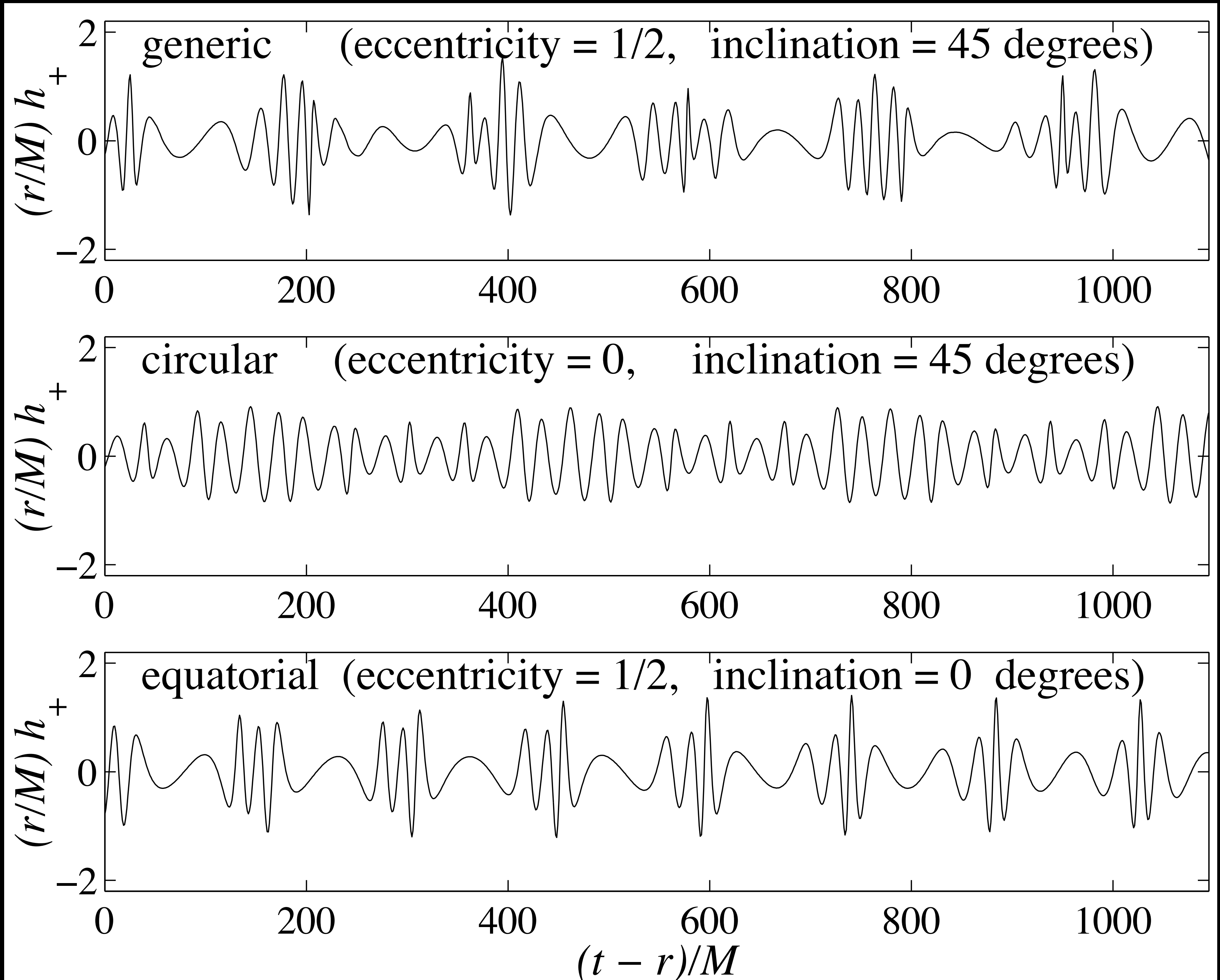
$$\left\langle \frac{dQ}{dt} \right\rangle = A \sum_{lmkn} \frac{\omega_{lmkn}}{|\omega_{lmkn}|} \left[(\tilde{Z}^{\mathrm{H}}_{lmkn})^* Z^{\mathrm{H}}_{lmkn} + \alpha_{lmkn} (\tilde{Z}^{\infty}_{lmkn})^* Z^{\infty}_{lmkn} \right]$$

due to scalar perturbations.
Here $\tilde{Z}^{\mathrm{H},\infty}_{lmkn}$ is as difficult to calculate as $Z^{\mathrm{H},\infty}_{lmkn}$.

History (frequency domain)

Ref.	$a \neq 0$	$e > 0$	$\iota \neq 0$	evolve
Cutler et al. (1994)		✓	(NA)	✓
Finn & Thorne (2000)	✓			✓
Shibata (1993)	✓		✓	
Shibata (1994)	✓	✓		
Hughes (2001)	✓		✓	✓
Glampedakis & Kennefick (2002)	✓	✓		✓
Drasco & Hughes (2005)	✓	✓	✓	

Comparing waveforms: $a = 0.9M,\ p = 4$



Radiation and orbit geometry

$$H = h_+ - ih_\times = \sum_{kn} H_{kn} e^{-i\omega_{kn}(t-r)}$$

$$\omega_{kn} = k\omega_\theta + n\omega_r$$

$H_{kn} \propto \delta_{n0}$ (circular orbits)

$H_{kn} \propto \delta_{0k}$ (equatorial orbits)

$H_{kn} \propto \delta_{0k}\delta_{n0}$ (circular-equatorial orbits)

Radial and polar voices

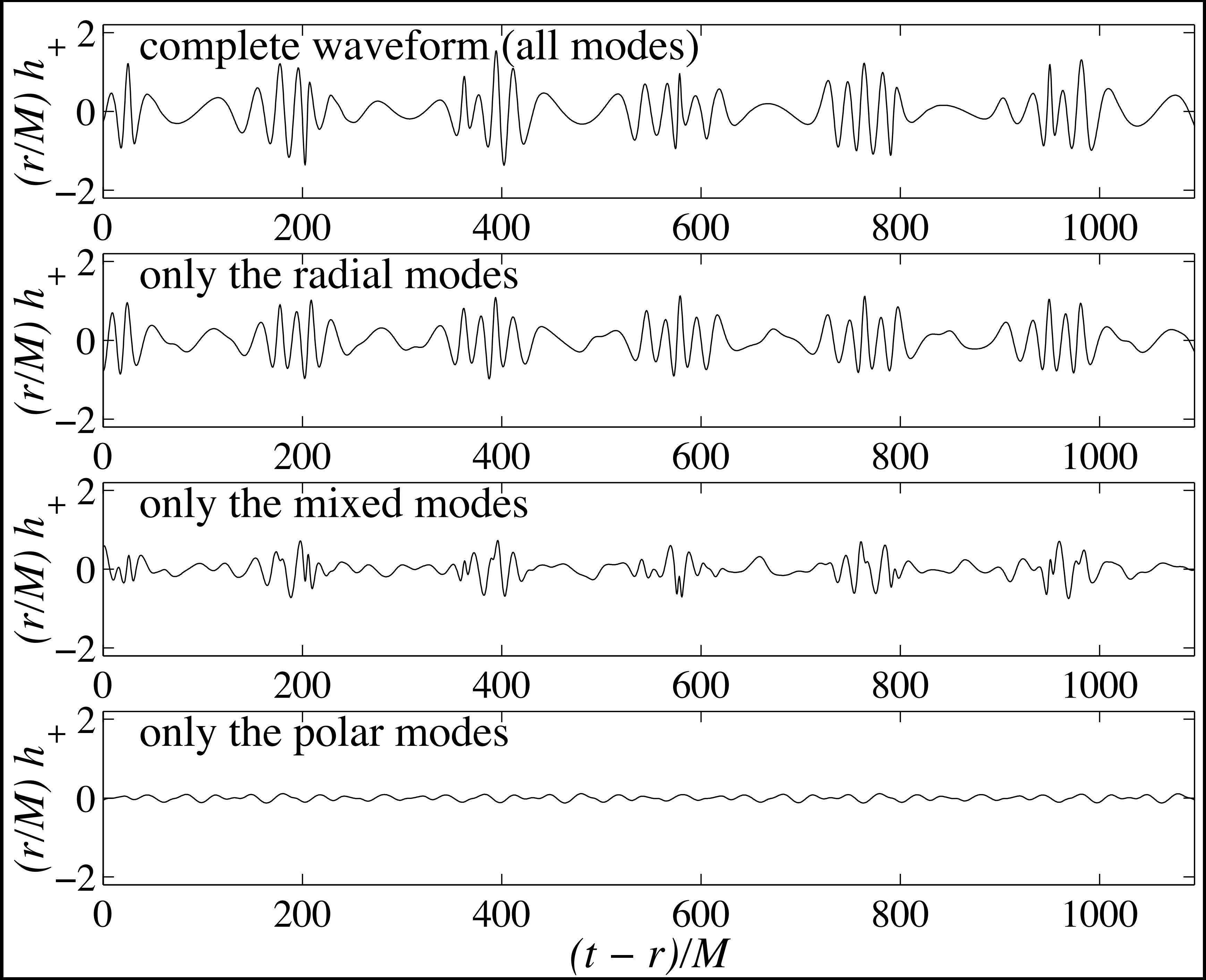
$$H = H_{\text{polar}} + H_{\text{radial}} + H_{\text{mixed}},$$

$$H_{\text{polar}} = \sum_{k \neq 0} H_{k0} e^{-i\omega_{k0} t}$$

$$H_{\text{radial}} = \sum_{n \neq 0} H_{0n} e^{-i\omega_{0n} t}$$

$$H_{\text{mixed}} = H_{00} + \sum_{k \neq 0} \sum_{n \neq 0} H_{kn} e^{-i\omega_{kn} t}$$

$e = 0.5,\ \iota = 45^\circ$



Radial and polar voices

$$e = 0.5,\ \iota = 45^\circ$$

	polar/total	radial/total	mixed/total
$\langle dE/dt \rangle$	1.2%	62%	37%
$\langle dL_z/dt \rangle$	0.77%	73%	26%

Increasing eccentricity excites radial harmonics
Increasing inclination excites polar hamonics (less so)

$a = 0.9M,\ p = 6,\ \theta_{\rm inc} = \iota$

e	$\theta_{\rm inc}$	$\frac{\langle dE/dt\rangle_{\rm radial}}{\langle dE/dt\rangle}$ (Radial)	$\frac{\langle dE/dt\rangle_{\rm polar}}{\langle dE/dt\rangle}$ (Polar)	$\frac{\langle dL_z/dt\rangle_{\rm radial}}{\langle dL_z/dt\rangle}$	$\frac{\langle dL_z/dt\rangle_{\rm polar}}{\langle dL_z/dt\rangle}$
0.1	20°	16%	8.3%	14%	5.0%
0.1	40°	13%	28%	14%	17%
0.1	60°	9.4%	48%	14%	29%
0.1	80°	6.7%	53%	20%	24%
0.3	20°	76%	1.9%	75%	1.3%
0.3	40°	58%	5.4%	67%	3.6%
0.3	60°	37%	5.9%	58%	2.7%
0.3	80°	17%	1.9%	51%	−2.4%
0.5	20°	90%	0.077%	93%	0.083%
0.5	40°	66%	0.24%	77%	0.18%
0.5	60°	39%	0.93%	58%	0.68%
0.5	80°	19%	1.6%	53%	−2.0%
0.7	20°	90%	0.057%	92%	0.10%
0.7	40°	67%	0.22%	77%	0.35%
0.7	60°	40%	0.19%	61%	−0.17%
0.7	80°	19%	0.16%	53%	−0.47%

Summed-spectra

$$\left\langle \frac{dX}{dt} \right\rangle_l^{\mathrm{H},\infty} = \sum_{mkn} \left\langle \frac{dX}{dt} \right\rangle_{lmkn}^{\mathrm{H},\infty}$$

$$\left\langle \frac{dX}{dt} \right\rangle_m^{\mathrm{H},\infty} = \sum_{lkn} \left\langle \frac{dX}{dt} \right\rangle_{lmkn}^{\mathrm{H},\infty}$$

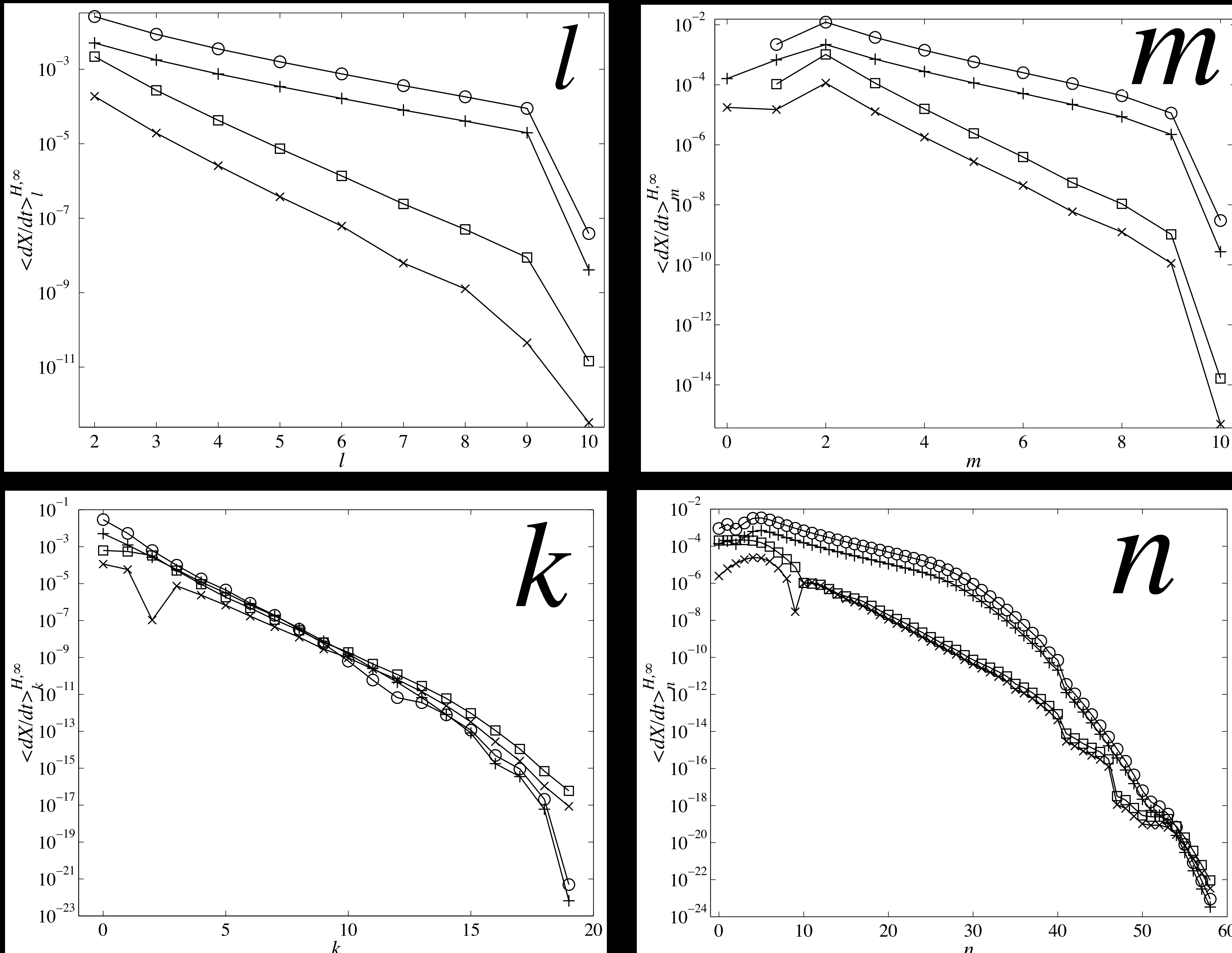
$$\left\langle \frac{dX}{dt} \right\rangle_n^{\mathrm{H},\infty} = \sum_{lmk} \left\langle \frac{dX}{dt} \right\rangle_{lmkn}^{\mathrm{H},\infty}$$

$$\left\langle \frac{dX}{dt} \right\rangle_k^{\mathrm{H},\infty} = \sum_{lmn} \left\langle \frac{dX}{dt} \right\rangle_{lmkn}^{\mathrm{H},\infty}$$

$$X = E, L_z$$

$$\left\langle \frac{dX}{dt} \right\rangle_\nu^{\mathrm{H},\infty} \propto \exp(-|\nu|\gamma_\nu^{\mathrm{H},\infty})$$

$$\nu = l, m, k, n$$



l
m
k
n
$<dX/dt>^{H,\infty}_l$
$<dX/dt>^{H,\infty}_m$
$<dX/dt>^{H,\infty}_k$
$<dX/dt>^{H,\infty}_n$

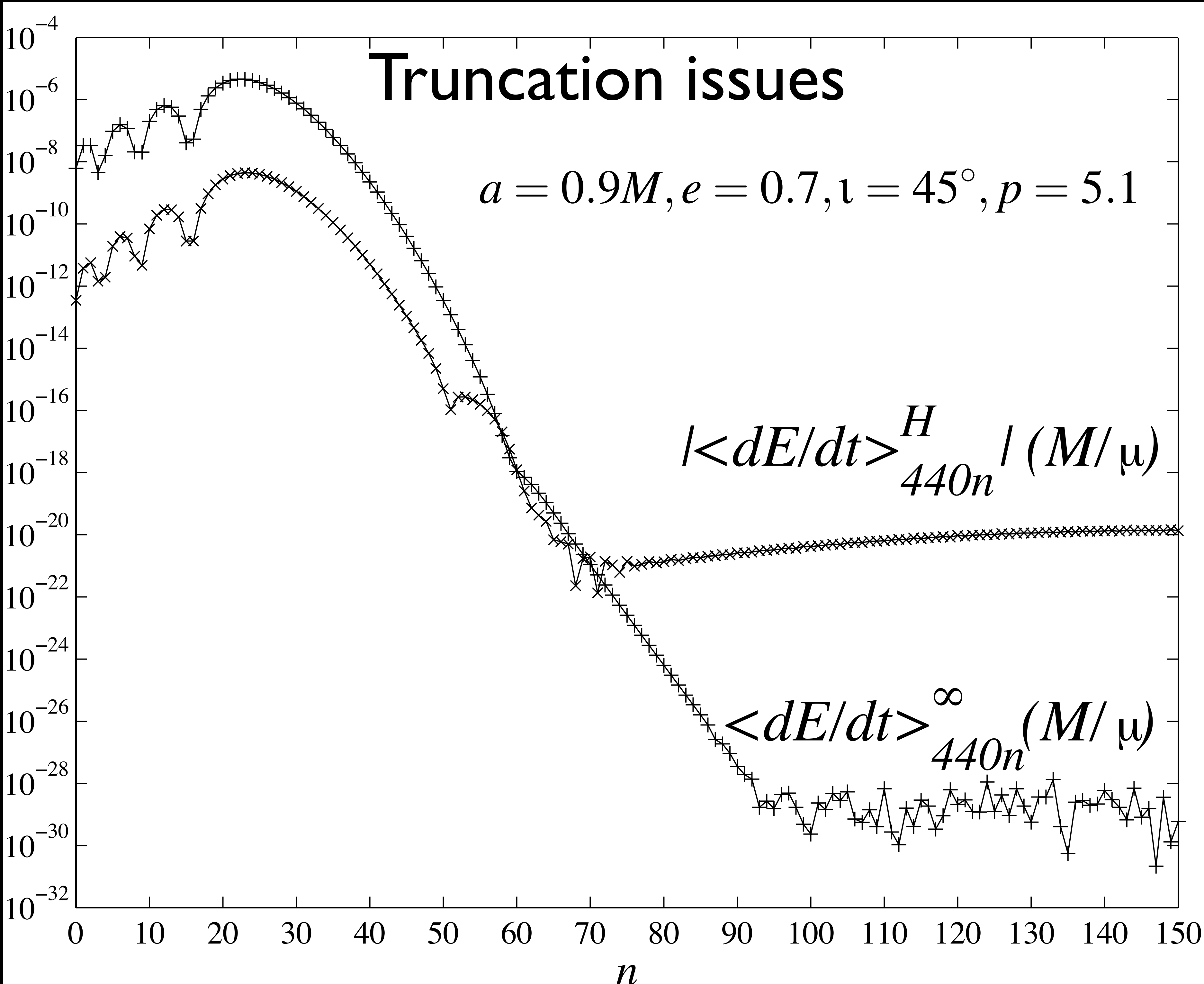
$a = 0.9M,\ p = 6,\ \theta_{\text{inc}} = \iota$

e	θ_{inc}	γ_l^{H}	γ_l^{∞}	γ_m^{H}	γ_m^{∞}	γ_k^{H}	γ_k^{∞}	γ_n^{H}	γ_n^{∞}
0.1	20°	3.2	1.4	2.4	1.0	4.6	4.2	5.8	4.1
0.1	40°	3.2	1.4	2.4	1.3	2.9	2.6	4.9	4.4
0.1	60°	3.1	1.4	2.4	1.6	2.2	2.0	1.6	2.0
0.1	80°	2.7	1.3	2.7	2.0	1.7	1.6	2.6	2.5
0.3	20°	2.9	1.2	2.3	0.92	4.2	3.7	2.6	2.2
0.3	40°	2.9	1.2	2.2	1.3	2.5	2.4	1.9	1.6
0.3	60°	2.7	1.2	2.3	1.4	1.8	1.8	1.5	1.6
0.3	80°	2.0	1.2	2.3	1.8	1.4	1.4	1.3	1.0
0.5	20°	2.7	1.0	2.2	0.81	4.2	3.9	1.1	1.0
0.5	40°	2.6	1.1	2.2	1.0	2.2	2.3	0.78	0.74
0.5	60°	2.6	1.1	2.1	1.3	1.5	1.7	0.72	0.64
0.5	80°	1.8	1.1	1.9	1.7	1.3	1.4	0.60	0.49
0.7	20°	2.9	1.4	2.3	0.92	3.1	3.4	0.40	0.45
0.7	40°	2.4	1.2	2.1	1.1	2.0	2.2	0.30	0.32
0.7	60°	1.8	1.4	2.1	1.4	1.3	1.5	0.27	0.26
0.7	80°	1.5	1.1	1.8	1.8	1.7	1.6	0.23	0.24

Polar

Radial

Future



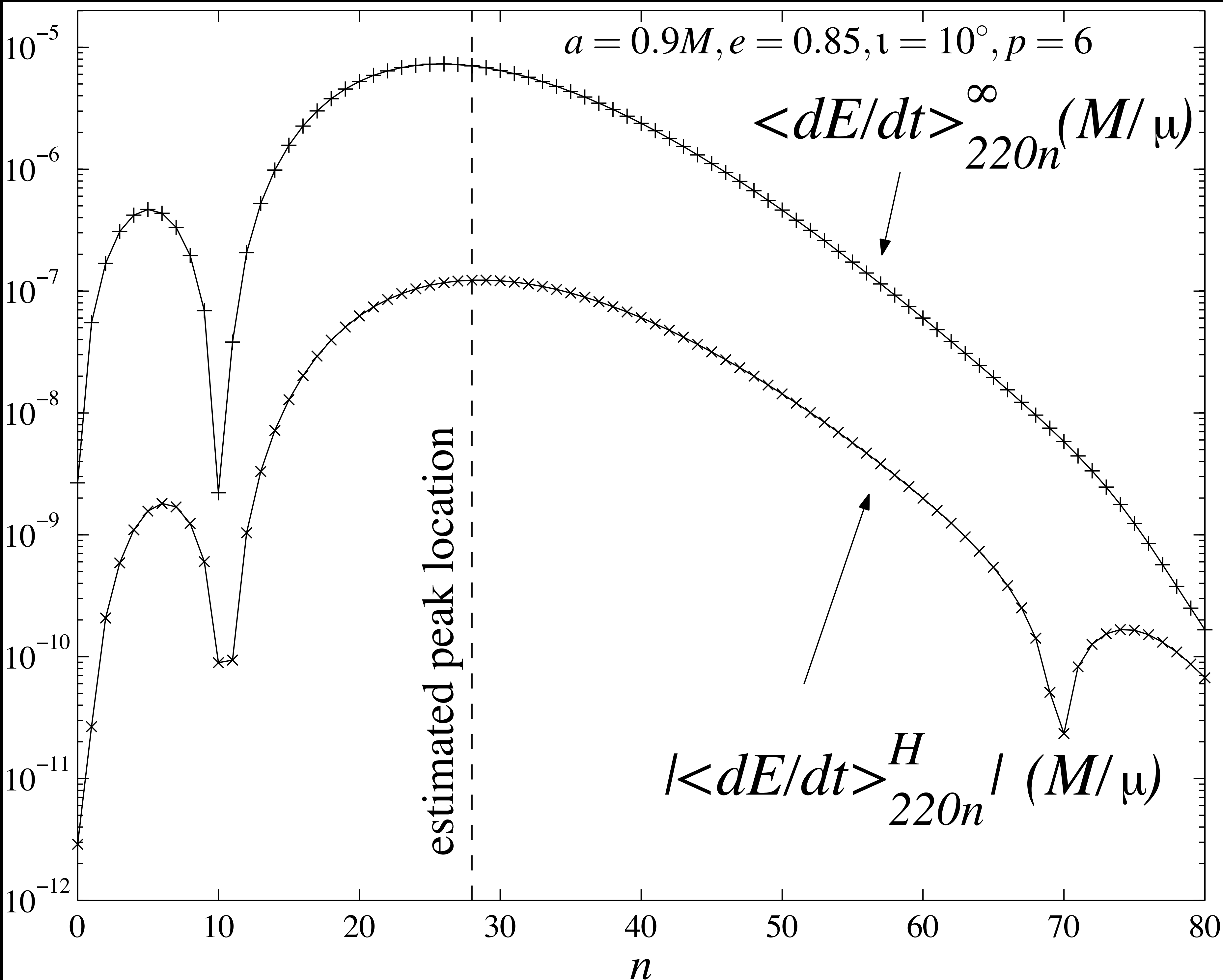
Truncation issues
$a = 0.9M, e = 0.7, \iota = 45^\circ, p = 5.1$
$|\langle dE/dt \rangle^{H}_{440n}|\,(M/\mu)$
$\langle dE/dt \rangle^{\infty}_{440n}(M/\mu)$
n

Making good guesses

- Gravitational wave spectrum for Keplarian waveforms
  Peters & Mathews, Phys. Rev. **131**, 435 (1963)

$$\left\langle \frac{dE}{dt} \right\rangle_n^{\mathrm{PM}} \propto \frac{n^4}{32}\Big\{\Big[J_{n-2}(ne) - 2eJ_{n-1}(ne) + \frac{2}{n}J_n(ne) + 2eJ_{n+1}(ne) - J_{n+2}(ne)\Big]^2 + (1-e^2)\left[J_{n-2}(ne) - 2J_n(ne) + J_{n+2}(ne)\right]^2 + \frac{4}{3n^2}[J_n(ne)]^2\Big\}$$

- $$n_{\mathrm{peak}} \approx \exp\left[\frac{1}{2} - \frac{3}{2}\ln(1-e)\right]$$



$a = 0.9M, e = 0.85, \iota = 10^\circ, p = 6$
$<dE/dt>^{\infty}_{220n}(M/\mu)$
$|<dE/dt>^{H}_{220n}|\ (M/\mu)$
estimated peak location
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
10^{-12}
0
10
20
30
40
50
60
70
80
n

Summary

- Generic snapshots are now available $e < 0.7$
- Resolve & implement Carter evolution
- Evolution will give generic adiabatic inspirals
- Optimization
- Evolution of initial conditions?
- immediate applications to astrophysics?